UNIVERSITÀ di VERONA

Runtime Complexity Analysis of Logically Constrained Rewriting

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University of Verona and University of Innsbruck

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Motivation: Runtime Analysis

```
mergesort = function
 | [] -> []
 | [x] -> [x]
 | x1 :: x2 :: xs ->
  let (l1,l2) = msplit (x1::x2::xs) in
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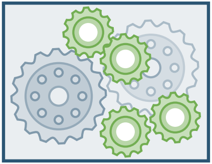
⟶



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- fully automatic worst-case runtime analysis
- support for full recursion, common data structures and types

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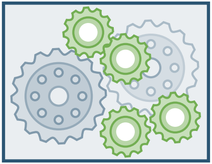

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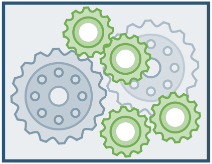

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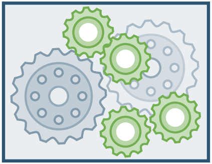

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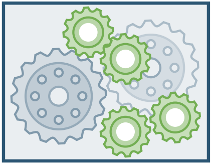
$\longrightarrow$  $\longrightarrow \mathcal{O}(n\log(n))$

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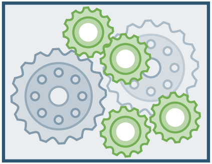

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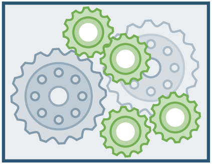

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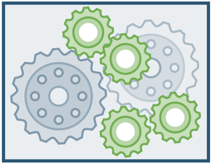

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- implementation in complexity tool $\mathrm{T_CT}$

Example 1: Integer Transition Systems

$$\mathsf{init}(x,y,z) \to \mathsf{sort}(x,y,z)$$

$$\mathsf{sort}(x,y,z) \to \langle \mathsf{ms}_0(x,u,v), \mathsf{ms}_1(x,u,v), \mathsf{ms}_2(x,u,v), \mathsf{ms}_3(x,u,v)\rangle$$
$$[x \geqslant 2 \wedge u \geqslant 0 \wedge v \geqslant 0 \wedge x+1 \geqslant 2u \wedge 2u \geqslant x \wedge x \geqslant 2v \wedge 2v+1 \geqslant x]$$

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- new $\mathsf{T_CT}$ version derives optimal $\mathcal{O}(n \log(n))$ bound
  (CoFloCo, KoAT, PUBS, and previous version of $\mathsf{T_CT}$ at best quadratic)

Example 2: Logic Programs

$$\mathsf{max\_length}(ls, m, l) \to \langle \mathsf{max}(ls, 0, m), \mathsf{len}(ls, l)\rangle$$

$$\mathsf{len}(xs, l) \to \mathsf{len}(t, l - 1)\ [xs \approx h :: t] \qquad \mathsf{len}([], 0) \to \langle\rangle$$

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- can use approach to analyze (constraint) logic programs
- new version of $\mathsf{T_CT}$ can handle LCTRSs corresponding to deterministic Prolog programs over integers and lists
- techniques known to extend to decomposable non-deterministic programs

J. Giesl, T. Ströder, P. Schneider-Kamp, F. Emmes, C. Fuhs.
Symbolic evaluation graphs and term rewriting: a general methodology for analyzing logic programs.
Proc. PPDP 2012, pp. 1–12, 2012.

Example 3: Simplification Systems

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int foo(int z) {
  int x = 4 * (z | 101);
  return -256 ^x;
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$\overset{*}{\dashrightarrow}$

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define i32 @foo(i32) #0 {
  %2 = or i32 %0, 101
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Instcombine →

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```
01011111010101011011010
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```

$$\mathsf{mul}(x, C_1) \to \mathsf{shl}(x, C_2)\ [\ \mathsf{log2}\ (C_1) = C_2 \land \mathsf{isPowerOf2}\ (C_1)\ ]$$

Expression simplifications in compilers

- e.g. in LLVM: multiplications to shifts, reordering bitwise operations, ...

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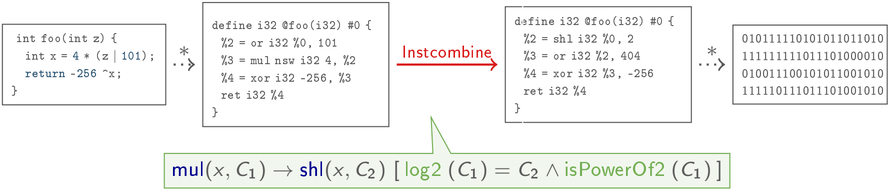

Expression simplifications in compilers

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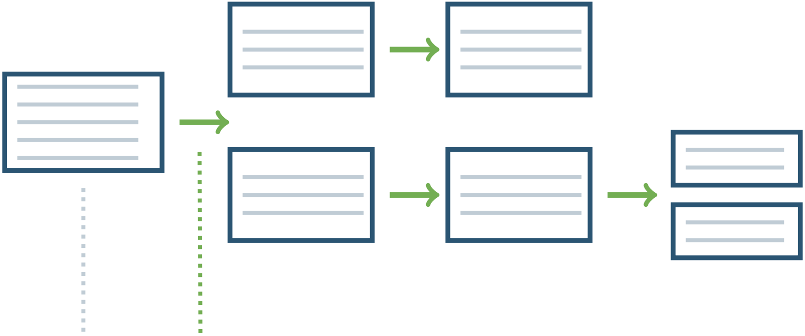
- can be modeled as LCTRS
- complexity crucial
  (current work is first step: derivational complexity needed)

How? Divide and Conquer



problem processor

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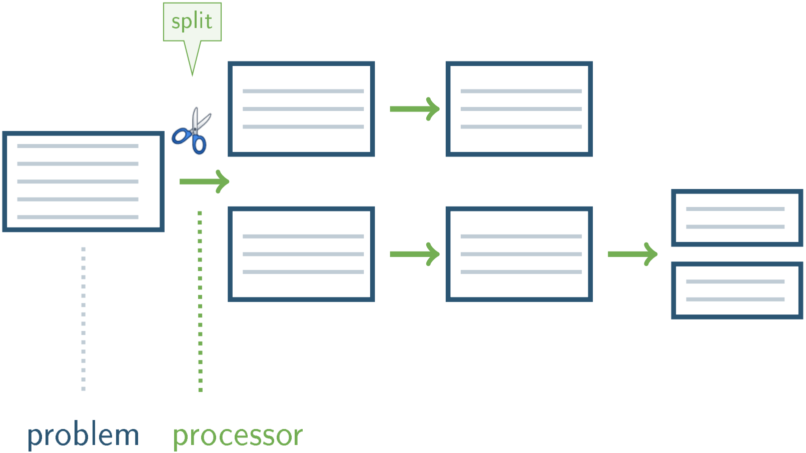

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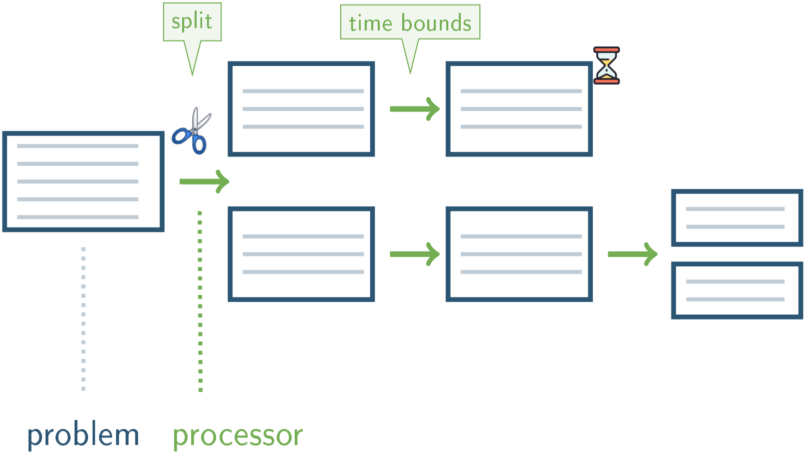

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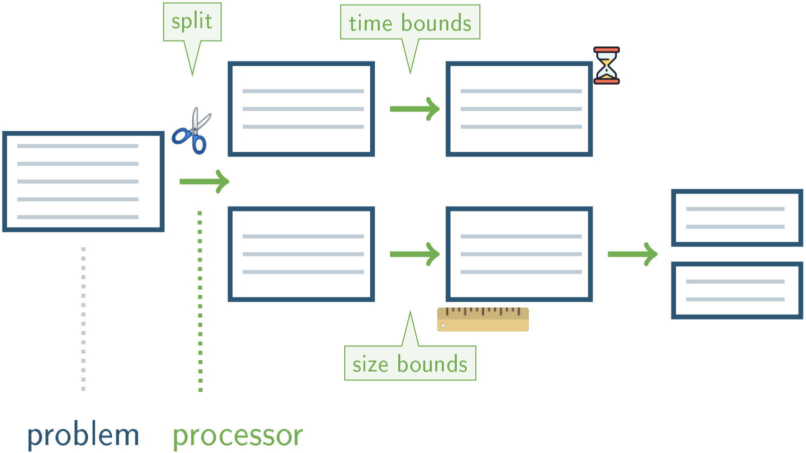

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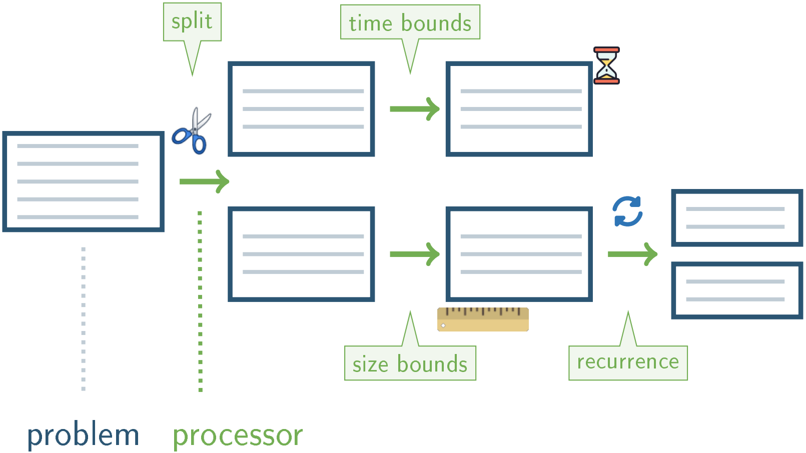

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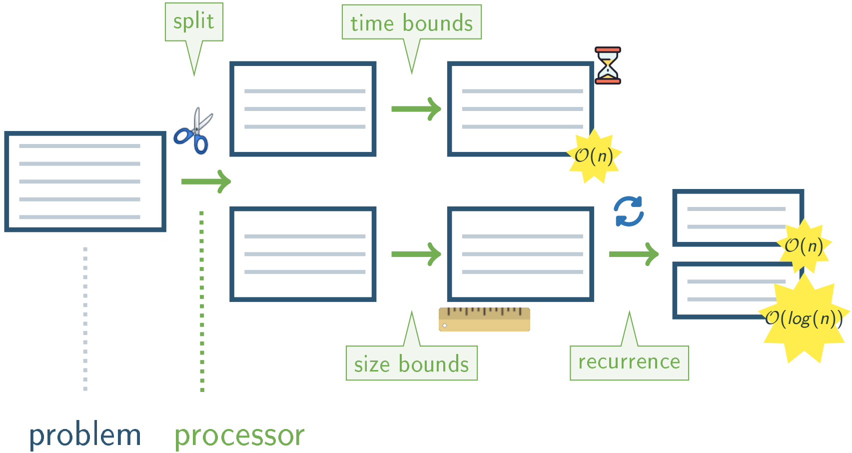

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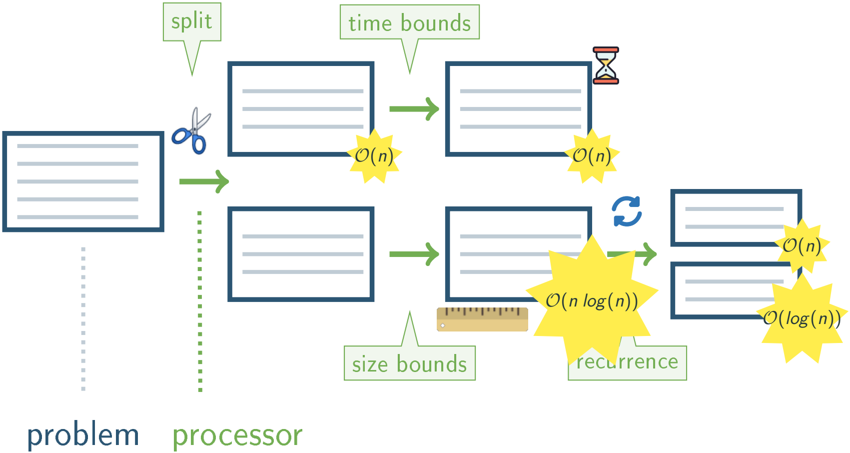

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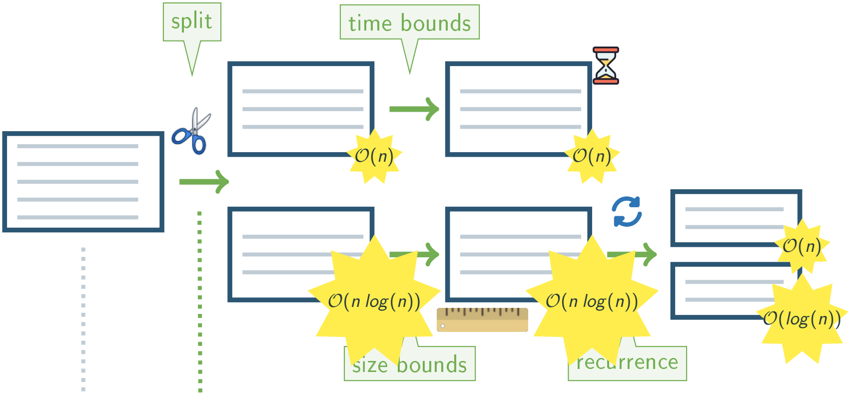

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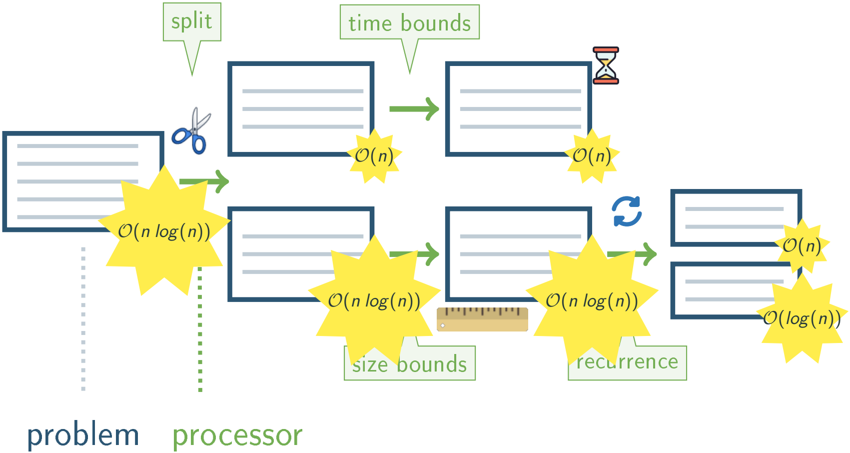

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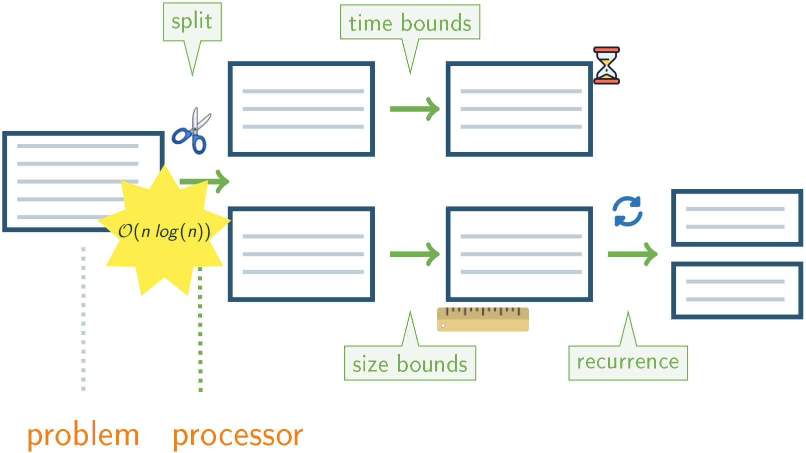

M. Avanzini and G. Moser.
A combination framework for complexity.
Inf. Comput., 248:22–55, 2016.

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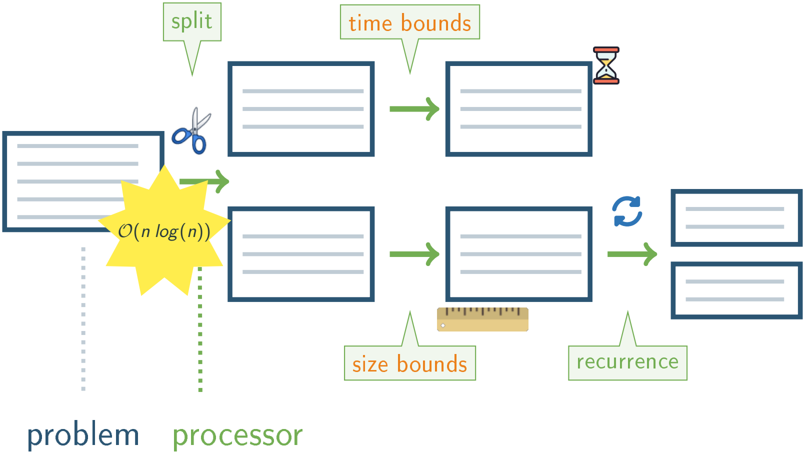

M. Brockschmidt, F. Emmes, S. Falke, C. Fuhs, and J. Giesl.
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Example

$$\boxed{\mathsf{init}^{\#}(x) \to \mathsf{f}^{\#}(x)}$$

$$\boxed{\mathsf{f}^{\#}(y) \to \langle \mathsf{f}^{\#}(y-1), \mathsf{g}^{\#}(y) \rangle\ [y \geqslant 0]}$$

$$\boxed{\mathsf{g}^{\#}(z) \to \mathsf{g}^{\#}(z/2)\ [z \geqslant 0]}$$

Definition (Dependency tuples)

- $t^{\#}$ is obtained from term t by marking root symbol by $\#$
- dependency tuple (DT) of $\ell \to r\ [c]$ is

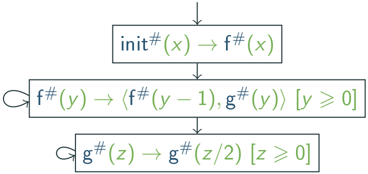
$$\ell^{\#} \to \langle r_1^{\#}, \ldots, r_k^{\#} \rangle\ [c]$$

  where $r_1, \ldots, r_k$ are all recursive calls in r
- set of dependency tuples of LCTRS $\mathcal{R}$ is denoted $\mathsf{DT}(\mathcal{R})$

Definition (Dependency graph)

- node set $\mathsf{DT}(\mathcal{R})$ for LCTRS $\mathcal{R}$
- edge from $s^{\#} \to \langle \ldots, t^{\#}, \ldots \rangle\ [\varphi]$ to $u^{\#} \to v^{\#}\ [\psi]$ if $t^{\#}\sigma \to_{\mathcal{R}}^{*} u^{\#}\tau$

Example



Contents

Bound expressions

$$\text{UB} ::= |x| \mid \text{UB} + \text{UB} \mid \text{UB} \cdot \text{UB} \mid \max(\text{UB}, \text{UB}) \mid \text{UB}^k \mid \log_k(\text{UB}) \mid \omega$$

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measure $|x| \in \mathbb{N}$ for all input variables x

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Time bounds and size bounds

for LCTRS $\mathcal{R}$, let $\rho\colon \ell \to r\ [c] \in \mathsf{DT}(\mathcal{R})$ and consider rewrite sequence:

$$\mathsf{init}^{\#}(x_1, \ldots, x_n) \to \ldots \xrightarrow[\rho]{\sigma_1}\to \ldots \xrightarrow[\rho]{\sigma_2}\to \ldots \xrightarrow[\rho]{\sigma_m}\to \ldots \qquad (\star)$$

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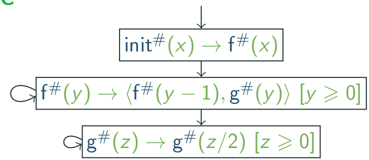

$T = 1$

$T = |x|$

$T = |x|^2$

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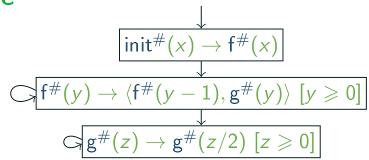

$T = 1$	$S(x) = \lvert x \rvert$
$T = \lvert x \rvert$	$S(y) = \lvert x \rvert$
$T = \lvert x \rvert^2$	$S(z) = \lvert x \rvert$

Definitions

given LCTRS $\mathcal{R}$,

- complexity problem is $P = (t_0, \mathcal{D}, \mathcal{R})$ for initial term t_0 and DTs $\mathcal{D}$

Processor Framework

Definitions

given LCTRS $\mathcal{R}$,

- complexity problem is $P = (t_0, \mathcal{D}, \mathcal{R})$ for initial term t_0 and DTs $\mathcal{D}$
- judgement $\vdash P\colon (T, S)$ states time bounds T and size bounds S for P

Processor Framework

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given LCTRS $\mathcal{R}$,

- complexity problem is $P = (t_0, \mathcal{D}, \mathcal{R})$ for initial term t_0 and DTs $\mathcal{D}$
- judgement $\vdash P : (T, S)$ states time bounds T and size bounds S for P
- processor Proc is inference rule on complexity judgements

$$\frac{\vdash P_1 : (T_1, S_1), \ldots, \vdash P_k : (T_k, S_k)}{\vdash P : (T, S)} \quad \text{Proc}$$

Processor Framework

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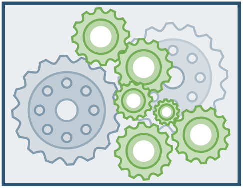
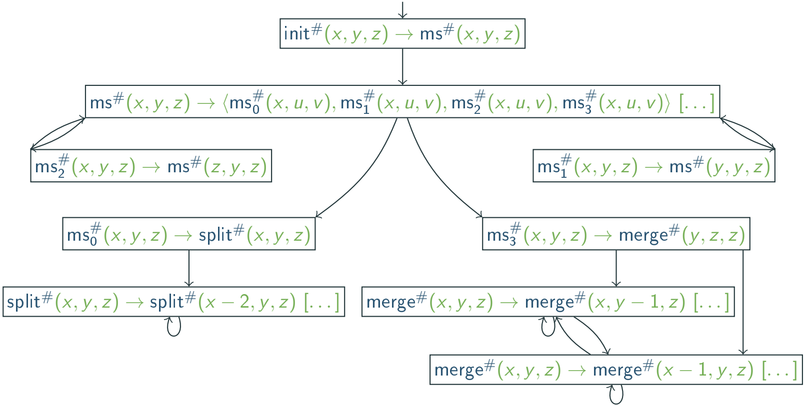
Processors



- interpretations
- time bounds
- size bounds
- splitting — new
- recurrence — new
- chaining, simplification

Example (Bounding mergesort)



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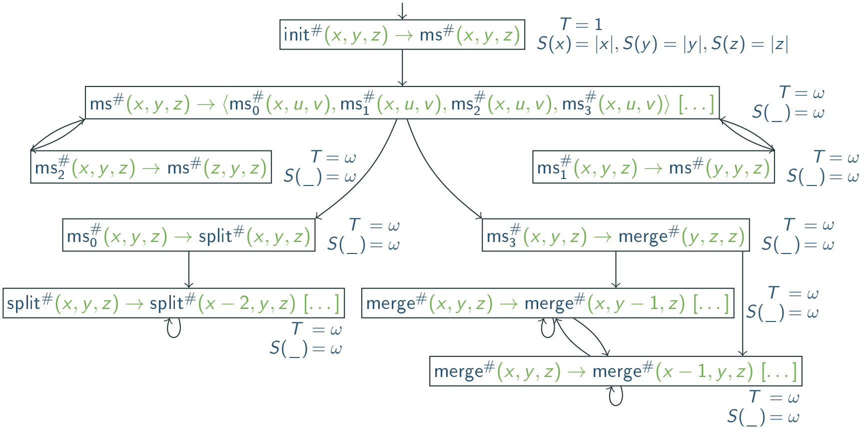
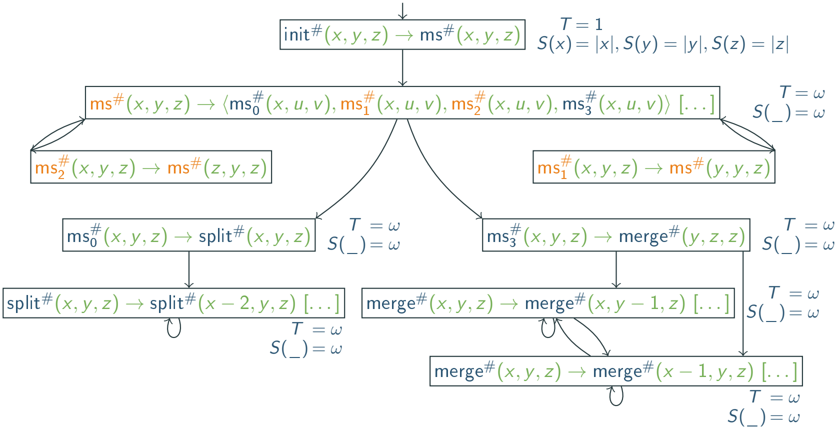


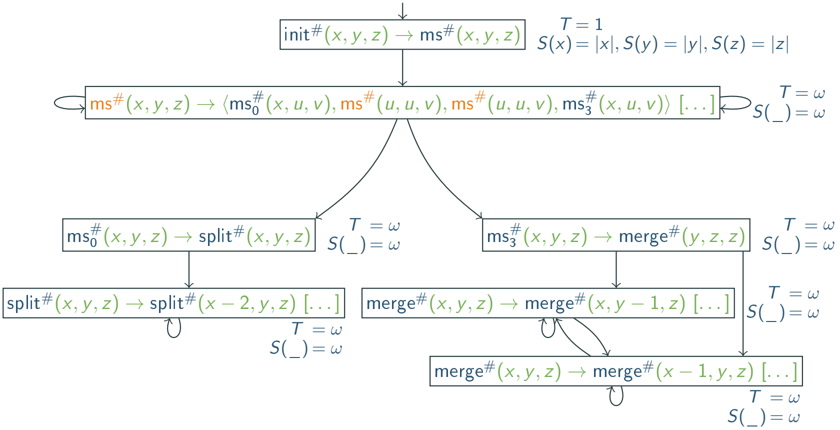
initial problem

Example (Bounding mergesort)



1 Chaining processor

Example (Bounding mergesort)



1 Chaining processor

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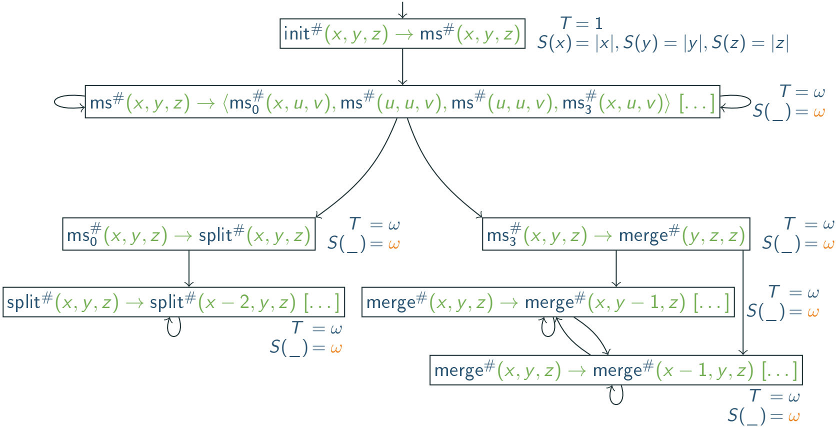
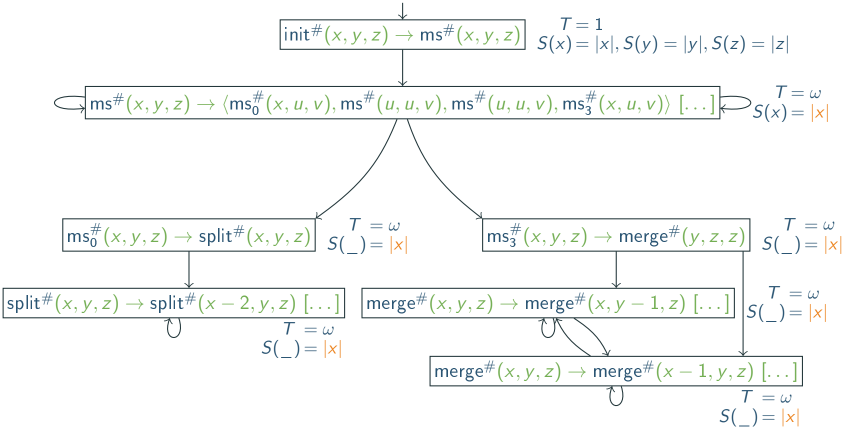


1. Chaining processor
2. Size bounds processor

Example (Bounding mergesort)



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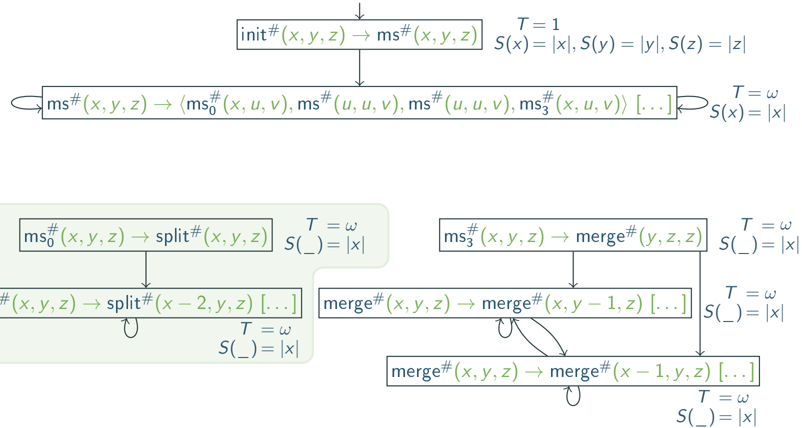
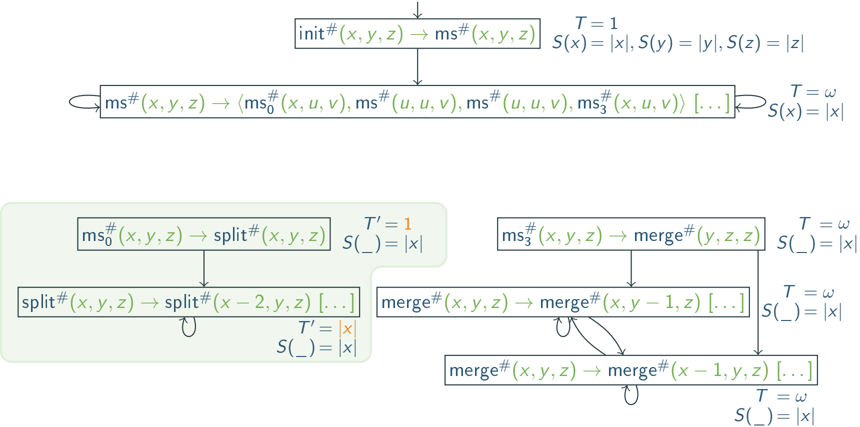


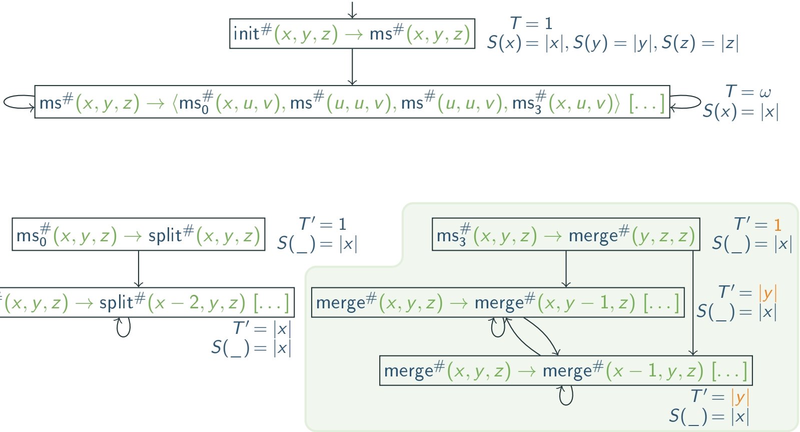
1. Chaining processor
2. Size bounds processor
3. Split processor

Example (Bounding mergesort)



1. Chaining processor
2. Size bounds processor
3. Split processor
4. Interpretation processor

Example (Bounding mergesort)



1. Chaining processor
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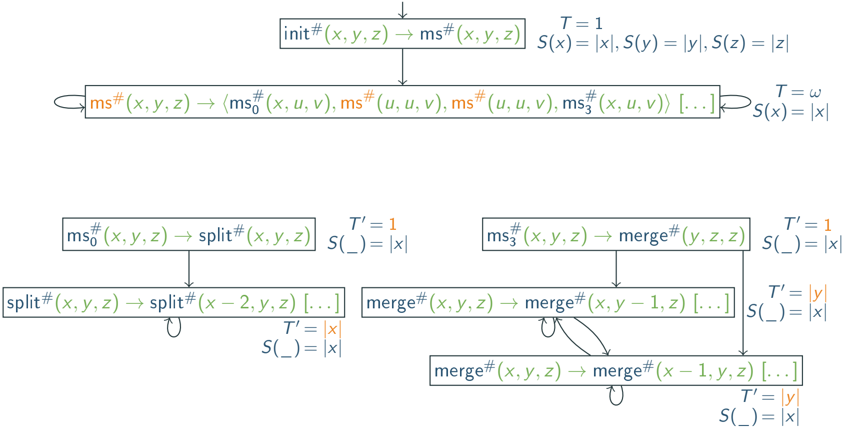
4. Interpretation processor
5. Time bounds processor+

Example (Bounding mergesort)



1. Chaining processor
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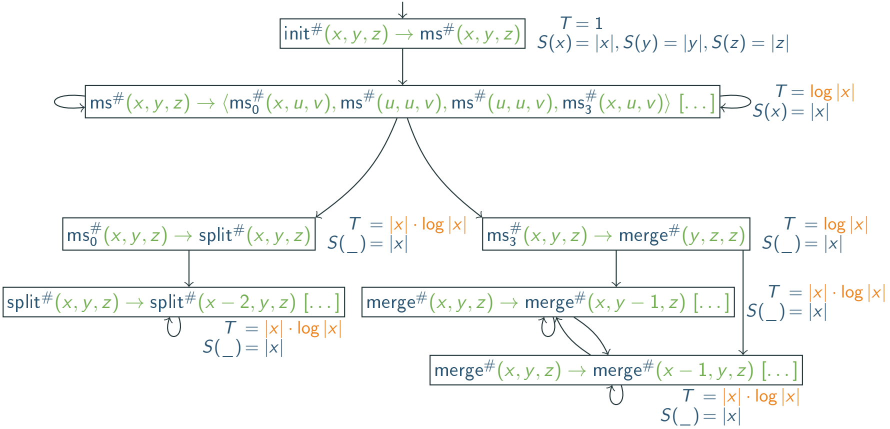
3. Split processor
4. Interpretation processor
5. Time bounds processor$^+$
6. Recurrence processor

Example (Bounding mergesort)



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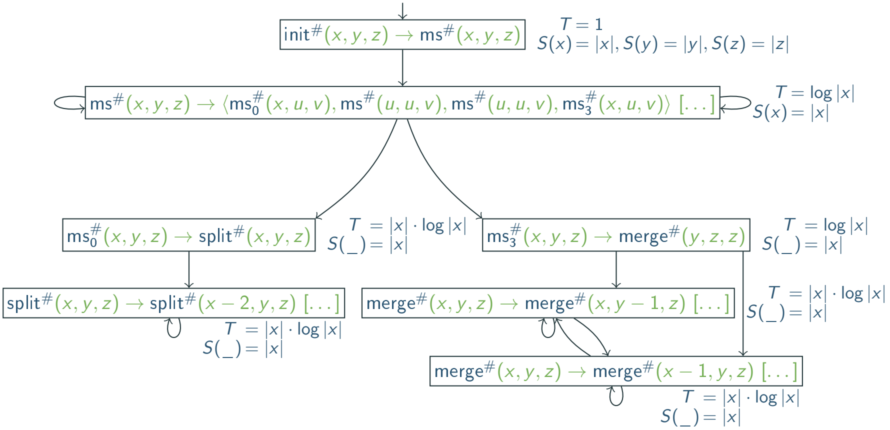

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5. Time bounds processor$^+$
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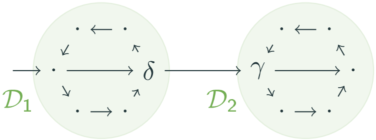
$$\sum T \in \mathcal{O}(|x| \cdot \log |x|)$$

Splitting Processor

Let $P = (t_0, \mathcal{D}, \mathcal{R})$ have dependency graph of shape



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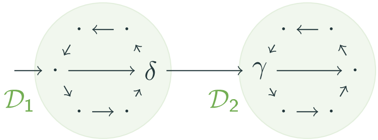

Then the following processor is sound

$$\frac{\vdash P \colon (T, S)}{} \ \text{Split}$$

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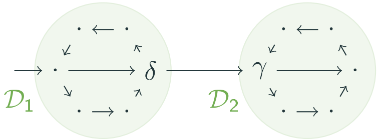

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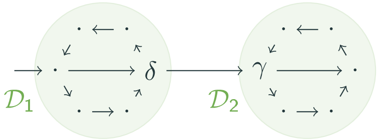

Then the following processor is sound, where $\gamma\colon \ell \to r\ [\psi]$

$$\frac{\vdash (t_0, \mathcal{D}_1, \mathcal{R})\colon (T_1, S_1) \quad \vdash (\ell, \mathcal{D}_2, \mathcal{R})\colon (T_2, S_2)}{\vdash P\colon (T, S)}\ \text{Split}$$

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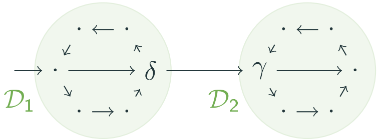

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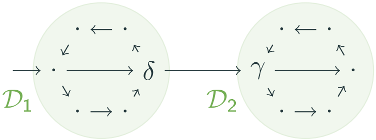

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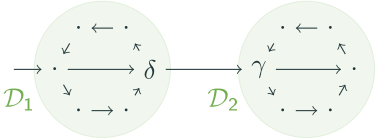

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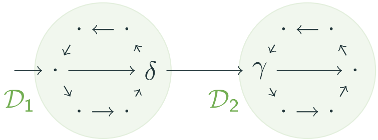

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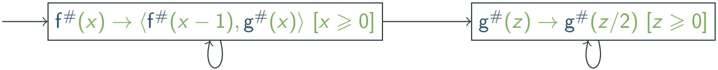

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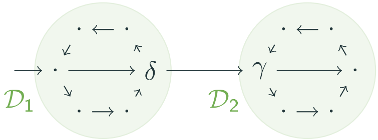

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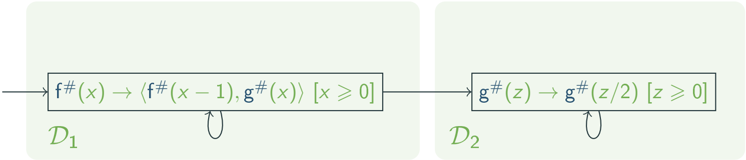

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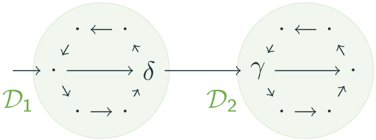

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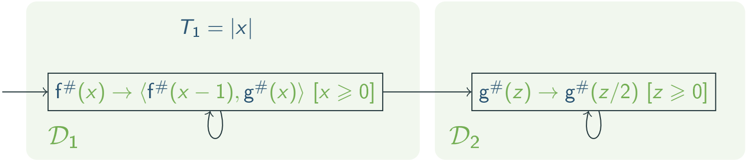

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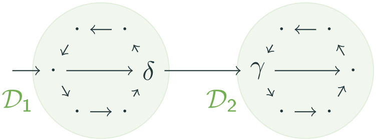
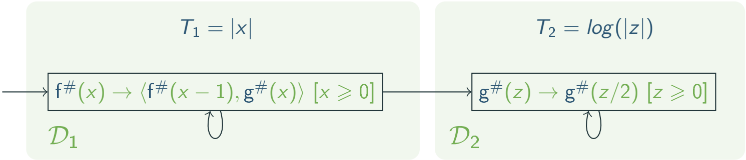


Then the following processor is sound, where $\gamma\colon \ell \to r\ [\psi]$

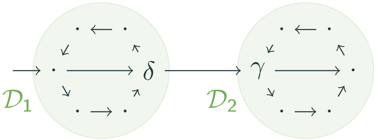
$$\frac{\vdash P\colon (T, S) \quad \vdash (t_0, \mathcal{D}_1, \mathcal{R})\colon (T_1, S_1) \quad \vdash (\ell, \mathcal{D}_2, \mathcal{R})\colon (T_2, S_2)}{\vdash P\colon (\lambda\rho. \begin{cases} T_1(\rho) & \text{if } \rho \in \mathcal{D}_1 \\ T(\delta) \cdot T_2(\rho)(S(\gamma, y_1), \ldots, S(\gamma, y_k)) & \text{if } \rho \in \mathcal{D}_2 \end{cases}, S)} \text{ Split}$$

Example



Splitting Processor

Let $P = (t_0, \mathcal{D}, \mathcal{R})$ have dependency graph of shape



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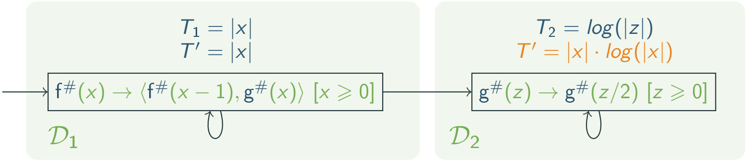
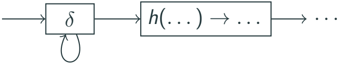


Recurrence Processor

For complexity problem $P = (f(\vec{x}), \mathcal{D}, \mathcal{R})$ of form



$$\delta : \quad f(\vec{x}) \to \langle f(\vec{r}), h(\vec{t}) \rangle \quad [\psi \wedge \vec{x} \geqslant \vec{b}]$$

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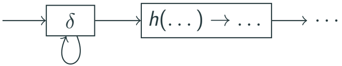

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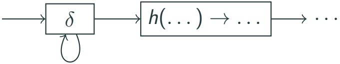

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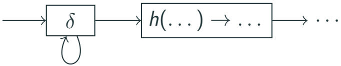

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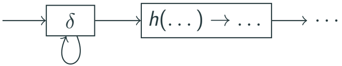

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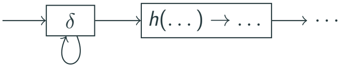
For complexity problem $P = (f(\vec{x}), \mathcal{D}, \mathcal{R})$ of form

form can be generalized



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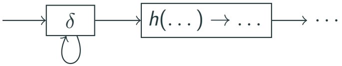

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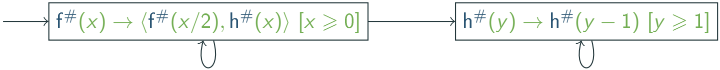

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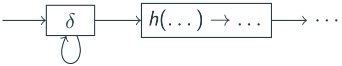

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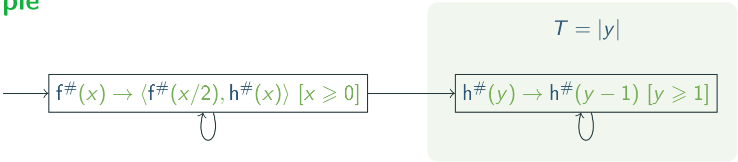

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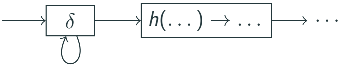

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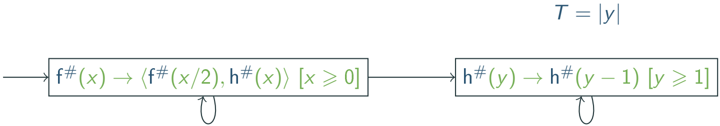

$F = |x| \cdot log(|x|)$ is solution to $f(x) = f(|x|/2) + |x|$, $f(1) = 1$

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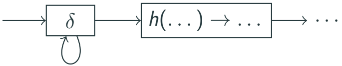

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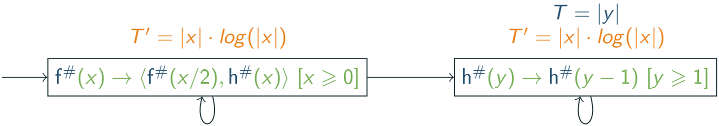

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