

The Semantics of Modal Predicate Logic II. Modal Individuals Revisited

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Abstract. We continue the investigations begun in [10]. We shall define a semantics that is build on a new notion of frame, called *coherence frames*. In these frames, objects are transcendental (world-independent), as in the standard constant-domain semantics. This will remove the asymmetry between constants and variables of the counterpart semantics of [10]. We demonstrate the completeness of (general) coherence frames with respect to first- and certain weak second-order logics as well as compare this notion of frame to counterpart frames as introduced in [10] and metaframe semantics, [13].

§1. Introduction. In [10] we have developed a semantics that is complete with respect to first- and weak second-order modal predicate logics. This semantics was in addition quite elementary, which was already a great step forward from the previous semantics by Ghilardi [6] and by Skvortsov and Shehtman [13]. Still, from a philosophical point of view this semantics left much to be desired. The introduction of counterpart relations, although in line with at least some philosophical ideas — notably by Lewis —, is not always very satisfactory, since it makes the notion of an object a derived one. The things we see become strictly world bound: there is no sense in which we can talk of say, the town hall of Berlin, rather than the town hall of Berlin at a particular point of time. The traditional semantics for modal predicate logic held the complete opposite view. Here, objects are transcendental entities. They are not world bound, since they do not belong to the worlds. The difference between these views becomes clear when we look at the way in which the formula $\Diamond\varphi(\vec{x})$ is evaluated. In the standard semantics, we simply go to some accessible world and see whether $\varphi(\vec{x})$ holds. In counterpart semantics, we would not only have to choose another world but also some counterparts for the things that we have chosen as values in this world. In the traditional semantics the question of counterparts does not arise because of the transcendental status of objects. We may view this as a limiting case of counterpart semantics, in which the counterpart relation always is the identity.

Note that the addition of *constant symbols* to the language introduces further complications. In counterpart-theoretical semantics, it is far from straightforward to interpret constant symbols, because we need to give an interpretation of these symbols across possible worlds that respects the counterpart relations in some appropriate sense. Variables on the other hand simply denote “objects” in the domain of a given world. In the case of traditional semantics this asymmetry appears in a similar fashion if one allows constant symbols to be non-rigid, as has been done e.g. in [5]. Then, variables denote transcendental entities, whereas constants denote something like *individual concepts*, i.e functions from possible worlds to a domain. Facing this dilemma, one solution to this is to completely move to a higher-order setting, where constants and variables can be of various higher types, e.g. type-0 constants denote objects, type-1 constants individual concepts etc. (cf. [4]). In this paper, we will follow a different approach, treating constants and variables in the same way, but assuming a more sophisticated notion of modal individual and identity-at-a-world.

It remains unsatisfactory having to choose between these competing semantics. Moreover, it would be nice if the difference between these semantics were better understood. Certainly, much research has been done into standard semantics and it is known to be highly incomplete if one aims for frame-completeness results.

Meeting

However, it is known that completeness with respect to models is as easy to show as in predicate logic but that if the language contains equality, different semantics have to be chosen for different theories/logics of identity (cf. e.g. [7]). The present paper developed from the insight that if the proper semantics is introduced, modal predicate logics with different logics of identity can be treated within the same semantical framework. We call this semantics *coherence semantics*. Completeness with respect to models is then uniformly shown for all modal predicate logics that are extensions of free quantified \mathbf{K} together with the predicate logical axioms of equality. We continue by investigating the relationships between coherence frames, counterpart frames and metaframe semantics, discuss the treatment of identity in each of the semantics as well as the interpretation of constant symbols and finally derive a completeness result for so-called cubic generalized metaframes.

§2. Preliminaries. The language has the following symbols. Following Scott [12] we shall work with nonobjectual (possibilist) quantifiers plus an existence predicate. This allows to eliminate the objectual (actualist) quantifiers (they are now definable), and straightens the theory considerably. The existence predicate is a unary predicate whose interpretation — unlike the identity symbol — is completely standard, i.e. doesn't have to meet extra conditions. Hence it can actually be suppressed in the notation, making proofs even more simple.

DEFINITION 1 (Symbols and Language). *The languages of modal predicate logic contain the following symbols.*

1. A denumerable set $V := \{x_i : i \in \omega\}$ of **object variables**.
2. A denumerable set $C := \{c_i : i \in \omega\}$ of **constants**.
3. A set Π of **predicate symbols** containing the unary **existence predicate** E .
4. **Boolean functors** \perp, \wedge, \neg .
5. **Quantifiers** \forall, \exists .
6. A set $M := \{\Box_\lambda : \lambda < \kappa\}$ of **modal operators**.

Furthermore, each symbol from Π has an arity, denoted by $\Omega(P)$. In particular, $\Omega(E) = 1$.

The variables are called $x_i, i \in \omega$. We therefore use x (without subscript!), y, y_j or z, z_k , as metavariables. We assume throughout that we have no function symbols of arity greater than 0. However, this is only a technical simplification. Notice that in [10] we even had no constants. This was so because the treatment of constants in the counterpart semantics is a very delicate affair, which we will discuss below. Moreover, for simplicity we assume that there is only one modal operator, denoted by \Box rather than \Box_0 . Nothing depends on this choice in fact. The standard quantifiers \forall and \exists are treated as abbreviations.

$$\begin{aligned} (\forall y)\varphi &:= \bigwedge y.E(y) \rightarrow \varphi \\ (\exists y)\varphi &:= \bigvee y.E(y) \wedge \varphi \end{aligned}$$

Moreover, $\Diamond\varphi$ abbreviates $\neg\Box\neg\varphi$. The sets of *formulae* and *terms* in this language are built in the usual way. Unless otherwise stated, equality (\doteq) is *not* a symbol of the language.

DEFINITION 2 (First-Order MPL). *A **first-order modal predicate logic** is a set L of formulae satisfying the following conditions.*

1. L contains all instances of axioms of first-order logic.
2. L is closed under all rules of first-order logic.
3. L contains all instances of axioms of the modal logic \mathbf{K} .
4. L is closed under the rule $\varphi/\Box\varphi$.
5. $\Diamond\bigvee y.\varphi \leftrightarrow \bigvee y.\Diamond\varphi \in L$.

To eliminate some uncertainties we shall note that the notions of free and bound occurrences of a variable are exactly the same as in ordinary first-order logic. A variable x occurs **bound** if this occurrence is in the scope of a quantifier $\bigwedge x$ or $\bigvee x$. We denote the simultaneous replacement

of the terms s_i for x_i ($i < n$) in χ by $[s_0/x_0, \dots, s_{n-1}/x_{n-1}]\chi$. Or, writing $\vec{s} = \langle s_i : i < n \rangle$ and $\vec{x} = \langle x_i : i < n \rangle$, we abbreviate this further to $[\vec{s}/\vec{x}]\chi$.

If the language contains equality, then the following is required of L .

$$\text{Eq1. } \bigwedge x.x \doteq x \in L.$$

$$\text{Eq2. } \bigwedge x.\bigwedge y.x \doteq y \rightarrow y \doteq x \in L.$$

$$\text{Eq3. } \bigwedge x.\bigwedge y.\bigwedge z.x \doteq y \wedge y \doteq z \rightarrow x \doteq z \in L.$$

$$\text{Eq4. } \bigwedge y_0.\bigwedge y_1.\dots.\bigwedge y_n.y_i \doteq y_n \rightarrow \{P(y_0, \dots, y_{n-1}) \leftrightarrow [y_n/y_i]P(y_0, \dots, y_{n-1})\} \in L, \text{ where } P \in \Pi, n = \Omega(P).$$

We start with a very basic semantics, standard constant–domain semantics. We assume no equality.

DEFINITION 3 (Frames, Structures and Models). A **predicate Kripke–frame** is a triple $\langle W, \triangleleft, U \rangle$, where W is a set (the set of **worlds**), $\triangleleft \subseteq W \times W$ a binary relation on W (the **accessibility relation**), and U a set (the **universe**). A **modal first–order structure** is a quadruple $\langle W, \triangleleft, U, \mathfrak{I} \rangle$, where $\langle W, \triangleleft, U \rangle$ is a predicate Kripke–frame and \mathfrak{I} a function mapping a predicate P to a function assigning to each world w an $\Omega(P)$ –relation on U and a constant symbol c to a member of U . \mathfrak{I} is called an **interpretation**. Further, \mathfrak{I}_w is the relativized interpretation function at w , which assigns to each $P \in \Pi$ the value $\mathfrak{I}(P)(w)$ and to each constant symbol c the value $\mathfrak{I}(c)(w)$. A **valuation** is a function $\beta : V \rightarrow U$. A **model** is a triple $\langle \mathfrak{F}, \beta, w \rangle$ such that \mathfrak{F} is a modal first–order structure, β a valuation into it and w a world of \mathfrak{F} .

As usual, $\gamma \sim_x \beta$ means that $\gamma(y) = \beta(y)$ for all $y \in V$ different from x . If P is a predicate symbol and $\langle t_1, \dots, t_{\Omega(P)} \rangle$ a Ω –tuple of terms, let ε be the function that assigns the tuple $\langle \varepsilon_0(t_0), \dots, \varepsilon_{\Omega(P)-1}(t_{\Omega(P)-1}) \rangle$, where $\varepsilon_i = \beta$ if $t_i \in V$ and $\varepsilon = \mathfrak{I}_w$ if $t_i \in C$.

DEFINITION 4 (Truth in a Model). Given a modal first–order structure $\mathfrak{F} = \langle W, \triangleleft, U, \mathfrak{I} \rangle$, a model $\langle \mathfrak{F}, \beta, w \rangle$, and a formula φ , we define $\langle \mathfrak{F}, \beta, w \rangle \models \varphi$ as follows.

$$\begin{aligned} \langle \mathfrak{F}, \beta, w \rangle \models P(\vec{t}) &\Leftrightarrow \varepsilon(\vec{t}) \in \mathfrak{I}_w(P) \\ \langle \mathfrak{F}, \beta, w \rangle \models \varphi \wedge \chi &\Leftrightarrow \langle \mathfrak{F}, \beta, w \rangle \models \varphi; \varphi \\ \langle \mathfrak{F}, \beta, w \rangle \models \neg \varphi &\Leftrightarrow \langle \mathfrak{F}, \beta, w \rangle \not\models \varphi \\ \langle \mathfrak{F}, \beta, w \rangle \models \bigvee x.\varphi &\Leftrightarrow \text{for some } \gamma \text{ with } \gamma \sim_x \beta : \langle \mathfrak{F}, \gamma, w \rangle \models \varphi \\ \langle \mathfrak{F}, \beta, w \rangle \models \diamond \varphi &\Leftrightarrow \text{exists } w' \text{ such that } w \triangleleft w' \text{ and } \langle \mathfrak{F}, \beta, w' \rangle \models \varphi \end{aligned}$$

§3. Completeness. Let L be a first–order MPL. We first show in analogy to the nonmodal case that there exists a Henkin–completion L^* . The proof presented below is a variation of proofs that can be found in the literature, confer e.g. [7]. The use of maximal consistent sets for completeness proofs in modal predicate logics goes back to [14].

DEFINITION 5. A logic L is called **Henkin–complete** if for all $\bigvee y.\chi$ there exists a constant c such that $\bigvee y.\chi \leftrightarrow [c/y]\chi \in L$.

THEOREM 6. Let L be a first–order MPL in the language \mathcal{L} . Then there exists a language \mathcal{L}^* extending \mathcal{L} by some constants and a conservative extension L^* of L which is Henkin–complete.

Proof. The proof is standard. We pick a formula $\bigvee y.\chi$ in the language. Suppose that there exists no constant c such that $\bigvee y.\chi \leftrightarrow [c/y]\chi \in L$. Then we expand the language by a new constant d and take the smallest logic L^+ containing L and the formula $\bigvee y.\chi \leftrightarrow [d/y]\chi$. We claim first that L^+ is a conservative extension of L . If not, there is a $\delta \in \mathcal{L}$ such that $\delta \in L^+ - L$. It does not contain the constant d . Now we have $L^+ \vdash \delta$, whence $L \vdash (\bigvee y.\chi \leftrightarrow [d/y]\chi) \rightarrow \delta$. However, by the rules of predicate logic, we also have $L \vdash \bigvee z.((\bigvee y.\chi \leftrightarrow [z/y]\chi) \rightarrow \delta)$. Distributing the quantifier and observing that d does not occur in δ we get $L \vdash (\bigvee z.(\bigvee y.\chi \leftrightarrow [z/y]\chi)) \rightarrow \delta$. However, $L \vdash \bigvee z.(\bigvee y.\chi \leftrightarrow [z/y]\chi)$. Consequently, $\delta \in L$. Contradiction.

Now, in a first step we add for all $\varphi = \bigvee y.\chi \in \mathcal{L}$ the axiom $\varphi \leftrightarrow [c_\varphi/y]\chi$. This gives the language \mathcal{L}^1 and the logic L^1 . We iterate this infinitely often. This yields the language \mathcal{L}^* and the logic L^* . By the previous proof, it is conservative, and by construction it is Henkin–complete. \dashv

Notice that this method of Henkin–closure does not work for the counterpart semantics of [10]. The reason is the asymmetry between variables and constants. Instead, a slightly different definition was used, where instead of constants variables were used as witnesses.

Let C_{L^*} be the set of constant terms of L^* . Now take L^* and define W_{L^*} to be the set of all maximal consistent L^* -sets. If $\Delta \in W_{L^*}$, the following interpretation is defined.

$$\mathfrak{I}_\Delta(P) := \{\langle c_i : i < \Omega(P) \rangle : P(\vec{c}) \in \Delta\}$$

This defines a first–order model on the world Δ . Finally, we put $\Delta \triangleleft \Sigma$ if for all $\Box\delta \in \Delta$ we have $\delta \in \Sigma$. \mathfrak{I}_{L^*} is defined by piecing the \mathfrak{I}_Δ together; it assigns to each world Δ the function \mathfrak{I}_Δ . Then we put

$$\mathbf{Can}_{L^*} := \langle W_{L^*}, \triangleleft, C_{L^*}, \mathfrak{I}_{L^*} \rangle$$

This is a modal first–order structure. The following is immediate from the definitions.

LEMMA 7. *Let L^* be Henkin–complete and Δ be a maximally L^* -consistent set. Then if $\bigvee y.\chi \in \Delta$, there is a constant d such that $[d/y]\chi \in \Delta$.*

LEMMA 8. *Let φ be a sentence and Δ a maximally L^* -consistent set. Then*

$$\langle \mathbf{Can}_{L^*}, \Delta \rangle \models \varphi \Leftrightarrow \varphi \in \Delta$$

Proof. The base case, $\varphi = P(\vec{c})$, follows trivially from the definition of \mathfrak{I}_Δ . The induction steps for \perp , \neg and \wedge are routine. Now, let $\varphi = \bigvee y.\chi$. Suppose that $\varphi \in \Delta$. Then, since L^* is Henkin–complete, there exists a constant d such that $\varphi \leftrightarrow [d/x]\chi \in L^*$. Consequently, $[d/x]\chi \in \Delta$. This is a sentence, and by induction hypothesis $\langle \mathbf{Can}_{L^*}, \Delta \rangle \models [d/x]\chi$. Hence, by definition, $\langle \mathbf{Can}_{L^*}, \Delta \rangle \models \bigvee y.\chi$. This argument is reversible. Finally, let $\varphi = \Diamond\chi$. Assume that $\langle \mathbf{Can}_{L^*}, \Delta \rangle \models \Diamond\chi$. Then there exists a Σ such that $\Delta \triangleleft \Sigma$ and $\langle \mathbf{Can}_{L^*}, \Sigma \rangle \models \chi$. By induction hypothesis, $\chi \in \Sigma$. By definition of \triangleleft , $\Diamond\chi \in \Delta$. Now assume $\Diamond\chi \in \Delta$. Put $\Delta_\Box := \{\delta : \Box\delta \in \Delta\}$. We claim that $\Delta_\Box \cup \{\chi\}$ is consistent. Otherwise, we have $\Delta_\Box \vdash_{L^*} \neg\chi$. There is a finite set, indeed a formula $\delta \in \Delta_\Box$ such that $\delta \rightarrow \neg\chi \in L^*$. It follows that $\Box\delta \rightarrow \Box\neg\chi \in L^*$. Since $\Box\delta \in \Delta$ we therefore have $\Box\neg\chi \in \Delta$, which contradicts the assumption that $\Diamond\chi \in \Delta$. Hence, $\Delta_\Box \cup \{\chi\}$ is consistent and is therefore contained in a maximally L^* -consistent set Σ . By induction hypothesis, $\langle \mathbf{Can}_{L^*}, \Sigma \rangle \models \chi$. By definition, $\Delta \triangleleft \Sigma$. So, $\langle \mathbf{Can}_{L^*}, \Delta \rangle \models \Diamond\chi$, as had to be shown. \dashv

Now we have given all the ingredients for a proof of the following well–known result. However, notice that as opposed to the proof given in [7], we did not explicitly use the Barcan formulae.

THEOREM 9. *Every modal predicate logic without equality is complete with respect to modal first–order structures.*

§4. Coherence Structures. Let us now see what happens if equality is introduced into the language. Evidently, if equality was just a member of Π instead of being a logical symbol, the previous proofs would go through. However, the interpretation of equality is an equivalence relation in each world. But this is not what is generally assumed. The interpretation of equality must be identity. Nonetheless, we must ask: identity of what? Think about the example of Hesperus and Phosphorus. As for the real world they are identical, but there are some people for whom they are not. Let George be such a person. Then there is a belief world of George’s in which Hesperus and Phosphorus are not identical. Many have argued that George’s beliefs are inconsistent. This is what comes out if we assume standard semantics. But we could turn this around in the following way. We say that equality does not denote identity of *objects* but of something else, which we shall call the *object trace*. We say that Hesperus and Phosphorus are different objects, which happen to have the same trace in this world, but nonidentical trace in each of George’s belief worlds. To make this distinction between object and object trace more acceptable we shall give a different example. Suppose someone owns a bicycle b and he has it repaired. The next day he picks it up; but then it has a different front wheel. Surely, he would consent to the statement that the bicycle he now has is that bicycle that he gave to the repair shop yesterday. But its front wheel isn’t. Let’s assume for simplicity that atoms are permanent, they will never cease to exist nor come into

existence. Next, let us assume (again simplifying things considerably) that the trace of an object is just the collection of atoms of which it consists. Then, while the object b continued to exist, its trace has changed from one day to the other. In order not to get confused with the problem of transworld identity, let us stress that we think of the objects as transcendental. b is neither a citizen of this world today nor of yesterday's world, nor of any other world. But its trace in this world does belong to this world. In fact, we may or may not assume that object traces are shared across worlds. Technically matters are simpler if they are not, but nothing hinges on that. So, in addition to the bicycle b we have two wheels w and w' , and the trace of b contained the object trace of w yesterday, and it contains the trace of w' today. In the light of these examples it seems sensible to distinguish an object from its trace. Of course, we are not committed to any particular view of traces and certainly do not want to assume that object traces are simply conglomerates of atoms.

Now, in the classical semantics, identity across worlds was a trivial matter. Objects were transcendental, and in using the same letter we always refer to the same object across worlds. However, identity is not relative to worlds. If Hesperus is the same object as Phosphorus in one world, it is the same in all worlds. The distinction between object and trace gets us around this problem as follows. Denote the objects by h and p ; further, let this world be w_0 and let w_1 be one of George's belief worlds. Then the traces of h and p are the same in this world, but different in w_1 . This solves the apparent problem. In our words, equality does not denote identity of two objects, but only identity of their traces in a particular world.

DEFINITION 10 (Coherence Frames and Structures). *By a **coherence frame** we understand a quintuple $\langle W, \triangleleft, U, T, \tau \rangle$ where $\langle W, \triangleleft, U \rangle$ is a predicate Kripke-frame, T a set, the set of **things**, and $\tau : U \times W \rightarrow T$ a function. We call τ the **trace function** and $\tau(o, w)$ the **trace of o in w** . An **interpretation** is a function \mathfrak{I} mapping each $P \in \Pi$ to a function from W to $U^{\Omega(P)}$ and each constant symbol c to a member of U . \mathfrak{I} is called **equivential** if for all $\vec{a}, \vec{b} \in U^{\Omega(P)}$ and $w \in W$, if $\tau(a_i, w) = \tau(b_i, w)$ for all $i < \Omega(P)$ then $\vec{a} \in \mathfrak{I}(P)(w)$ iff $\vec{b} \in \mathfrak{I}(P)(w)$. A **coherence structure** is a sextuple $\langle W, \triangleleft, U, T, \tau, \mathfrak{I} \rangle$ where $\langle W, \triangleleft, U, T, \tau \rangle$ is a coherence frame and \mathfrak{I} an equivential interpretation.*

This notion of equivalence is perhaps a curious one. It says that the basic properties of objects cannot discriminate between objects of equal trace. So, if Pierre believes that London is beautiful and Londres is not, we have two objects which happen to have the same trace in this world. Hence they must share all properties *in this world*. So, London and Londres can only be both beautiful or both ugly. This seems very plausible indeed. From a technical point of view, however, the fact that they cannot simply have different properties is a mere stipulation on our part. However, it is conceivable that there are basic predicates that are actually intensional, which would mean that they fail the substitution under (extensional) equality.

An alternative setup for strictly extensional basic predicates is the following. An interpretation is a function assigning to predicates in a world not tuples of objects but tuples of things. Then an object has a property iff its trace does. This approach is certainly more transparent because it attributes the fact that an object bears a property only to the fact that its trace does. Yet, technically it amounts to the same.

DEFINITION 11 (Coherence Models). *A **coherence model** is a triple $\langle \mathfrak{C}, \beta, w \rangle$, where \mathfrak{C} is a coherence frame, $\beta : V \rightarrow U$ a valuation, $w \in W$ and ε as in Definition 4. We define the truth of a formula inductively as follows.*

$$\begin{aligned}
\langle \mathfrak{C}, \beta, w \rangle \models P(\vec{t}) &\Leftrightarrow \varepsilon(\vec{t}) \in \mathfrak{I}_w(P) \\
\langle \mathfrak{C}, \beta, w \rangle \models s \doteq t &\Leftrightarrow \tau(\varepsilon(s), w) = \tau(\varepsilon(t), w) \\
\langle \mathfrak{C}, \beta, w \rangle \models \chi \wedge \varphi &\Leftrightarrow \langle \mathfrak{C}, \beta, w \rangle \models \chi; \varphi \\
\langle \mathfrak{C}, \beta, w \rangle \models \neg \varphi &\Leftrightarrow \langle \mathfrak{C}, \beta, w \rangle \not\models \varphi \\
\langle \mathfrak{C}, \beta, w \rangle \models \bigvee x. \varphi &\Leftrightarrow \text{for some } \gamma \text{ with } \gamma \sim_x \beta : \langle \mathfrak{C}, \gamma, w \rangle \models \varphi \\
\langle \mathfrak{C}, \beta, w \rangle \models \diamond \varphi &\Leftrightarrow \text{there is } w' \text{ such that } w \triangleleft w' \text{ and } \langle \mathfrak{C}, \beta, w' \rangle \models \varphi
\end{aligned}$$

It is a matter of straightforward verification to see that if L is a modal predicate logic and \mathfrak{C} a coherence frame, then $\mathfrak{C} \models L$. Notice that the fourth postulate for equality holds in virtue of the special clause for equality and the condition that the interpretation must be equivalential. For if $\langle \mathfrak{C}, \beta, w \rangle \models y_i \doteq y_n$, then $\tau(\beta(y_i), w) = \tau(\beta(y_n), w)$. So, if $\langle \mathfrak{C}, \beta, w \rangle \models P(y_0, \dots, y_{n-1})$ for $P \in \Pi$, then $\langle \beta(y_i) : i < n \rangle \in \mathfrak{I}(P)(w)$. Let $\beta' \sim_{y_i} \beta$ be such that $\beta'(y_i) = \beta(y_n)$. By equivalentiality, $\langle \beta'(y_i) : i < n \rangle \in \mathfrak{I}(P)(w)$. This means that $\langle \mathfrak{C}, \beta', w \rangle \models P(y_0, \dots, y_{n-1})$, and so $\langle \mathfrak{C}, \beta, w \rangle \models [y_n/y_i]P(y_0, \dots, y_{n-1})$.

The difference with the counterpart semantics is that we have disentangled the quantification over objects from the quantification over worlds. Moreover, objects exist independently of worlds. Each object leaves a trace in a given world, though it need not exist there. Furthermore, two objects can have the same trace in any given world without being identical. However, identity of two objects holds in a world iff they have the same trace in it. If we also have function symbols, the clauses for basic predicates and equality will have to be generalized in the obvious direction.

The last thing we have to do is to define the object traces in the canonical structure. To do this, let $\Delta \in W_{L^*}$ and c a constant. Then put $[c]_\Delta := \{c \in C_{L^*} : c \doteq d \in \Delta\}$. Now set

$$\tau_{L^*}(c, \Delta) := \langle [c]_\Delta, \Delta \rangle$$

Then let

$$T_{L^*} := \{ \langle [c]_\Delta, \Delta \rangle : \Delta \in W_{L^*}, c \in C_{L^*} \}$$

Finally, put

$$\mathfrak{Coh}_{L^*} := \langle W_{L^*}, \triangleleft, C_{L^*}, T_{L^*}, \tau_{L^*}, \mathfrak{I}_{L^*} \rangle$$

This is a coherence structure. For by Eq4, \mathfrak{I} is equivalential, as is easily checked.

LEMMA 12. *Let φ be a sentence and Δ a maximally L^* -consistent set. Then*

$$\langle \mathfrak{Coh}_{L^*}, \Delta \rangle \models \varphi \Leftrightarrow \varphi \in \Delta$$

THEOREM 13. *Every modal predicate logic with or without equality is complete with respect to coherence structures.*

§5. Coherence Structures and Counterpart Structures. Counterpart frames and structures were introduced in Kracht and Kutz [10]. They generalize the functor-semantic of Ghilardi. Call a relation $R \subseteq M \times N$ a **CE-relation** (CE stands for ‘counterpart existence’) if for all $x \in M$ there exists a $y \in N$ such that $x R y$ and, likewise, for all $y \in N$ there exists an $x \in M$ such that $x R y$. This is a slight adaptation of the definition of that paper to take care of the fact that we now deal with possibilist quantifiers plus an existence predicate as opposed to ‘proper’ free-logical quantifiers. Furthermore, we shall make more explicit the world dependence of the universes.

DEFINITION 14 (Counterpart Frames and Structures). *A quadruple $\langle W, T, \mathcal{U}, \mathcal{C} \rangle$ is a **counterpart frame** if $W, T \neq \emptyset$ are non-empty sets, \mathcal{U} a function assigning to each member v of W a non-empty subset \mathcal{U}_v of T (its **domain**) and finally, \mathcal{C} a function assigning to each pair of worlds v, w a set $\mathcal{C}(v, w)$ of CE-relations from \mathcal{U}_v to \mathcal{U}_w . A quintuple $\langle W, T, \mathcal{U}, \mathcal{C}, \mathfrak{I} \rangle$ is called a **counterpart structure** if $\langle W, T, \mathcal{U}, \mathcal{C} \rangle$ is a counterpart frame and \mathfrak{I} an interpretation, that is, a function assigning to each $w \in W$ and to each n -ary predicate letter a subset of \mathcal{U}_w^n .*

We say that v **sees** w in \mathfrak{F} if $\mathcal{C}(v, w) \neq \emptyset$. A **valuation** is a function η which assigns to every possible world v and every variable an element from the universe \mathcal{U}_v of v . We write η_v for the valuation η at v . A **counterpart model** is a quadruple $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{I}, \eta, w \rangle$, where \mathfrak{F} is a counterpart frame, \mathfrak{I} an interpretation, η a valuation and $w \in W$. Note that interpretations in counterpart frames differ from interpretations in coherence frames in that they don’t assign values to constants. That is to say, unless otherwise stated, when working with counterpart frames we assume that the language does not contain constants.

Let $v, w \in W$ be given and ρ a CE-relation from \mathcal{U}_v to \mathcal{U}_w . We write $\eta \xrightarrow{\rho} \tilde{\eta}$ if for all $x \in V$: $\langle \eta_v(x), \tilde{\eta}_w(x) \rangle \in \rho$. In the context of counterpart frames, $\tilde{\eta} \sim_x^v \eta$ denotes a local x -variant at the

domain of world v , i.e., $\tilde{\eta}$ is a valuation that may differ from η only in the values that it assigns to the variable x at world v .

DEFINITION 15 (Truth in a Counterpart Model). *Let $\varphi(\vec{y})$ and $\psi(\vec{z})$ be modal formulae with the free variables y_0, \dots, y_{n-1} and z_0, \dots, z_{m-1} , respectively. Let \mathfrak{F} be a counterpart frame, \mathfrak{I} an interpretation, v a possible world and let η be a valuation. We define:*

$$\begin{aligned} \langle \mathfrak{F}, \mathfrak{I}, \eta, v \rangle \models x_i \doteq x_j &\Leftrightarrow \eta_v(x_i) = \eta_v(x_j) \text{ in } \mathcal{U}_v \\ \langle \mathfrak{F}, \mathfrak{I}, \eta, v \rangle \models R(\vec{y}) &\Leftrightarrow \langle \eta_v(y_0), \dots, \eta_v(y_{n-1}) \rangle \in \mathcal{J}_v(R) \\ \langle \mathfrak{F}, \mathfrak{I}, \eta, v \rangle \models \neg\varphi &\Leftrightarrow \langle \mathfrak{F}, \mathfrak{I}, \eta, v \rangle \not\models \varphi \\ \langle \mathfrak{F}, \mathfrak{I}, \eta, v \rangle \models \varphi \wedge \chi &\Leftrightarrow \langle \mathfrak{F}, \mathfrak{I}, \eta, v \rangle \models \varphi; \chi \\ \langle \mathfrak{F}, \mathfrak{I}, \eta, v \rangle \models \Diamond\varphi(\vec{y}) &\Leftrightarrow \text{there are } w \in W, \rho \in \mathcal{C}(v, w) \text{ and } \tilde{\eta} : \eta \xrightarrow{\rho} \tilde{\eta} \text{ such that } \langle \mathfrak{F}, \mathfrak{I}, \tilde{\eta}, w \rangle \models \varphi(\vec{y}) \\ \langle \mathfrak{F}, \mathfrak{I}, \eta, v \rangle \models \bigvee x.\varphi(x) &\Leftrightarrow \text{there is } \tilde{\eta} \sim_x^v \eta \text{ such that } \langle \mathfrak{F}, \mathfrak{I}, \tilde{\eta}, v \rangle \models \varphi(x) \end{aligned}$$

As usual, $\mathfrak{F} \models \varphi$ means that for all interpretations \mathfrak{I} , all valuations η and worlds v , it holds that $\langle \mathfrak{F}, \mathfrak{I}, \eta, v \rangle \models \varphi$.

The intuition behind counterpart frames is that objects do not exist; the only things that exist are the object traces (which belong to the domains of the worlds), and the counterpart relations. However, the notion of an object is still definable, even though it shall turn out that counterpart frames can have very few objects in this sense.

DEFINITION 16. *Let $\mathfrak{F} = \langle W, T, \mathcal{U}, \mathcal{C} \rangle$ be a counterpart frame. An **object** is a function $f : W \rightarrow T$ such that (i) $f(v) \in \mathcal{U}_v$ for all $v \in W$, (ii) for each pair $v, w \in W$ with $\mathcal{C}(v, w) \neq \emptyset$ there is $\rho \in \mathcal{C}(v, w)$ such that $\langle f(v), f(w) \rangle \in \rho$.*

So, objects are constructed using the counterpart relation. If the trace b in world w is a counterpart of the trace a in world v , then there may be an object leaving trace a in v and trace b in w . If not, then not. However, there are frames which are not empty and possess no objects. Here is an example. Let $W = \{v\}$, $T = \{a, b\}$, $\mathcal{U}_v = \{a, b\}$, and $\mathcal{C}(v, v) = \{\rho\}$ with $\rho = \{\langle a, b \rangle, \langle b, a \rangle\}$. It is easy to see that this frame has no objects. The crux is that we can only choose one trace per world, but when we pass to an accessible world, we must choose a counterpart. This may become impossible the moment we have cycles in the frame.

Counterpart frames show a different behaviour than coherence frames. As we have shown above, for each modal predicate logic there exists an adequate structure. However, counterpart structures satisfy a formula that is actually not generally valid, when one thinks of the quantifiers as quantifying over intensional rather than extensional (trace-like) objects.

PROPOSITION 17. *Let \mathfrak{F} be a counterpart frame, x and y variables not occurring in \vec{z} . Then for all formulae $\varphi(x, \vec{z})$*

$$\mathfrak{F} \models \bigwedge x. \bigwedge y. (x \doteq y) \rightarrow (\varphi(x, \vec{z}) \leftrightarrow \varphi(y, \vec{z}))$$

Proof. It is clear that we can restrict our attention to formulae of the type $\varphi(x, \vec{z}) = \Diamond\chi(x, \vec{z})$. Let η be a valuation and v a world. Assume that $\langle \mathfrak{F}, \mathfrak{I}, \eta, v \rangle \models x \doteq y$. Then $\eta_v(x) = \eta_v(y)$. We will show that $\langle \mathfrak{F}, \mathfrak{I}, \eta, v \rangle \models \Diamond\chi(x, \vec{z}) \rightarrow \Diamond\chi(y, \vec{z})$. Suppose therefore that $\langle \mathfrak{F}, \mathfrak{I}, \eta, v \rangle \models \Diamond\chi(x, \vec{z})$. Then there exists a world w , a $\rho \in \mathcal{C}(v, w)$ and a valuation $\tilde{\eta}$ such that $\eta \xrightarrow{\rho} \tilde{\eta}$ and $\langle \mathfrak{F}, \mathfrak{I}, \tilde{\eta}, w \rangle \models \chi(x, \vec{z})$. Now define η' by $\eta'_w(y) := \tilde{\eta}_w(x)$, and $\eta'_{w'}(y') := \tilde{\eta}_{w'}(y')$ for all w' and y' such that either $w' \neq w$ or $y' \neq y$. Then $\langle \mathfrak{F}, \mathfrak{I}, \eta', w \rangle \models \chi(y, \vec{z})$. Furthermore, for all variables y' : $\langle \eta_v(y'), \eta'_w(y') \rangle \in \rho$. For if $y' \neq y$ this holds by definition of η and choice of $\tilde{\eta}$. And if $y' = y$ we have $\eta_v(x) = \eta_v(y)$, so that $\rho \ni \langle \eta_v(x), \tilde{\eta}_w(x) \rangle = \langle \eta_v(y), \eta'_w(y) \rangle$. It follows that $\eta \xrightarrow{\rho} \eta'$ and therefore that $\langle \mathfrak{F}, \mathfrak{I}, \eta, v \rangle \models \Diamond\chi(y, \vec{z})$, as had to be shown. \dashv

From the previous theorem we deduce that also the following holds.

$$(\ddagger) \quad \bigwedge x. \bigwedge y. (x \doteq y \wedge \Diamond\top) \rightarrow \Diamond(x \doteq y)$$

Namely, take

$$\bigwedge x. \bigwedge y. x \doteq y. \rightarrow \Diamond(x \doteq z) \leftrightarrow \Diamond(y \doteq z)$$

By the above theorem, this is generally valid. Substituting x for z we get

$$\bigwedge x. \bigwedge y. x \doteq y. \rightarrow \diamond(x \doteq x) \leftrightarrow \diamond(y \doteq x)$$

Applying standard laws of predicate logic yields (\ddagger) . We remark here that the logics defined in the literature (for example Ghilardi [6], Skvortsov and Shehtman [13] and Kracht and Kutz [10]), differ from modal predicate logics as defined here only in the additional laws of equality that they assume.

The modal Leibniz law of [10] allows for simultaneous substitution of all free occurrences of x by y in $\diamond\chi$ (denoted by $\diamond\chi(y//x)$, provided that $x \doteq y$ is true.

$$\bigwedge x. \bigwedge y. x \doteq y. \rightarrow \diamond\chi(x) \rightarrow \diamond\chi(y//x)$$

Now, notice that in a modal predicate logic as defined above, the rule of replacing constants for universally quantified variables is valid. In counterpart frames this creates unexpected difficulties. For, suppose we do have constants and that they may be substituted for variables. Then we may derive from (\ddagger) , using the substitution of c for x and d for y :

$$(c \doteq d \wedge \diamond\top) \rightarrow \diamond(c \doteq d)$$

Since a constant has a fixed interpretation in each world, this means that if two constants are equal in a world and there exists some accessible possible world, then there will also be some accessible world in which they are equal. This is not generally valid. What is happening here is a shift from a *de re* to a *de dicto* interpretation. If we follow the traces of the objects, the formula is valid, but if we substitute intensional objects, namely constants, it becomes refutable. Notice, that this situation is also reflected in the way, non-rigid constants are treated in [5]. There, the two possible readings of the above formula, the *de dicto* and *de re* reading, are distinguished by actually binding the interpretation of the constants to the respective worlds by using the term-binding λ -operator.

Applied to Hesperus and Phosphorus, this means that if they are equal, then there is a belief world of George's in which they are equal. However, if George believes that they are different, this cannot be the case. So, the counterpart semantics cannot handle constants correctly—at least not in a straightforward way, i.e., without restricting the possible values of constants in accessible worlds. This paradox is avoided in Kracht and Kutz [10] by assuming that the language actually has *no* constants.

§6. Objectual Counterpart Structures. The connection between coherence frames and counterpart frames is not at all straightforward. Since the logic of a counterpart frame is a first-order modal predicate logic, one might expect that for every counterpart frame there is a coherence frame having the same logic. This is only approximately the case. We will show in Theorem 24 that for every counterpart *structure* there is a coherence structure having the same theory. This is not generally true for frames. However, adopting a modification of coherence frames proposed by Melvin Fitting in [3], namely balanced coherence frames (in [3] the corresponding frames are called Riemann FOIL frames), it can indeed be shown that for every counterpart frame there is a balanced coherence frame with the same theory.

Let us begin by elucidating some of the connections between counterpart and coherence frames. Note again that since counterpart structures as defined above don't interpret constants, we have to assume that the language does not contain constants.

First, fix a coherence structure $\mathfrak{C} = \langle W, \triangleleft, U, T, \tau, \mathfrak{J} \rangle$. We put $U_v := \{\tau(o, v) : o \in U\}$. This defines the domains of the world. Next, for $v, w \in W$ we put $\rho(v, w) := \{\langle \tau(o, v), \tau(o, w) \rangle : o \in U\}$ and $\mathcal{C}(v, w) := \emptyset$ if $v \triangleleft w$ does not obtain; otherwise, $\mathcal{C}(v, w) := \{\rho(v, w)\}$. Finally, $\langle \tau(a_i, w) : i < \Omega(P) \rangle \in \mathfrak{J}(P)(w)$ iff $\langle a_i : i < \Omega(P) \rangle \in \mathfrak{J}(P)(w)$. Then $\langle W, T, \mathcal{U}, \mathcal{C}, \mathfrak{J} \rangle$ is a counterpart structure. We shall denote it by $CP(\mathfrak{C})$. Notice, that there is at most one counterpart relation between any two worlds.

Conversely, let a counterpart structure $\mathfrak{N} = \langle W, T, \mathcal{U}, \mathcal{C}, \mathfrak{J} \rangle$ be given. We put $v \triangleleft w$ iff $\mathcal{C}(v, w) \neq \emptyset$. $U := T$. Let O be the set of all objects $o : W \rightarrow T$. Further, $\tau(o, w) := o(w)$. This defines

a coherence frame if the set of objects is non-empty. Finally, $\langle o_i : i < \Omega(P) \rangle \in \mathfrak{J}(P)(w)$ iff $\langle o_i(w) : i < n \rangle \in \mathfrak{J}(P)(w)$. It is easy to see that this is an equivalential interpretation. So, $\langle W, \triangleleft, O, U, \tau, \mathfrak{J} \rangle$ is a coherence structure, which we denote by $CH(\mathfrak{N})$.

Unfortunately, the logical relation between these two types of structures is more than unclear, not the least since the notion of satisfaction in them is different. Moreover, the operations just defined are not inverses of each other. For example, as we have already seen, there exist counterpart structures with nonempty domains which have no objects. In this case $CP(CH(\mathfrak{N})) \not\cong \mathfrak{N}$. Also let \mathfrak{C} be the following coherence frame. $W := \{v, w, x, y\}$, $T := \{1, 2, 3, 4, 5, 6\}$, $U = \{a, b\}$, $\triangleleft = \{\langle v, w \rangle, \langle w, x \rangle, \langle x, y \rangle\}$. Finally, $\tau(a, -) : v \mapsto 1, w \mapsto 2, x \mapsto 4, y \mapsto 5$, $\tau(b, -) : v \mapsto 1, w \mapsto 3, x \mapsto 4, y \mapsto 6$. Generating the counterpart frame we find that 2 and 3 are counterparts of 1, and 5 and 6 are counterparts of 4. Hence, there are more objects in the counterpart frame than existed in the coherence frame, for example the function $v \mapsto 1, w \mapsto 2, x \mapsto 4, y \mapsto 6$.

DEFINITION 18. *Let \mathfrak{N} be a counterpart frame. A sequence $\langle (w_i, t_i) : i < n \rangle$ is called a **thread** if (1) for all $i < n$: $w_i \in W$, $t_i \in \mathcal{U}_{w_i}$, and (2) for all $i < n - 1$: $w_i \triangleleft w_{i+1}$ and $\langle t_i, t_{i+1} \rangle \in \rho$ for some $\rho \in \mathcal{C}(w_i, w_{i+1})$. \mathfrak{N} is **rich in objects** if for all threads there exists an object o such that $o(w_i) = t_i$ for all $i < n$.*

Notice that if \triangleleft is cycle free then \mathfrak{N} is automatically object rich. Otherwise, we must only be able to come back to the same thing when going in a cycle. This is a rather strict condition. Nevertheless, we can use unravelling to produce an equivalent cycle free structure from a given one, which is then object rich. Additionally, we can ensure that between any two worlds there is at most one counterpart relation. We call counterpart frames that satisfy the condition $|\mathcal{C}(v, w)| \leq 1$ for all worlds $v, w \in W$ **Lewisian counterpart frames**.

THEOREM 19. *For every counterpart structure \mathfrak{N} there exists a Lewisian counterpart structure \mathfrak{N}' rich in objects such that \mathfrak{N} and \mathfrak{N}' have the same theory.*

Proof. Let $\mathfrak{N} = \langle W, T, \mathcal{U}, \mathcal{C}, \mathfrak{J} \rangle$ be a counterpart structure. A path in \mathfrak{N} is a sequence $\pi = \langle w_0, \langle w_i, \rho_i : 0 < i < n \rangle \rangle$ such that $\rho_i \in \mathcal{C}(w_{i-1}, w_i)$ for all $0 < i < n$. We let $e(\pi) := w_{n-1}$ and $r(\pi) = \rho_{n-1}$ and call these, respectively, the **end point** and **end relation** of π . Let W' be the set of all paths in \mathfrak{N} and $T' := T$. Further, let $\mathcal{U}'_\pi := \mathcal{U}_{e(\pi)}$ and for two paths π and μ put $\mathcal{C}'(\pi, \mu) := r(\mu)$ if $r(\mu) \in \mathcal{C}(e(\pi), e(\mu))$ and empty otherwise. Finally, let P be an n -ary predicate letter. Then $\mathfrak{J}'(P)(\pi) := \mathfrak{J}(P)(e(\pi))$. Now let $\mathfrak{N}' = \langle W', T', \mathcal{U}', \mathcal{C}', \mathfrak{J}' \rangle$. This is a Lewisian counterpart structure and clearly rich in objects. The following can be verified by induction. If β is a valuation on \mathfrak{N} , and w a world, and if β' is a valuation on \mathfrak{N}' and π a path such that $e(\pi) = w$ and $\beta'_\pi(x_i) = \beta_w(x_i)$, then $\langle \mathfrak{N}, \beta, w \rangle \models \varphi$ iff $\langle \mathfrak{N}', \beta', \pi \rangle \models \varphi$ for all φ . The theorem now follows; for given β and w , β' and π satisfying these conditions can be found, and given β' and π also β and w satisfying these conditions can be found. \dashv

Notice by the way that in the propositional as well as the second-order case this theorem is false. This is so because the interpretation of a predicate in π must be identical to that of μ if the two have identical end points. If this is not the case, the previous theorem becomes false. However, if we are interested in characterizing MPLs by means of models, it follows from the above result that we can restrict ourselves in the discussion to Lewisian counterpart structures that are rich in objects.

But we can also strike the following compromise. Let us keep the counterpart semantics as it is, but interpret formulae in a different way. Specifically, let us define the following.

DEFINITION 20 (Objectual Counterpart Interpretation). *We say that $\mathfrak{M} = \langle W, T, \mathcal{U}, \mathcal{C}, \mathfrak{J}, \beta, v \rangle$ is an **objectual counterpart model**, if $\mathfrak{F} = \langle W, T, \mathcal{U}, \mathcal{C} \rangle$ is a counterpart frame as before, \mathfrak{J} is an **objectual interpretation**, that is, a counterpart interpretation that additionally assigns objects to constant symbols, β an **objectual valuation** in \mathfrak{F} , i.e., a function that assigns to each variable an object, and v a world. In this context, $\varepsilon_v(o) := \beta_v(o)$ if o is a variable and $\varepsilon_v(o) = \mathfrak{J}(o)(v)$ if o is a constant symbol.*

Write $\beta \rightarrow_{v,w}^{\vec{y}} \beta$ if for some $\rho \in \mathcal{C}(v,w)$ we have $\langle \beta_v(x_i), \beta_w(x_i) \rangle \in \rho$ for all $x_i \in \vec{y}$. Furthermore, write $\beta \rightarrow_{v,w}^{\vec{y}} \gamma$ if for some $\rho \in \mathcal{C}(v,w)$ we have $\langle \beta_v(x_i), \gamma_w(x_i) \rangle \in \rho$ for all $x_i \in \vec{y}$, where γ is an objectual valuation. Terms t_i denote either variables or constants, \vec{y} tuples of variables and \vec{c} tuples of constants. The symbol \models^* is called the **weak objectual truth-relation** and is defined thus:

$$\begin{aligned} \langle \mathfrak{F}, \mathfrak{I}, \beta, v \rangle \models^* t_i \doteq t_j &\Leftrightarrow \varepsilon_v(t_i) = \varepsilon_v(t_j) \\ \langle \mathfrak{F}, \mathfrak{I}, \beta, v \rangle \models^* R(\vec{t}) &\Leftrightarrow \langle \varepsilon_v(t_0), \dots, \varepsilon_v(t_{n-1}) \rangle \in \mathcal{I}_v(R) \\ \langle \mathfrak{F}, \mathfrak{I}, \beta, v \rangle \models^* \neg\varphi &\Leftrightarrow \langle \mathfrak{F}, \mathfrak{I}, \beta, v \rangle \not\models^* \varphi \\ \langle \mathfrak{F}, \mathfrak{I}, \beta, v \rangle \models^* \varphi \wedge \chi &\Leftrightarrow \langle \mathfrak{F}, \mathfrak{I}, \beta, v \rangle \models^* \varphi; \chi \\ \langle \mathfrak{F}, \mathfrak{I}, \beta, v \rangle \models^* \diamond\varphi(\vec{y}, \vec{c}) &\Leftrightarrow \text{there is } \beta \rightarrow_{v,w}^{\vec{y}} \gamma \text{ such that } \langle \mathfrak{F}, \mathfrak{I}, \gamma, w \rangle \models^* \varphi(\vec{y}, \vec{c}) \\ \langle \mathfrak{F}, \mathfrak{I}, \beta, v \rangle \models^* \forall y.\varphi(y, \vec{c}) &\Leftrightarrow \text{there is } \tilde{\beta} \sim_y \beta \text{ such that } \langle \mathfrak{F}, \mathfrak{I}, \tilde{\beta}, v \rangle \models^* \varphi(y, \vec{c}) \end{aligned}$$

The **strong objectual truth-relation** \models^\dagger is like \models^* except for the clause for \diamond which is given by:

$$\langle \mathfrak{F}, \mathfrak{I}, \beta, v \rangle \models^\dagger \diamond\varphi(\vec{y}, \vec{c}) \Leftrightarrow \text{there is } \beta \rightarrow_{v,w}^{\vec{y}} \beta \text{ and } \langle \mathfrak{F}, \mathfrak{I}, \beta, w \rangle \models^\dagger \varphi(\vec{y}, \vec{c})$$

These interpretations remove the asymmetry between variables and constants in the sense that constants and variables are now assigned the same kind of values. However, while the strong objectual interpretation brings us very close to coherence semantics, the weak interpretation still bears essential properties of counterpart semantics, namely that we may move via a counterpart relation to a new object. More precisely we have the following:

PROPOSITION 21. *The rule of substituting constants for universally quantified variables is valid in the strong objectual interpretation, i.e. for every counterpart frame $\mathfrak{F} = \langle W, T, \mathcal{U}, \mathcal{C} \rangle$ it holds that*

$$\mathfrak{F} \models^\dagger \bigwedge x.\varphi \rightarrow [c/x]\varphi.$$

Furthermore, there is an objectual counterpart model \mathfrak{M} such that

$$\mathfrak{M} \not\models^\dagger \bigwedge x.\bigwedge y.(x \doteq y) \rightarrow (\varphi(x, \vec{z}) \leftrightarrow \varphi(y, \vec{z}))$$

Both claims are false for the weak objectual interpretation.

Proof. For the first claim suppose that β is an objectual valuation, \mathfrak{I} an objectual interpretation, v a world and that $\langle \mathfrak{F}, \mathfrak{I}, \beta, v \rangle \models^\dagger \bigwedge x.\varphi$. We only need to consider the case where $\varphi = \diamond\psi$. We then have that for every x -variant $\tilde{\beta}$: $\langle \mathfrak{F}, \mathfrak{I}, \tilde{\beta}, v \rangle \models^\dagger \diamond\psi(x)$. I.e., for every object $o = \tilde{\beta}(x)$ there is $\tilde{\beta} \rightarrow_{v,w}^x \tilde{\beta}$ and $\langle \mathfrak{F}, \mathfrak{I}, \tilde{\beta}, w \rangle \models^* \psi(x)$. Now $\mathfrak{I}(c) = \tilde{\beta}(x)$ for some x -variant $\tilde{\beta}$ of β from which the claim follows immediately.

For the second claim, fix the following simple model \mathfrak{M} . Let $W = \{v, w\}$, $T = \{a, b, b'\}$, $\mathcal{U}(v) = \{a\}$, $\mathcal{U}(w) = \{b, b'\}$, $\mathcal{C}(v, w) = \rho = \{\langle a, b \rangle, \langle a, b' \rangle\}$ and $\mathfrak{I}(P)(w) = \{b\}$ and $\beta(x) = o$ with $o(v) = a$ and $o(w) = b$ and $\beta(y) = o'$ with $o'(v) = a$ and $o'(w) = b'$. o and o' are the only objects in this model. It should be obvious that $\mathfrak{M} \models^\dagger x \doteq y \wedge \diamond P(x)$ while $\mathfrak{M} \not\models^\dagger \diamond P(y)$.

Consider now the weak objectual interpretation. Take the model just defined and assume furthermore that $\mathfrak{I}(c) = o'$. Then clearly $v \models^* \bigwedge x.\diamond P(x)$ while $w \not\models^* P(c)$, which shows that the rule is not valid. That the formula in the second claim is still valid under the weak objectual interpretation should be clear. \dashv

The following is also straightforward.

THEOREM 22. *Let $\mathfrak{N} = \langle W, T, \mathcal{U}, \mathcal{C}, \mathfrak{I} \rangle$ be a counterpart structure rich in objects, v a world and let β be an objectual valuation and $\tilde{\beta}$ a counterpart valuation such that $\beta_v(x_i) = \tilde{\beta}_v(x_i)$ for all variables. Then for all φ :*

$$\langle \mathfrak{N}, \beta, v \rangle \models^* \varphi \Leftrightarrow \langle \mathfrak{N}, \tilde{\beta}, v \rangle \models \varphi$$

The proof is by induction on φ . The two relations differ only with respect to formulae of the form $\diamond\chi$. Here, object richness assures that for each choice of counterparts in the successor worlds

an object exists. (Actually, for that we only need that every element of a domain is the trace of some object.)

§7. A Logic That is Coherence Frame Incomplete. In this section we will give an axiomatization of a modal predicate logic that is the logic of a single counterpart frame having two distinct counterpart relations. This logic turns out not to be valid on any coherence frame. So fix the following counterpart frame \mathfrak{F} : let $W = \{w\}$ be the set of worlds, $U_w = \{a, b\}$ be the universe of w and $C(w, w) = \{f, g\}$ the set of counterpart relations from w to itself, where $f : a \rightarrow a, b \rightarrow b$ and $g : a \rightarrow b, b \rightarrow a$. Notice that this frame is not object rich, for $\langle a, b \rangle$ is a thread in \mathfrak{F} but there is no object that leaves both a and b as trace in w .

\mathfrak{F} is a frame for the logic L axiomatized as follows:

- (a) The T schema: $p \rightarrow \diamond p$.
- (b) The alt_2 schema: $\diamond p \wedge \diamond q \wedge \diamond r \rightarrow \diamond(p \wedge q) \vee \diamond(p \wedge r) \vee \diamond(q \wedge r)$
- (c) The 4 schema: $\diamond \diamond p \rightarrow \diamond p$
- (d) $\forall x_0, x_1, x_2. (x_0 = x_1 \vee x_1 = x_2 \vee x_0 = x_2)$ (There exist at most two things)
- (e) $\exists x_0, x_1. \neg(x_0 = x_1)$ (There exist at least 2 things)
- (f) $\forall x_0, x_1. x_0 = x_1 \rightarrow \Box(x_0 = x_1)$ (Necessity of Identity)
- (g) $\forall x_0, x_1. \neg(x_0 = x_1) \rightarrow \Box \neg(x_0 = x_1)$ (Necessity of Distinctness)
- (h) $\forall x_0, x_1. P(x_0) \wedge \neg P(x_1) \rightarrow \Box(P(x_0) \wedge \neg P(x_1) \vee \neg P(x_0) \wedge P(x_1))$
- (i) $\forall x_0, x_1. P(x_0) \wedge \neg P(x_1) \rightarrow \diamond(P(x_0) \wedge \neg P(x_1)) \wedge \diamond(\neg P(x_0) \wedge P(x_1))$
- (j) Axioms and rules of MPLs as given in Definition 2.

This may or may not be closed under second-order substitutions. It is arguable from a philosophical point of view whether this defines a genuine logic, because it makes *existence assumptions* about e.g. the number of objects in the world. Nevertheless it surely defines a logic, say \mathbf{L} , in the technical sense of definition 2. The point we are about to make is that \mathbf{L} has no counterpart models with just one counterpart relation between worlds. This may be seen as follows:

Axioms (f) and (g) say that counterpart relations are injective functions. The propositional reduct of this logic is the logic $K.alt_2.T.4$ which possesses exactly two Kripke frames, namely the 1-point reflexive frame and the 2-point frame, where accessibility is universal.

Postulates (d) and (e) say that every world is inhabited by exactly two objects.

Now (h) says in this situation that we may not go to another world, because (h) can be easily falsified in the two point frame (P is true of exactly one object in, say, world u , but true of both in world v), but not in the one point frame (because of (d)–(g)). So, the two objects may only swap their roles. So we have just one world. Finally, (i) forces the frame to have at least two counterpart relations.

It follows that whereas L is characterized by its canonical coherence model, compare Theorem 13, there is no coherence frame in which L is valid, that is to say, L is coherence frame incomplete. For an object in a coherence frame cannot leave two distinct traces in one world, which is required according to (i). We will show in Section 9 how the notion of coherence frame can be modified to gain the same expressive power as counterpart semantics without moving to the full second-order semantics.

§8. Passing Back and Forth. By the previous theorem, we can introduce the notion of an object into counterpart frames, which then makes them rather similar to coherence frames. However, counterpart structures with object valuations are still different from coherence structures. A different approach is to embed the language of counterpart structures into the language of coherence structures in the following way. Write \blacklozenge for the possibility operator, which has the

definition given by \models^* . Now define a translation as follows.

$$\begin{aligned}
(x \doteq y)^\gamma &:= x \doteq y \\
P(\vec{y})^\gamma &:= P(\vec{y}) \\
(\neg\varphi)^\gamma &:= \neg\varphi^\gamma \\
(\varphi \wedge \chi)^\gamma &:= \varphi^\gamma \wedge \chi^\gamma \\
(\bigvee x.\varphi)^\gamma &:= \bigvee x.\varphi^\gamma \\
(\blacklozenge\varphi(y_0, \dots, y_{n-1}))^\gamma &:= \bigvee z_0 \dots \bigvee z_{n-1}. \bigwedge_{i < n} z_i \doteq y_i \wedge \blacklozenge\varphi(\vec{z}/\vec{y})^\gamma
\end{aligned}$$

Here, y_i ($i < n$) are the free variables of φ and z_i ($i < n$) distinct variables not occurring in φ . This is actually unique only up to renaming of bound variables. Further, notice that $\bigwedge_{i < n}$ denotes a finite conjunction, not a quantifier. This translation makes explicit the fact that variables inside a \blacklozenge are on a par with bound variables. (In linguistics, one speaks of \blacklozenge as an *unselective binder*.) Notice now that

$$\begin{aligned}
&(\bigwedge x_0. \bigwedge x_1. x_0 \doteq x_1 \rightarrow (\blacklozenge\varphi(x_0) \leftrightarrow \blacklozenge\varphi(x_1)))^\gamma \\
&= \bigwedge x_0. \bigwedge x_1. x_0 \doteq x_1 \rightarrow ((\blacklozenge\varphi(x_0))^\gamma \leftrightarrow (\blacklozenge\varphi(x_1))^\gamma) \\
&= \bigwedge x_0. \bigwedge x_1. x_0 \doteq x_1 \rightarrow ((\bigvee x_3)(x_3 \doteq x_0 \wedge \blacklozenge\varphi(x_3)^\gamma) \leftrightarrow ((\bigvee x_3)(x_3 \doteq x_1 \wedge \blacklozenge\varphi(x_3)^\gamma))
\end{aligned}$$

This principle is actually valid in coherence structures. For it is a substitution instance of the following theorem of predicate logic.

$$\bigwedge x_0. \bigwedge x_1. x_0 \doteq x_1 \rightarrow ((\bigvee x_3. x_3 \doteq x_0 \wedge \varphi) \leftrightarrow (\bigvee x_3. x_3 \doteq x_1 \wedge \varphi))$$

PROPOSITION 23. *Let \mathfrak{N} be a counterpart structure and x a world. Then for any φ :*

$$\langle \mathfrak{N}, x \rangle \models \varphi^\gamma \quad \Leftrightarrow \quad \langle \mathfrak{N}, x \rangle \models \varphi$$

In object rich structures also \models and \models^* coincide, which makes all four notions the same. So, while in counterpart structures formulae φ and φ^γ are equivalent, they are certainly not equivalent when interpreted in coherence structures.

From the completeness result of this paper, the following can be established.

THEOREM 24. *For every counterpart structure \mathfrak{N} there exists a coherence structure \mathfrak{C} such that for any formula φ and any world x :*

$$\langle \mathfrak{N}, x \rangle \models^* \varphi \quad \Leftrightarrow \quad \langle \mathfrak{C}, x \rangle \models \varphi$$

We sketch a construction that yields such a coherence structure. We perform the construction only for the frame, the valuation is straightforward to construct. The construction is in stages and a somewhat more complicated unravelling. In each stage we construct a coherence structure \mathfrak{H}_n , and a map h_n from the set of worlds of \mathfrak{H}_n to the set of worlds of \mathfrak{N} . At stage 0, we take the disjunction of all frames $\langle \{x\}, \triangleleft_0, U, T, \rangle$, where w is a world of \mathfrak{C} , $\triangleleft_0 = \emptyset$, $U := T = \mathfrak{U}_x$. Finally, $\tau(o, x) := o$. Put $h_0(x) := x$. At Stage $n + 1$, take a world w without successor in \mathfrak{H}_n and a successor relation $\rho \in \mathcal{C}(h_n(w), v)$ for some v , and a function γ from objects of $h_n(w)$ to ρ -counterparts in v . Add a new successor world w_γ to \mathfrak{H}_n . This is the first step. It defines the set W_{n+1} and \triangleleft_{n+1} . Let Γ be the set of all choice functions γ . Now $U_{n+1} := U_n \times \Gamma$, $T_{n+1} := T_n$. $\tau_{n+1}(\langle o, \gamma \rangle, u) := \tau(o, u)$ for all $u \neq w_\gamma$; $\tau_{n+1}(\langle o, \gamma \rangle, w_\delta) := \delta(\tau_n(o, w))$, for all $\gamma, \delta \in \Gamma$ which are choice functions of $\mathcal{C}(w, v)$. Now h_{n+1} extends h_n by $h_{n+1}(w_\gamma) := v$, where v is the target world of γ .

§9. Balanced Coherence Frames and Worldline Semantics. In his [11], Gerhard Schurz introduced a semantics, called *worldline semantics*, in the context of analyzing Hume's is-ought thesis, i.e. the logical problem whether one may infer ethical value (normative) statements from factual (descriptive) statements. This semantics is very close to the coherence semantics defined in this paper. A worldline frame is a quintuple $\langle W, R, L, U, Df \rangle$, where W is a set of worlds, R the accessibility relation, $U \neq \emptyset$ a non-empty set of *possible objects*, $\emptyset \neq L \subseteq \{l : W \rightarrow U\}$ a set of functions from possible worlds to possible objects (members of L are called *worldlines*), $Df : W \rightarrow U$ a *domain function* such that $Df(w) := D_w \subseteq U_w$, where $U_w := \{d \in U : \exists l \in$

$L(l(w) = d)$ (the set of term extensions at world w) and $L_w = \{l \in L : \exists d \in D_w(l(w) = d)\}$ (the set of worldlines with extension in D_w). An interpretation V into a worldline frame is a function such that $V(t) \in L$ for any term t and $V_w(Q) \subseteq U_w^n$ for any n -ary predicate Q . If V is a worldline interpretation denote by $V[l/x]$ the interpretation that is like V except that it assigns worldline l to the variable x . Since, unlike Schurz, we assume that free logical quantifiers are a defined notion, we suppose in the following that $D_w = U_w$ for all w . Then, in particular, $Df(w) = U_w$ and $L_w = L$ for all w , which means that Df can be omitted.

Then the truth relation in worldline semantics can be defined as follows:

DEFINITION 25 (Truth in Worldline Semantics). *Let $\mathfrak{F} = \langle W, R, L, U \rangle$ be a worldline frame, V an interpretation and w a world, define*

$$\begin{aligned} \langle \mathfrak{F}, V, w \rangle \models P(\vec{t}) &\Leftrightarrow V_w(\vec{t}) \in V_w(P) \\ \langle \mathfrak{F}, V, w \rangle \models s \doteq t &\Leftrightarrow V(s)(w) = V(t)(w) \\ \langle \mathfrak{F}, V, w \rangle \models \chi \wedge \varphi &\Leftrightarrow \langle \mathfrak{F}, V, w \rangle \models \chi; \varphi \\ \langle \mathfrak{F}, V, w \rangle \models \neg \varphi &\Leftrightarrow \langle \mathfrak{F}, V, w \rangle \not\models \varphi \\ \langle \mathfrak{F}, V, w \rangle \models \bigvee x. \varphi &\Leftrightarrow \text{for some } l \in L : \langle \mathfrak{F}, V[l/x], w \rangle \models \varphi \\ \langle \mathfrak{F}, V, w \rangle \models \diamond \varphi &\Leftrightarrow \text{there is } w' \text{ such that } w \triangleleft w' \text{ and } \langle \mathfrak{F}, V, w' \rangle \models \varphi \end{aligned}$$

It should be rather clear that the main difference between worldline and coherence semantics is terminological. While in worldline semantics one quantifies over worldlines and evaluates predicates and identity statements with respect to the value of a worldline at a particular world, in coherence semantics we quantify over modal individuals without specifying their internal structure, but assume a trace function that maps an individual at a world to its trace. So we can give the following translation. Given a coherence model $\langle \mathfrak{F}, \mathfrak{I}, \beta, w \rangle$ based on the coherence frame $\langle W, \triangleleft, U, T, \tau \rangle$, define a worldline model $\langle \mathfrak{G}, V, w' \rangle$ based on the worldline frame $\langle W', R, L, U' \rangle$ as follows. Set $W' := W$, $R := \triangleleft$, $U' = T$, and $w' = w$. Further, given $u \in U$, define $f_u : W' \rightarrow U'$ by letting $f_u(w) := t$, if $\tau(u, w) = t$. Then set $L := \{f_u : u \in U\}$. Then, for $v \in W'$, we have $U'_v := \{t \in U' : \exists l \in L(l(v) = t)\} = \{t \in T : \exists u \in U(\tau(u, v) = t)\}$. Call \mathfrak{G} the *worldline companion* of \mathfrak{F} .

PROPOSITION 26. *For every coherence frame \mathfrak{F} and its worldline companion \mathfrak{G} and for all φ :*

$$\mathfrak{F} \models \varphi \text{ iff } \mathfrak{G} \models \varphi$$

In particular, if a logic L is frame complete with respect to coherence frames, it is frame complete with respect to worldline frames.

PROOF. The proof is by a rather straightforward structural induction on φ . The only task is to define appropriate interpretations, so we only consider the atomic cases. Fix a coherence frame \mathfrak{F} and its worldline companion \mathfrak{G} . Suppose $\varphi = P(x_0, \dots, x_{n-1})$ and that $\langle \mathfrak{F}, \mathfrak{I}, \beta, w \rangle \not\models \varphi$ for some equivalential interpretation \mathfrak{I} , valuation β and world w . Define

$$\langle t_0, \dots, t_{n-1} \rangle \in V_w(P) \text{ iff } t_i = \tau(w, u_i) \text{ and } \langle u_0, \dots, u_{n-1} \rangle \in \mathfrak{I}_w(P) \quad (*)$$

and $\beta_w(x_i) = V_w(x_i)$. Then, clearly, $\langle \mathfrak{G}, V, w \rangle \not\models P(x_0, \dots, x_{n-1})$. Conversely, given an interpretation V and $\langle \mathfrak{G}, V, w \rangle \not\models \varphi$, we can define an interpretation \mathfrak{I} and a valuation β as in (*) such that \mathfrak{I} is equivalential and $\langle \mathfrak{F}, \mathfrak{I}, \beta, w \rangle \not\models \varphi$. The case $\varphi = (x_i \doteq x_j)$ is treated in the same way. \dashv

Since Schurz showed, among other things, that worldline semantics provides for the same class of frame complete logics in the absence of extra equality axioms as standard constant domain semantics, it follows that the same holds for coherence frames. This means that while coherence frames allow for a more natural treatment of non-rigid designation for example, unlike counterpart frames, they don't enlarge the class of frame complete logics unless one moves to the full second-order semantics as we will do in paragraph 12. But there is a way around this problem. Instead of introducing algebras of admissible interpretations, we can assume that certain worlds are copies of each other. The basic idea of incorporating an equivalence relation between worlds, which assumes that predicates are always interpreted in the same way in equivalent worlds, into a new notion of

frame is due to Melvin Fitting, compare his [3]. Here we use a slightly different approach, namely add a first-order bisimulation to coherence frames. To be precise, call a relation $\mathcal{E} \subseteq W \times W$ a **world-mirror**, if \mathcal{E} is an equivalence relation and whenever $v\mathcal{E}w$ and $v \triangleleft u_1$, there is a u_2 such that $w \triangleleft u_2$ and $u_1\mathcal{E}u_2$. Intuitively, two mirrored worlds v and w may be understood as a situation seen from two different perspectives (because v and w may have “different histories”, but have the “same future”).

DEFINITION 27 (Balanced Coherence Frames). *A **balanced coherence frame** is a pair $\langle \mathfrak{F}, \mathcal{E} \rangle$ where $\mathfrak{F} = \langle W, \triangleleft, U, T, \tau \rangle$ is a coherence frame and \mathcal{E} is a world-mirror on W . An interpretation \mathfrak{I} is called **balanced**, if it is equivalential and $\langle u_0, \dots, u_{n-1} \rangle \in \mathfrak{I}_v(P)$ iff $\langle u_0, \dots, u_{n-1} \rangle \in \mathfrak{I}_w(P)$ for all n -ary relations P and for all worlds v, w such that $v\mathcal{E}w$. A **balanced coherence model** is a quintuple $\langle \mathfrak{F}, \mathcal{E}, \mathfrak{I}, \beta, w \rangle$, where $\langle \mathfrak{F}, \mathcal{E} \rangle$ is a balanced coherence frame, \mathfrak{I} a balanced interpretation, β a valuation and w a world.*

The next theorem gives the connection between counterpart frames and balanced coherence frames.

THEOREM 28. *For every counterpart frame \mathfrak{F} there exists a balanced coherence frame \mathfrak{F}^* such that for all formulae φ :*

$$\mathfrak{F} \models \varphi \iff \mathfrak{F}^* \models \varphi^\gamma.$$

PROOF. Fix a counterpart frame $\mathfrak{F} = \langle W, T, \mathcal{U}, \mathcal{C} \rangle$. Let $\mathfrak{F}' = \langle W', T', \mathcal{U}', \mathcal{C}' \rangle$ be the unravelled Lewisian counterpart frame defined as in Theorem 19. We define a balanced coherence frame $\mathfrak{F}^* = \langle W^*, \triangleleft, U^*, T^*, \tau, \mathcal{E} \rangle$ from \mathfrak{F}' . Let $W^* = W'$, $T^* = T'$ and define $\pi \triangleleft \nu \iff \mathcal{C}'(\pi, \nu) \neq \emptyset$. Since \mathfrak{F}' is rich in objects, there is an object $o : W' \rightarrow T$ for every thread in \mathfrak{F}' . Define U^* as the set of all objects in \mathfrak{F}' and set $\tau(o, w) = t \iff o(w) = t$. Finally, for $\pi, \nu \in W^*$, set $\pi\mathcal{E}\nu \iff e(\pi) = e(\nu)$, where $e(\pi), e(\nu)$ again denote the endpoints of $e(\pi)$ and $e(\nu)$, respectively. Clearly, \mathcal{E} is a world-mirror for it is an equivalence relation and if $\pi \triangleleft \mu$ and $\pi\mathcal{E}\nu$, there is a path μ' such that $r(\mu) = r(\mu')$ and $\nu \triangleleft \mu'$, hence $\mu\mathcal{E}\mu'$.

Now suppose that $\mathfrak{F} \not\models \varphi$. Then there is a valuation β , an interpretation \mathfrak{I} and a world v such that $\langle \mathfrak{F}, \mathfrak{I}, \beta, v \rangle \not\models \varphi$. By Theorem 19, $\langle \mathfrak{F}', \mathfrak{I}', \beta', \nu \rangle \not\models \varphi$, where $\mathfrak{I}'(P)(\pi) := \mathfrak{I}(P)(e(\pi))$ for all worlds π , $\beta'_\pi(x_i) := \beta_w(x_i)$ if $e(\pi) = w$ and ν is a world in \mathfrak{F}' such that $e(\nu) = v$. Set $\mathfrak{I}^*(P)(\pi) := \{ \langle o_0, \dots, o_{n-1} \rangle \in (U^*)^n : \langle o_0(e(\pi)), \dots, o_{n-1}(e(\pi)) \rangle \in \mathfrak{I}'(P)(\pi) \}$ and choose an object valuation β^* on \mathfrak{F}^* such that $\beta^*(x_i)(\nu) = \beta'(x_i)$. Such a valuation exists because \mathfrak{F}' is rich in objects. Furthermore, \mathfrak{I}^* is a balanced interpretation, so $\langle \mathfrak{F}^*, \mathfrak{I}^*, \beta^*, \nu \rangle$ defines a balanced coherence model. We claim that $\mathfrak{M}^* = \langle \mathfrak{F}^*, \mathfrak{I}^*, \beta^*, \nu \rangle \not\models \varphi^\gamma$. The atomic and Boolean cases are trivial and the quantificational case follows from object richness. So consider the case $\varphi = \diamond\psi(y_0, \dots, y_{n-1})$. We have to show that $\mathfrak{M}^* \models \bigwedge z_0 \dots \bigwedge z_{n-1} . (\bigwedge_{i < n} z_i \doteq y_i \rightarrow \Box \neg \psi(\vec{z}/\vec{y})^\gamma)$. Choose objects o_0, \dots, o_{n-1} and a \vec{z} -variant $\widetilde{\beta}^*$ such that $o_i(\pi) = \widetilde{\beta}^*_\pi(z_i) = \beta^*_\pi(y_i)$ for all i . Then $\langle \mathfrak{F}^*, \mathfrak{I}^*, \widetilde{\beta}^*, \nu \rangle \models \bigwedge_{i < n} z_i \doteq y_i$. Assume further that for some $\pi \in W^*$ we have $\nu \triangleleft \pi$, i.e. that there is $\rho \in \mathcal{C}'(\nu, \pi)$. Then, since \mathfrak{F}' is a Lewisian counterpart frame, ρ is unique and hence $\langle o_i(\nu), o_i(\pi) \rangle \in \rho$ for all i . Then, by assumption, we have $\langle \mathfrak{F}, \mathfrak{I}, \widetilde{\beta}, w \rangle \not\models \psi$, where $e(\pi) = w$ and $\beta_\pi(z_i) = \beta^*_\pi(z_i)$, so by induction it follows that $\langle \mathfrak{F}^*, \mathfrak{I}^*, \widetilde{\beta}^*, \pi \rangle \models \neg \psi(\vec{z}/\vec{y})^\gamma$.

Conversely, suppose that $\mathfrak{F}^* \not\models \varphi^\gamma$. Again we consider only the case of $\varphi^\gamma = \bigvee_{i < n} z_i . (\bigwedge_{i < n} z_i \doteq y_i \wedge \diamond \psi(\vec{z}/\vec{y})^\gamma)$ and assume that we can find a countermodel for ψ if a countermodel for ψ^γ is given. Pick a balanced interpretation \mathfrak{I}^* , an object valuation β^* and a world ν such that $\langle \mathfrak{F}^*, \mathfrak{I}^*, \beta^*, \nu \rangle \not\models \varphi^\gamma$. Let \mathfrak{M}' be the Lewisian counterpart model $\langle \mathfrak{F}', \mathfrak{I}', \beta', \nu \rangle$ where \mathfrak{F}' is as above, $\mathfrak{I}'(P)(\pi) = \{ \langle t_0, \dots, t_{n-1} \rangle \in (T')^n : \exists o_i \in U^* \text{ with } o_i(\pi) = t_i (i < n) \wedge \langle o_0, \dots, o_{n-1} \rangle \in \mathfrak{I}^*(P)(\pi) \}$ for all worlds π and $\beta'_\nu(y_i) = \beta^*(y_i)(\nu)$. We claim that $\mathfrak{M}' \models \Box \neg \psi$, which implies by Theorem 19 that $\mathfrak{F} \not\models \varphi$. So suppose there is a world π such that $\nu \triangleleft \pi$ in \mathfrak{F}' , i.e. that there is $\delta \in \mathcal{C}(e(\nu), e(\pi))$ and $\delta = r(\pi)$. Suppose $\widetilde{\beta}'$ is a counterpart valuation such that $\widetilde{\beta}' : \beta' \xrightarrow{\delta} \widetilde{\beta}'$. By object richness there are objects $u_i \in U^*(i < n)$ such that $u_i(\nu) = \beta'_\nu(y_i)$ and $u_i(\pi) = \widetilde{\beta}'_\pi(y_i)$. Hence there is an

objectual \bar{z} -variant $\widetilde{\beta}^*$ such that $\widetilde{\beta}^*(z_i)(\nu) = \beta^*(y_i)(\nu)$ for all i . Then $\langle \mathfrak{F}^*, \mathfrak{J}^*, \widetilde{\beta}^*, \nu \rangle \models \bigwedge_{i < n} z_i \doteq y_i$ whence $\langle \mathfrak{F}^*, \mathfrak{J}^*, \widetilde{\beta}^*, \pi \rangle \models \neg \psi^\gamma(\bar{z}/\bar{y})$. But $\widetilde{\beta}^*(z_i)(\pi) = \widetilde{\beta}'_\pi(y_i)$ and the claim follows. \dashv

This result has interesting consequences. For example, since counterpart semantics is frame complete with respect to all first-order extensions QL of canonical propositional modal logics L (compare [6]), the same holds true for the translation QL^γ with respect to balanced coherence frames. Now we noted above that coherence frames per se characterize the same logics as standard constant domain semantics if no extra equality axioms are involved. But it is known that already rather simple canonical propositional logics have frame incomplete predicate extensions. In [2] it is shown that to complete frame incomplete MPLs by adding appropriate axioms, one needs mixed de re formulae rather than substitution instances of purely propositional formulae. So the above result gives a hint on where the source for frame incompleteness with respect to standard semantics is to be found. In particular, note that the translation $.\gamma$ leaves propositional formulae untouched, whereas de re formulae of the form $\Diamond\varphi(y_0, \dots, y_{n-1})$ are transformed into formulae $\bigvee z_0 \dots \bigvee z_{n-1} \bigwedge_{i < n} z_i \doteq y_i \wedge \Diamond\varphi(\bar{z}/\bar{y})^\gamma$, which are de re formulae involving equality. So what we need if we want to use standard possible worlds semantics to characterize a large class of logics via frame completeness are two things: firstly, a different understanding of the modal operator as given by $.\gamma$, and, secondly, the assumption that certain worlds are copies of each other.

§10. Varieties of Equality. Scott [12] proposes various kinds of identity. The first is the one we have discussed so far, namely **identity in trace**. The second, stronger notion, is the **inherent identity in trace**, which we shall denote by \doteq^* . Two objects satisfy this at a world if they are identical in trace at all subsequent worlds. The third is the **global identity in trace**, which we denote by \approx . Two objects are globally identical in trace at a world if their traces are identical in all worlds that can be reached from this world by either moving forward or backward along the relations. The fourth is **strong identity in trace**, denoted by \approx^+ . Two objects are strongly identical in trace if they have identical trace in all worlds. The fifth is **identity as object**, denoted by \equiv . This is the numerical identity of objects. The semantics can be formally defined as follows. Denote by $T(w)$ the set of all worlds which are accessible from w in a series of steps. More formally, we define this as follows.

DEFINITION 29 (Transit). *Let $\mathfrak{C} = \langle W, \triangleleft, U, T, \tau, \mathfrak{J} \rangle$ be a coherence frame. Then define $v \triangleleft^n w$ inductively by (a) $v \triangleleft^0 w$ iff $v = w$, (b) $v \triangleleft^{n+1} w$ iff there is a $u \in W$ such that $v \triangleleft^n u \triangleleft w$. Further, put $v \triangleleft^* w$ iff there is an n such that $v \triangleleft^n w$. Define $T(v) := \{w : v \triangleleft^* w\}$, and $Z(v) := \{w : v(\triangleleft \cup \triangleleft^\sim)^* w\}$.*

Here, $R^\sim := \{\langle y, x \rangle : \langle x, y \rangle \in R\}$ is the converse relation of R .

DEFINITION 30 (Equality in Coherence Models). *Let $\langle \mathfrak{C}, \beta, v \rangle$, be a coherence model.*

$$\begin{aligned} \langle \mathfrak{C}, \beta, v \rangle \models x \doteq y & \quad :\Leftrightarrow \quad \tau(\beta(x), v) = \tau(\beta(y), v) \\ \langle \mathfrak{C}, \beta, v \rangle \models x \doteq^* y & \quad :\Leftrightarrow \quad \text{for all } w \in T(v) : \tau(\beta(x), w) = \tau(\beta(y), w) \\ \langle \mathfrak{C}, \beta, v \rangle \models x \approx y & \quad :\Leftrightarrow \quad \text{for all } w \in Z(v) : \tau(\beta(x), w) = \tau(\beta(y), w) \\ \langle \mathfrak{C}, \beta, v \rangle \models x \approx^+ y & \quad :\Leftrightarrow \quad \text{for all } w \in W : \tau(\beta(x), w) = \tau(\beta(y), w) \\ \langle \mathfrak{C}, \beta, v \rangle \models x \equiv y & \quad :\Leftrightarrow \quad \beta(x) = \beta(y) \end{aligned}$$

As it turns out, although all these notions are different semantically, we can only distinguish simple identity in trace from the other relations, that is to say, the latter four cannot be defined by means of modal axioms in the standard modal language using \doteq . Metatheoretically, the interrelations between \approx , \approx^+ , \doteq^* and \equiv can—besides the usual axioms for identity (reflexivity, symmetry, transitivity)—be given by the following postulates.

$$\begin{aligned} \bigwedge x. \bigwedge y. x \equiv y \rightarrow x \approx^+ y \\ \bigwedge x. \bigwedge y. x \approx^+ y \rightarrow x \approx y \\ \bigwedge x. \bigwedge y. x \approx y \rightarrow x \doteq^* y \\ \bigwedge x. \bigwedge y. x \doteq^* y \rightarrow x \doteq y \\ \bigwedge x. \bigwedge y. x \doteq^* y \rightarrow \Box(x \doteq^* y) \end{aligned}$$

Global identity in trace implies strong identity in trace if the frame is connected. A frame is called **cyclic** if for all v : $T(v) = Z(v)$. S5–frames are cyclic. Tense frames are also cyclic. (Notice that we have not defined $T(v)$ for polymodal frames. See however [9] for a definition.) Inherent identity in trace implies global identity in cyclic frames.

§11. Objects in Metaframes. Shehtman and Skvortsov have introduced in [13] the metaframe semantics and shown that it is complete for all modal predicate logics. Their results were stated and proved for superintuitionistic logics and extensions of S4. However, by removing some of the category theoretic definitions one can generalize these results easily to arbitrary modal predicate logic.

DEFINITION 31. Σ denotes the category of finite ordinals and functions between them.

DEFINITION 32 (Metaframe). A **general metaframe** M is a contravariant functor from the category Σ into the category of general frames. In particular, for every n , $M(n) = \langle F_n, \triangleleft_n, \mathbb{F}_n \rangle$ is a general frame, and for each $\sigma : m \rightarrow n$, $M(\sigma)$ is a p –morphism from $M(n)$ to $M(m)$. A **metaframe** is a contravariant functor from Σ into the category of Kripke–frames. We call the members of F_n n –**points**.

The idea is this. $M(0)$ represents the frame of possible worlds and $M(n)$ for $n > 0$ represents n –tuples over worlds. The arrows are needed to be able to identify the worlds and the tuples. For example, there is a unique map $0_n : 0 \rightarrow n$ for each n . Consequently, we have a map $M(0_n) : M(n) \rightarrow M(0)$. Thus, for each $a \in M(n)$, the **world** of a is $M(0_n)(a)$. Further, there is a (unique) natural embedding $i_{n,n+1} : n \rightarrow n+1 : i \mapsto i$. Hence, we define a **projection** of $a \in M(n+1)$ onto $M(n)$ by $M(i_{n,n+1})(a)$. We shall write $a \downarrow b$ if $a \in M(n+1)$ for some $n \in \omega$ and $b = M(i_{n,n+1})(a)$. Further, write $p_i^n : 1 \rightarrow n$ for the unique map sending 0 to i and if $\sigma : m \rightarrow n$, write $x^\sigma := \langle x_{\sigma(0)}, x_{\sigma(1)}, \dots, x_{\sigma(m-1)} \rangle$.

DEFINITION 33 (Interpretation). Let M be a general metaframe. An **interpretation** on M is a function ξ assigning to each predicate letter P an internal set of $M(\Omega(P))$, i.e. $\xi(P) \in \mathbb{F}_{\Omega(P)}$. For $a \in M(n)$ and $\sigma : m \rightarrow n$, $m = \Omega(P)$ we define

$$\begin{aligned} \langle M, \xi, a \rangle \models P(x^\sigma) & :\Leftrightarrow M(\sigma)(a) \in \xi(P) \\ \langle M, \xi, a \rangle \models x_i \dot{=} x_j & :\Leftrightarrow M(p_i^n)(a) = M(p_j^n)(a) \\ \langle M, \xi, a \rangle \models \neg \chi & :\Leftrightarrow \langle M, \xi, a \rangle \not\models \chi \\ \langle M, \xi, a \rangle \models \varphi \wedge \chi & :\Leftrightarrow \langle M, \xi, a \rangle \models \varphi; \chi \\ \langle M, \xi, a \rangle \models \diamond \chi & :\Leftrightarrow \text{exists } b \triangleright_n a : \langle M, \xi, b \rangle \models \chi \\ \langle M, \xi, a \rangle \models \bigvee x_n \cdot \chi & :\Leftrightarrow \text{exists } b \downarrow a : \langle M, \xi, b \rangle \models \chi \end{aligned}$$

We can identify $M(0)$ with the sets of worlds, $M(1)$ with the sets of objects, $M(2)$ with the sets of pairs of objects, and so on. Now, the definitions will not suffice to define a MPL from a metaframe unless it satisfies a further condition. Let $\sigma : m \rightarrow n$. Write σ^+ for the unique function from $m+1$ to $n+1$ such that $\sigma^+ \upharpoonright m = \sigma$ and $\sigma^+(m) = n$. Further, for $m \leq n$, $i_{m,n} : m \rightarrow n : j \mapsto j$ is the unique inclusion.

DEFINITION 34 (Modal Metaframe). A metaframe satisfies the **lift property** if for all $\sigma : m \rightarrow n$ and $a \in M(n)$, $b \in M(m+1)$ such that $M(\sigma)(a) = M(i_{m,m+1})(b) = d \in M(m)$ there exists $a \in M(n+1)$ such that

$$a = M(i_{n,n+1})(c) \text{ and } b = M(\sigma^+)(c)$$

A metaframe is a **modal metaframe** if it satisfies the lift property.

Shehtman and Skvortsov give a canonical procedure to obtain a modal metaframe from a modal predicate logic, see also Bauer [1]. Let L be given. We then let $M(n)$ be the set of all complete n –types $\Gamma = \Gamma(x_0, \dots, x_{n-1})$. They form a frame, where $\Gamma \triangleleft_n \Delta$ iff for all $\Box \varphi \in \Gamma$ we have $\varphi \in \Delta$. Further, if $\sigma : m \rightarrow n$, then $M(\sigma) : M(n) \rightarrow M(m)$ is defined by

$$M(\sigma)(\Delta) := \{ \chi : [x_{\sigma(0)}/x_0, x_{\sigma(1)}/x_1, \dots, x_{\sigma(m-1)}/x_{m-1}] \chi \in \Delta \}$$

Actually, the definition of truth in a model can be changed somewhat. First, it can be shown that if the free variables of χ are in the set $\{x_i : i < n\}$, we have

$$\langle M, \xi, a \rangle \models \chi \quad \Leftrightarrow \quad \text{for all } b \downarrow a : \langle M, \xi, b \rangle \models \chi$$

In this form we see that the truth of a formula is upward persistent. (The above-mentioned conditions on metaframes are such that this holds.)

Now write $a \sim_i c$ for the following. Let $\sigma : n \rightarrow n : i \mapsto n-1, n-1 \mapsto i, j \mapsto j$ ($j \notin \{i, n-1\}$). (If i is not less than $n-1$, this map is the identity.) Then $a \sim_i c$ iff there exist b such that $M(\sigma)(a) \downarrow b$ and $M(\sigma)(c) \downarrow b$. Then we have

$$\langle M, \xi, a \rangle \models \bigvee x_i. \chi \quad \Leftrightarrow \quad \text{there is } c \sim_i a : \langle M, \xi, c \rangle \models \chi$$

This is the form that we shall use later on. (It is closer in spirit to cylindrification.)

This construction is highly abstract. We shall demonstrate this construction with a very simple example. Our language contains only equality. Suppose that we have a logic L which contains $\varphi \leftrightarrow \Box\varphi$ for every formula φ . Then we have $\Gamma \triangleleft_n \Delta$ iff $\Gamma = \Delta$ for all n -types Γ and Δ . So, the relations are trivial. Suppose also that the logic contains the sentence saying that there are exactly three objects. We shall calculate the cardinalities of the $M(n)$. There is exactly one 0-type Γ_0 , since the logic is POST-complete. There exists exactly one 1-type, since all objects are indistinguishable. There exist 2 2-types, namely the type containing $x_0 \doteq x_1$ and the other containing $x_0 \not\equiv x_1$. The general formula is as follows. Let n be given. Choose a function $f : \{0, 1, \dots, n-1\}$ into the set $\{0, 1, 2\}$. Then for this function the associated type is

$$t_f := \{x_i \doteq x_j : f(i) = f(j), i, j < n\} \cup \{\neg(x_i \doteq x_j) : f(i) \neq f(j), i, j < n\}$$

Now let $f \approx g$ iff there is a permutation $\pi : \{0, 1, 2\} \rightarrow \{0, 1, 2\}$ such that $f = \pi \circ g$. Obviously, $t_f = t_g$ iff $f \approx g$. (Case 1) $f(i) = f(j)$ for all $i, j < n$. There are 3 functions, all representing the same type. (Case 2) The image of f has at least two members. There are $3^n - 3$ many such functions. Each type is represented by six functions. This gives $(3^{n-1} - 1)/2$ functions. In total we have $(3^{n-1} + 1)/2$ functions. The series is

$$1, 2, 5, 14, 41, 122, \dots$$

It is clear that the objects in such a frame are very difficult to recover. For this reason, Shehtman and Skvortsov define a *cartesian metaframe* (see below in Section 13 for a definition). This is a metaframe in which for each 0-type Γ , the set of all n -types Δ with sentential reduct Γ is isomorphic to the n -fold cartesian product of the set of 1-types with sentential reduct Γ . Moreover, the projections are the maps $M(i_i^n)$, where $i_i^n : 1 \rightarrow n : 0 \mapsto i$. As Bauer [1] shows, each metaframe that allows to fuse types (a condition which we shall not define here) has a logically equivalent cartesian metaframe. The canonical metaframe defined above satisfies this condition. Thus, every logic is complete with respect to cartesian metaframes. Still, these proofs are very tedious. In the next section we shall show how the present results allow to prove completeness with respect to metaframe semantics using the previous completeness result.

§12. Going Second Order. In Kracht and Kutz [10] we have defined a notion of second order modal logics. Although they technically correspond to Π_1^1 -formulae, we shall nevertheless call them second order logics. To be precise we shall describe them as logics over a slightly different language. Namely, while before we had a set Π of predicate symbols, now we assume to have predicate variables of any given arity.

DEFINITION 35 (Symbols and Language). *The languages of **second order modal predicate logic**, abbreviated collectively by **MPL**², contain the following symbols.*

1. A denumerable set $V := \{x_i : i \in \omega\}$ of **object variables**.
2. A denumerable set $C := \{c_i : i \in \omega\}$ of **constants**.
3. For each $n \in \omega$, a denumerable set $PV^n := \{P_i^n : i \in \omega\}$ of **predicate variables**.
4. **Boolean functors** \perp, \wedge, \neg .

5. **Quantifiers** \forall, \wedge .
6. A set $M := \{\Box_\lambda : \lambda < \kappa\}$ of **modal operators**.

Furthermore, $\Omega(P_i^n) = n$ for all $n, i \in \omega$.

There may also be predicate constants but we have chosen to do without them for the purpose of simplicity, with the exception of equality. As before, we deal with only one modal operator. The generalization is obvious. The following substitution principle has first been discussed by Steven Kleene in his [8].

DEFINITION 36 (Second Order Substitution). *Let φ and χ be formulae of \mathbf{MPL}^2 and P be a predicate variable. Then $[\chi/P]\varphi$ denotes the formula that is obtained by replacing every occurrence of $[\vec{y}/\vec{x}]P$ by $[\vec{y}/\vec{x}]\chi$, where bound variables get replaced in the usual (first-order) way to prevent accidental capture.*

We shall describe this substitution principle in a little more detail. Notice that χ can have free variables that are not among the variables $x_0, \dots, x_{\Omega(P)-1}$. Let \vec{z} be these variables. Then we replace φ by a bound variant φ' , in which all variables of φ occurring in \vec{z} are replaced by suitable variables not occurring in either φ or χ . Next, we perform the replacement of any occurrence of $[\vec{y}/\vec{x}]P$ for some variables \vec{y} by $[\vec{y}/\vec{x}]\chi$. This time, no bound variant needs to be chosen. For example, let $\varphi = \forall x_2. \wedge x_0. \wedge x_1. P_0^1(x_2) \wedge P_1^2(x_0, x_2) \rightarrow P_1^2(x_1, x_0)$. Let $\chi = \forall x_1. P_0^3(x_2, x_1, x_0)$. Suppose that we want to replace P_1^2 by χ . Then since x_2 occurs free in χ , we shall replace bound occurrences of x_2 in φ by x_4 . This gives

$$\varphi' = \forall x_4. \wedge x_0. \wedge x_1. P_0^1(x_2) \wedge P_1^2(x_0, x_4) \rightarrow P_1^2(x_1, x_0)$$

Finally, we must replace $P_1^2(x_0, x_4)$ by $[x_4/x_1]\chi = \forall x_1. P_0^3(x_2, x_1, x_0)$ and $P_1^2(x_1, x_0)$ by $[x_1/x_0, x_0/x_1]\chi = \forall x_3. P_0^3(x_2, x_3, x_1)$.

$$[\chi/P_1^2]\varphi = \forall x_4. \wedge x_0. \wedge x_1. P_0^1(x_2) \wedge \forall x_1. P_0^3(x_2, x_1, x_0) \rightarrow \forall x_3. P_0^3(x_2, x_3, x_1)$$

This language does not contain an existence predicate constant, but this is only for convenience. There are no further complications in introducing predicate constants as well, but we have omitted them here to keep the notation reasonably simple. The following definition is analogous to Skvortsov and Shehtman [13].

DEFINITION 37 (Second Order MPL). *A **second order MPL** is a set L of \mathbf{MPL}^2 -formulae satisfying the following conditions.*

1. L contains all instances of axioms of first-order logic.
2. L is closed under all rules of first-order logic.
3. L is closed under second-order substitution.
4. L contains all instances of axioms of the modal logic \mathbf{K} .
5. L is closed under the rule $\varphi/\Box\varphi$.
6. $\Diamond\forall y. \varphi \leftrightarrow \forall y. \Diamond\varphi \in L$.

Clearly, a second order MPL can be viewed as a special sort of a first-order MPL, by taking Π to be the union of the sets PV^m . This allows us to speak, for example, of the Henkin-closure of L . However, these languages are technically distinct, since the predicate variables are not interpreted in the structure. Their value is not fixed in the structure, just like the value of an object variable is not fixed in a first-order structure. This means that technically we get a different notion of structure. However, the way we get these structures is by abstracting them from the corresponding first-order structures. Thus, we begin with a second order MPL L and interpret it as a first-order MPL, also called L , for which we then build the canonical structure \mathbf{Can}_{L^*} . Starting with this structure we shall define the second-order structure for L^* .

DEFINITION 38 (Second Order Coherence Frame). *A **second order coherence frame** is a triple $\langle W, \triangleleft, U \rangle$, where $\langle W, \triangleleft \rangle$ is a Kripke-frame and U a set.*

Given a second order coherence frame, we call an n -**point** a member of $W \times U^n$; an n -**set** a subset of $W \times U^n$. Let $p = \langle v, \vec{a} \rangle$ and $q = \langle w, \vec{c} \rangle$ be n -points. We write $p \sim_k q$ if $a_i = c_i$ for all $i \neq k$.

$$\mathbb{V}_k(A) := \{q : \text{exists } p \in A : p \sim_k q\}$$

(If $k < n$ does not obtain, then we may put $\mathbb{V}_k(A) := A$.) Next, let $\sigma : m \rightarrow n$. Then we define $\widehat{\sigma}$ on n -points as follows.

$$\widehat{\sigma}(\langle v, \vec{a} \rangle) := \langle v, \langle a(\sigma(i)) : i < m \rangle \rangle$$

If p is an n -point, $\widehat{\sigma}(p)$ is an m -point. So, $\widehat{\cdot}$ is a contravariant functor from Σ into the set of functions from points to points. This is also directly verified. If $\tau : \ell \rightarrow m$ then

$$\begin{aligned} \widehat{\sigma \circ \tau}(p) &= \langle v, \langle a(\sigma \circ \tau(i)) : i < \ell \rangle \rangle \\ &= \widehat{\sigma}(\langle v, \langle a(\tau(i)) : i < \ell \rangle \rangle) \\ &= \widehat{\tau}(\widehat{\sigma}(p)) \\ &= \widehat{\tau} \circ \widehat{\sigma}(p) \end{aligned}$$

For an m -set A we put

$$\mathbb{C}_\sigma(A) := \{p : \widehat{\sigma}(p) \in A\}$$

It is directly verified that \mathbb{C} is covariant, that is, $\mathbb{C}_{\sigma \circ \tau} = \mathbb{C}_\sigma \circ \mathbb{C}_\tau$. For we have for an ℓ -set A :

$$\begin{aligned} \mathbb{C}_{\sigma \circ \tau}(A) &= \{p : \widehat{\sigma \circ \tau}(p) \in A\} \\ &= \{p : \widehat{\tau}(\widehat{\sigma}(p)) \in A\} \\ &= \{p : \widehat{\sigma}(p) \in \mathbb{C}_\tau(A)\} \\ &= \mathbb{C}_\sigma(\mathbb{C}_\tau(A)) \\ &= \mathbb{C}_\sigma \circ \mathbb{C}_\tau(A) \end{aligned}$$

And finally we set

$$\blacklozenge A := \{\langle w, \vec{a} \rangle : \text{exists } v \triangleright w : \langle v, \vec{a} \rangle \in A\}$$

DEFINITION 39 (n -Complexes and Towers). An n -**complex** over a 2nd order coherence frame is a set \mathbb{C}_n of n -sets closed under intersection, complement, the operations \mathbb{V}_k , \blacklozenge and \mathbb{C}_σ for every $\sigma : n \rightarrow n$. A **tower** is a sequence $\langle \mathbb{C}_n : n \in \omega \rangle$ such that \mathbb{C}_n is an n -complex for every n , and for every $\sigma : n \rightarrow m$, $\mathbb{C}_\sigma : \mathbb{C}_n \rightarrow \mathbb{C}_m$.

DEFINITION 40 (Generalized Second Order Coherence Frames). A **generalized second order coherence frame** is a quadruple $\mathfrak{S} = \langle W, \triangleleft, U, \mathcal{T} \rangle$, where $\langle W, \triangleleft, U \rangle$ is a second order coherence frame and $\mathcal{T} = \langle \mathbb{C}_i : i \in \omega \rangle$ a tower over it. A **valuation** into \mathfrak{S} is a pair ξ and β of mappings, where ξ is defined on all predicate variables, and $\xi(P_i^m) \in \mathbb{C}_m$ for all $m, i \in \omega$ and β assigns to each $x_i \in V$ a member of U .

$$\begin{aligned} \langle \mathfrak{S}, \xi, \beta, v \rangle \models P_i^m(\vec{y}) &:\Leftrightarrow \langle \beta(y_i) : i < m \rangle \in \xi(P_i^m) \\ \langle \mathfrak{S}, \xi, \beta, v \rangle \models \neg \chi &:\Leftrightarrow \langle \mathfrak{S}, \xi, \beta, v \rangle \not\models \chi \\ \langle \mathfrak{S}, \xi, \beta, v \rangle \models \varphi \wedge \chi &:\Leftrightarrow \langle \mathfrak{S}, \xi, \beta, v \rangle \models \varphi; \chi \\ \langle \mathfrak{S}, \xi, \beta, v \rangle \models \bigvee y. \chi &:\Leftrightarrow \text{for some } \beta' \sim_y \beta : \langle \mathfrak{S}, \xi, \beta', v \rangle \models \chi \\ \langle \mathfrak{S}, \xi, \beta, v \rangle \models \diamond \chi &:\Leftrightarrow \text{for some } w \triangleright v : \langle \mathfrak{S}, \xi, \beta, w \rangle \models \chi \end{aligned}$$

We write $\mathfrak{S} \models \varphi$ if for all valuations ξ, β and all worlds v : $\langle \mathfrak{S}, \xi, \beta, v \rangle \models \varphi$.

Notice that Shehtman and Skvortsov define the truth of a formula at an n -point. We can do the same here. Namely, we set

$$\langle \mathfrak{S}, \xi, \langle v, \vec{a} \rangle \rangle \models \varphi$$

iff for any valuation β such that $\beta(x_i) = a_i$ for all $i < n$ we have

$$\langle \mathfrak{S}, \xi, \beta, v \rangle \models \varphi$$

An inductive definition can be given as well. We can also define a valuation on a metaframe in the following way. We say that a function $\beta : V \rightarrow \bigcup_n M(n)$ is a **valuation** if $\beta(x_n) \in M(n)$, and if $\beta(x_{n+1}) \downarrow \beta(x_n)$ for every n . Thus, $\beta(x_n)$ is an n -point which is the projection of the $n+1$ -point $\beta(x_{n+1})$.

We shall show that second order MPLs are complete with respect to this semantics. So, let L be a second order MPL. We understand it as a first-order MPL, which we denote by the same letter. Then we can construct the canonical first-order coherence structure $\mathbf{Can}_{L^*} = \langle W_{L^*}, \triangleleft, C_{L^*}, \mathcal{J}_{L^*} \rangle$. We shall now define a second order canonical *frame* from it. This construction is completely general. First, observe that we can transport the notion of n -point as well as the satisfaction of a formula at an n -point to frames (and first-order coherence frames).

Take a modal (first-order) structure $\mathfrak{S} = \langle W, \triangleleft, U, \mathcal{J} \rangle$. Let φ be a formula such that the free variables occurring in it are contained in $\{x_i : i < n\}$. Then we write $[\varphi]_n$ for the set of n -points satisfying φ . Formally,

$$[\varphi]_n := \{p : \langle \mathfrak{S}, p \rangle \models \varphi\}$$

Now set

$$\mathbb{C}_n := \{[\varphi]_n : fvar(\varphi) \subseteq \{x_i : i < n\}\}$$

Finally, we put $\mathcal{T} := \langle \mathbb{C}_n : n \in \omega \rangle$. Now let

$$\mathfrak{S}^2 := \langle W, \triangleleft, U, \mathcal{T} \rangle$$

LEMMA 41. \mathfrak{S}^2 is a second order generalized coherence frame. Furthermore, with $\xi(P) := \mathcal{J}(P)$ we have for every n -point p and every formula φ with free variables in $\{x_i : i < n\}$:

$$\langle \mathfrak{S}, p \rangle \models \varphi \iff \langle \mathfrak{S}^2, \xi, p \rangle \models \varphi$$

Proof. We need to verify that \mathcal{T} is a tower. This follows from the following equations. (In the last clause, $\sigma : n \rightarrow m$.)

$$\begin{aligned} [\neg\chi]_n &= -[\chi]_n \\ [\varphi \wedge \chi]_n &= [\varphi]_n \cap [\chi]_n \\ [\bigvee x_k \cdot \chi]_n &= \bigvee_k([\chi]_n) \\ [\diamond\chi]_n &= \blacklozenge[\chi]_n \\ [[\vec{x}^\sigma / \vec{x}]\chi]_n &= \mathbf{C}_\sigma([\chi]_m) \end{aligned}$$

Only the last clause needs comment.

$$\begin{aligned} [[\vec{x}^\sigma / \vec{x}]\chi]_n &= \{p \in U \times W^n : \langle \mathfrak{S}, p \rangle \models [\vec{x}^\sigma / \vec{x}]\chi\} \\ &= \{p \in U \times W^n : \langle \mathfrak{S}, \hat{\sigma}(p) \rangle \models \chi\} \\ &= \{p \in U \times W^n : \hat{\sigma}(p) \in [\chi]_m\} \\ &= \mathbf{C}_\sigma([\chi]_m) \end{aligned}$$

The second claim is immediate to verify. □

THEOREM 42 (Second Order Completeness). *Let L be a second order modal logic without equality with Henkin-closure L^* and φ a formula. Then $\mathbf{Can}_{L^*}^2 \models \varphi$ iff $\varphi \in L$. It follows that L is complete with respect to second order generalized coherence frames.*

Proof. We pass to the Henkin-closure L^* of the (first-order MPL) L . Let ξ_{L^*} be defined by

$$\xi_{L^*}(P_i^m) := [P_i^m(x_0, \dots, x_{m-1})]_m$$

Then, by first order completeness and Lemma 41 we get

$$\langle \mathbf{Can}_{L^*}^2, \xi_{L^*} \rangle \models \varphi \iff \varphi \in L^*$$

We have to show that if $\varphi \in L^*$ then for every valuation ξ into $\mathbf{Can}_{L^*}^2$ we have

$$\langle \mathbf{Can}_{L^*}^2, \xi \rangle \models \varphi$$

For this establishes $\mathbf{Can}_{L^*}^2 \models \varphi$ in case $\varphi \in L$. If $\varphi \notin L$ then we have anyway

$$\langle \mathbf{Can}_{L^*}^2, \xi_{L^*} \rangle \not\models \varphi$$

by first-order completeness and Lemma 41. Now for the proof of the claimed fact. Assume that ξ is given. By definition of the tower \mathcal{T}_{L^*} there exists for every predicate variable P a formula $\pi_P(\vec{x})$ such that

$$\xi(P) = \xi_{L^*}(\pi_P(\vec{x}))$$

Let $pvar(\varphi)$ denote the set of predicate variables occurring in φ . Define

$$\varphi_* := [\pi_P/P : P \in pvar(\varphi)]\varphi$$

This formula is unique up to renaming of bound variables. Then, by induction, it is verified that

$$\langle \mathbf{Can}_{L^*}^2, \xi, \Delta \rangle \models \varphi \quad \Leftrightarrow \quad \langle \mathbf{Can}_{L^*}^2, \xi_{L^*}, \Delta \rangle \models \varphi_*$$

Since $\varphi \in L$ and L is closed under second order substitution we have $\varphi_* \in L$ as well. Hence the right hand side obtains, and therefore the left hand side is true as well. This is what we had to prove. \dashv

This construction of retracting the valuation ξ and adding the tower of definable sets is applicable to any first order coherence frame.

§13. Cartesian Metaframes. In this section we shall use the previous completeness proof to derive a very simple completeness proof for the metaframe semantics. Shehtman and Skvortsov give the following definition.

DEFINITION 43 (Cartesian Metaframes). *A metaframe M is called **cartesian** if the following holds.*

1. *There is a set U and a family $\{W_u : u \in U\}$ of nonempty and pairwise disjoint sets such that $M(0) = U$ and $M(n) = \bigcup_{u \in U} (W_u)^n$ for every n . We write $a \triangleleft_1 b$ iff there are $u, v \in U$ such that $a \in W_u$, $b \in W_v$, $u \triangleleft_0 v$ and $\langle u, a \rangle \triangleleft_1 \langle v, b \rangle$.*
2. *$\langle v, \vec{a} \rangle \triangleleft_n \langle w, \vec{c} \rangle$ iff*
 - (a) $v \triangleleft_0 w$
 - (b) $a_i \triangleleft_1 c_i$ for all $i < n$ and
 - (c) for all $i < j < n$: if $a_i = a_j$ then also $c_i = c_j$.
3. *For every $\sigma : n \rightarrow m$, $m, n > 0$, $M(\sigma) = \hat{\sigma} : \vec{a} \mapsto \langle a(\sigma(i)) : i < n \rangle$. For $\sigma : 0 \rightarrow n$, $n > 0$, $M(\sigma) : \vec{a} \mapsto u$, where u is such that $\vec{a} \in (W_u)^n$.*

For $\sigma : n \rightarrow 0$, there is no definition of $M(\sigma)$ given. One possibility is to choose an object $u^* \in W_u$ for every $u \in U$ and then let $M(\sigma) : u \mapsto (u^*)^n$. It is an approximation of the idea that the elements of the n th frame are n -tuples. While cartesian metaframes assume that the n -tuples are tuples of things, we shall offer another variant, where the idea is that the tuples are in fact tuples of objects.

DEFINITION 44 (Cubic Metaframe). *A metaframe M is called **cubic** if the following holds.*

1. *There are sets U and W such that $M(0) = U$ and $M(n) = U \times W^n$ for every n .*
2. *$\langle u, \vec{a} \rangle \triangleleft_n \langle v, \vec{c} \rangle$ iff $\vec{c} = \vec{a}$ and $u \triangleleft_0 v$.*
3. *For every $\sigma : n \rightarrow m$, $M(\sigma) = \hat{\sigma} : \langle u, \vec{a} \rangle \mapsto \langle u, \langle a(\sigma(i)) : i < n \rangle \rangle$.*

It is first of all to be checked that the above requirements define a contravariant functor from Σ to the class of generalized frames. (a) $M(n)$ is a general frame, as is easily seen. (b) for each $\sigma : m \rightarrow n$, $M(\sigma)$ is a p-morphism from $M(n)$ to $M(m)$. Namely, suppose that $p = \langle v, \vec{a} \rangle \triangleleft_n \langle w, \vec{c} \rangle = q$. Then $\vec{c} = \vec{a}$ and $v \triangleleft_0 w$. Hence $\langle a(\sigma(i)) : i < m \rangle = \langle c(\sigma(i)) : i < m \rangle$, and so $\hat{\sigma}(p) \triangleleft_m \hat{\sigma}(q)$. Second, suppose that $\hat{\sigma}(p) \triangleleft_m q'$. Then $q' = \langle w', c'(i) : i < m \rangle$ for some w' such that $v \triangleleft_0 w'$ and $c'(i) = a(\sigma(i))$ for each $i < m$. So, put $q := \langle w', \vec{a} \rangle$. Then $p \triangleleft_n q$ and $M(\sigma)(q') = q$. Third, let $A \in \mathcal{J}(m)$. Then $C_\sigma(A) \in \mathcal{J}(n)$, by definition of towers. This proves that $M(\sigma)$ is a p-morphism. (c) For each $\sigma : m \rightarrow n$ and $\tau : n \rightarrow q$, $M(\tau \circ \sigma) = M(\sigma) \circ M(\tau)$. But by previous calculations, $M(\tau \circ \sigma)(p) = \hat{\sigma} \circ \hat{\tau}(p) = M(\sigma) \circ M(\tau)(p)$.

PROPOSITION 45. *For every cubic metaframe M there exists a semantically equivalent cartesian metaframe N .*

Proof. Let M be a cubic metaframe. Put $N(0) := M(0)$ and $W_u := \{u\} \times W$ for all $u \in U$ and $\langle u, a \rangle \triangleleft_1 \langle v, b \rangle$ iff $u \triangleleft_0 v$ and $a = b$. Then $N(n) := \{\langle u, a_i \rangle : i < n\} : a_i \in W\}$ for all n and

$\vec{a} \triangleleft_n \vec{c}$ iff there are u and v such that $u \triangleleft_0 v$ and $a_i = \langle u, o_i \rangle$, $c_i = \langle v, o_i \rangle$ for some $o \in W$. The p -morphisms $M(\sigma)$ are straightforwardly defined. \dashv

The reason that this works is a construction that we have used before: the trace of an object at u may be the pair consisting of u and the object itself. It is easy to establish a bijective correspondence between second order generalized coherence frames and cubic generalized metaframes. Given a second order coherence frame $\langle W, \triangleleft, U, \mathcal{J} \rangle$, we simply define $M(n) := \langle W \times U^n, \triangleleft_n, \mathcal{J}(n) \rangle$, where $\langle v, \vec{a} \rangle \triangleleft_n \langle w, \vec{c} \rangle$ iff $\vec{a} = \vec{c}$ and $v \triangleleft w$. This is a cubic generalized metaframe. Conversely, let M be a cubic generalized metaframe, with $M(n) = \langle W \times U^n, \triangleleft_n, \mathbb{M}_n \rangle$ for every n . Then let $\triangleleft := \triangleleft_0$ and $\mathcal{J}(n) := \mathbb{M}_n$. Then it is easily checked that $\langle W, \triangleleft, U, \mathcal{J} \rangle$ is a second order general coherence frame. As an immediate consequence we get the following

THEOREM 46. *Every second order MPL without equality is complete with respect to cubic generalized metaframes.*

Notice that the way this result has been obtained is by abstraction from the first-order case, rather than the first-order case being an application of the second-order case. Completeness with respect to cartesian generalized metaframes now follows, given that cubic metaframes are special cartesian metaframes.

§14. Equality. Let us turn to the treatment of equality in coherence frames and metaframes. We have argued earlier that two objects may be equal in one world and different in another. Equality in a world has been regulated by the trace function. The most direct way to account for equality between objects is therefore to add the trace function into the generalized coherence frame. Another way is to add equality as a predicate constant whose interpretation is an equivalence relation on U in each individual world. Thus, we add a constant $\Delta \in \mathbb{C}_2$ such that the following holds. Write $a \Delta_w b$ iff $\langle w, a, b \rangle \in \Delta$.

1. $a_0 \Delta a_0$ for all $w \in W$, $a_0 \in U$.
2. If $a_0 \Delta_w a_1$ then $a_1 \Delta_w a_0$, for all $w \in W$, $a_0, a_1 \in U$.
3. If $a_0 \Delta_w a_1$ and $a_1 \Delta_w a_2$ then $a_0 \Delta_w a_2$ for all $w \in W$, $a_0, a_1, a_2 \in U$.

Additionally, a valuation must satisfy the following property. Call $A \in \mathbb{C}^n$ **equivalential** if for all $p = \langle w, \vec{a} \rangle \in A$, $q = \langle w, \vec{b} \rangle$ such that $a_i \Delta_w b_i$ for all $i < n$ then $q \in A$. Then we require that for every predicate P , $\xi(P)$ must be an equivalential set. However, notice that equivalential sets are not closed under \blacklozenge !

It may be disappointing to see that we have not been able to reduce $\dot{=}$ to simple identity. However, there is to our knowledge no semantics under which this is so (and for reasons given below it is not to be expected either). Let us look for example at metaframes. In a metaframe, $\langle M, \xi, a \rangle \models x_i \dot{=} x_j$ if $pr_{n,i}(a) = pr_{n,j}(a)$. Furthermore, a frame interpreting $\dot{=}$ (called an $m^{\dot{=}}$ -**metaframe**) must satisfy the following requirement:

$$(0^\sharp) \quad \text{For all } n \text{ and } a, b \in M(n) \quad a = b \text{ iff } pr_{n,i}(a) = pr_{n,i}(b).$$

This condition effectively eliminates the distinction between object and trace. However, in general metaframes the possibility of distinct developments for identical objects still exists: let a be in $M(2)$. Think of a as the pair $\langle a_0, a_1 \rangle$. If $pr_{2,0}(a) = pr_{2,1}(a)$, then $a_0 = a_1$. Now, accessibility is a relation between pairs, so if $a \triangleleft_2 b = \langle b_0, b_1 \rangle$, we may or may not have $b_0 = b_1$. If we move to cartesian metaframes, the situation is different, however. For now, if $a_0 \triangleleft_1 b_0$ and $a_1 \triangleleft_1 b_1$ then from $a_0 = a_1$ we expect $a \triangleleft_2 \langle b_0, b_0 \rangle, \langle b_1, b_1 \rangle, \langle b_1, b_0 \rangle$ as well. Shehtman and Skvortsov make some maneuvers to avoid this consequence.

First, let us look at a definition of cartesian metaframes and assume that the clause that $a_i = a_j$ implies $c_i = c_j$ was not there. Then the following principle is valid. If $\varphi(y, \vec{z})$ is a formula such that x_0 and x_1 do not occur in \vec{z} then

$$M \models \bigwedge x_0. \bigwedge x_1. x_0 \dot{=} x_1 \rightarrow (\blacklozenge[x_0/y]\varphi(y, \vec{z}) \leftrightarrow \blacklozenge[x_1/y]\varphi(y, \vec{z}))$$

For let ξ be a valuation and $a \in M(n)$. Assume that $\langle M, \xi, a \rangle \models x_0 \doteq x_1$. Then we have $pr_{n,0}(a) = pr_{n,1}(a)$. Let $\alpha_0 : n - 1 \rightarrow n : i \mapsto i + 1$ and $\alpha_1 : n - 1 \rightarrow n : 0 \mapsto 0, i \mapsto i + 1 (i \neq 0)$. Put $a_0 := M(\alpha_0)(a)$, $a_1 := M(\alpha_1)(a)$. Intuitively, a_0 is a reduced by its 0th coordinate, a_1 is a reduced by its 1st coordinate. From this it follows that $a_0 = a_1$. Assume next that $\langle M, \xi, a \rangle \models \Diamond[x_0/y]\varphi(y, \vec{z})$. Then there is a b such that $a \triangleleft_n b$ and $\langle M, \xi, b \rangle \models [x_0/y]\varphi(y, \vec{z})$. Put $b_0 := M(\alpha_0)(b)$ and $b_1 := M(\alpha_1)(b)$. We have $a_0 \triangleleft_{n-1} b_0$, since $M(\alpha_0)$ is a p-morphism, and likewise $a_1 \triangleleft_{n-1} b_1$. Since x_0 is not free in $[x_1/y]\varphi(y, \vec{z})$ we have $\langle M, \xi, b_1 \rangle \models [x_0/y]\varphi(y, \vec{z})$. From this follows $\langle M, \xi, a_1 \rangle \models \Diamond[x_0/y]\varphi(y, \vec{z})$, and so $\langle M, \xi, a \rangle \models \Diamond[x_1/y]\varphi(y, \vec{z})$. Likewise for the other direction.

So we find, as indicated, that without the clause, metaframes imitate counterpart semantics. However, Shehtman and Skvortsov have added it. Thereby they avoid counterpart semantics, but there is a price to be paid.

LEMMA 47. *Let M be a cartesian $m^=$ -metaframe. Then*

$$M \models \bigwedge x_0. \bigwedge x_1. x_0 \doteq x_1 \rightarrow \Box(x_0 \doteq x_1).$$

So, neither of the alternatives is completely general. It turns out that metaframe semantics could have been saved in the same way as coherence semantics, namely by adding a constant interpreting equality. This seems to be necessary. If we do not treat equality in this way, we must assume that the interpretation of identity is an equivalence relation. Shehtman and Skvortsov have shown that the condition (0^\sharp) makes the semantics less general: there are formulae which are not generally valid but valid in all metaframes satisfying (0^\sharp) . This indicates that equating objects and object traces even done in metaframes à la Shehtman and Skvortsov cannot eliminate the problems of identity.

§15. Conclusion. In this paper we have defined a new semantics for modal predicate logic, namely coherence frames. Coherence frames differ from counterpart frames in that variables are interpreted in the same way as constants, namely by objects. We have shown completeness both for first-order and for second-order MPLs with respect to generalized coherence frames. From this we have derived a completeness theorem for second order MPLs with respect to generalized metaframe semantics. In fact, completeness with respect to cubic generalized metaframes is obtained rather directly.

The proposal of distinguishing between an object and its trace is certainly a very far reaching one but not without justification. Many philosophers have argued that there may exist different identity criteria for objects (see van Leeuwen [15] for a review of these ideas). A statue is not the same as the material it is made of. Hence, though perhaps trace identical, the two are not the same objects. There are predicates that are sensitive to this difference (again see [15]). These predicates reject the postulate Eq4. It goes beyond the scope of this paper to review the possibilities that coherence structures offer in this respect.

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