

# Engineering of ontologies with Description Logics

## 3. description logics

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# Outline

- 1 A fragment of FOL
- 2 Syntax of basic Description Logics
- 3 Semantics of basic Description Logics
- 4 Reasoning
- 5 More expressive DLs
- 6 Less expressive DLs

Many application domains can be modelled without the full expressivity of FOL.

- E.g., Knowledge bases widely used in bio-medicine are simple taxonomies.

Description Logics are a family of logics:

- (typically) fragments of FOL
- (typically) decidable satisfiability, validity, entailment problems
- underlying the W3C Web Ontology Language

## Example

- A juvenile disease affects only children or teenagers
- Children and teenagers are not adults
- A person is either a child, a teenager, or an adult
- Juvenile arthritis is a kind of arthritis and a juvenile disease
- Every kind of arthritis damages some joint

Formalisation in FOL:

- $\forall x.(\text{JuvDisease}(x) \rightarrow \forall y.(\text{Affects}(x, y) \rightarrow \text{Child}(y) \vee \text{Teenager}(y)))$
- $\forall x.(\text{Child}(x) \vee \text{Teenager}(x) \rightarrow \neg \text{Adult}(x))$
- $\forall x.(\text{Person}(x) \rightarrow (\text{Child}(x) \vee \text{Teenager}(x) \vee \text{Adult}(x)))$
- $\forall x.(\text{JuvArthritis}(x) \rightarrow \text{Arthritis}(x) \wedge \text{JuvDisease}(x))$
- $\forall x.(\text{Arthritis}(x) \rightarrow \exists y.(\text{Damages}(x, y) \wedge \text{Joint}(y)))$

Some patterns:

- $\forall x.(\text{C}(x) \rightarrow \text{D}(x))$
- $\text{D}(x)$  is a formula with one free variable, that can be identified with formulas with quantifiers:
  - ▶  $\forall x.(\text{C}(x) \rightarrow \forall y.(\text{R}(x, y) \rightarrow \text{E}(y)))$
  - ▶  $\forall x.(\text{C}(x) \rightarrow \exists y.(\text{R}(x, y) \wedge \text{E}(y)))$
- $\text{C}(x)$  is a formula with one free variable, that can be identified with formulas with quantifiers (not in this example):
  - ▶  $\forall x.(\forall y.(\text{R}(x, y) \rightarrow \text{E}(y)) \rightarrow \text{D}(x))$
  - ▶  $\forall x.(\exists y.(\text{R}(x, y) \wedge \text{E}(y)) \rightarrow \text{D}(x))$

## Variables are redundant

- $\forall x.(\text{Child}(x) \vee \text{Teenager}(x) \rightarrow \neg \text{Adult}(x))$

- ▶ Outermost universal quantification on variable  $x$ .
- ▶ Variable  $x$  is free in both  $\text{Child}(x) \vee \text{Teenager}(x)$  and  $\neg \text{Adult}(x)$ .

In DL:

$\text{Child} \sqcup \text{Teenager}$  is subset of  $\neg \text{Adult}$

- $\forall x.(\text{Arthritis}(x) \rightarrow \exists y.(\text{Damages}(x, y) \wedge \text{Joint}(y)))$

- ▶  $\exists y.(\text{Damages}(x, y) \wedge \text{Joint}(y))$  introduces one fresh variable  $y$ .
- ▶  $\text{Damages}(x, y)$  uses the free (in subformula) variable  $x$ , and the fresh variable  $y$ .
- ▶  $\text{Joint}(y)$  uses the fresh variable  $y$ .

In DL:

$\text{Arthritis}$  is subset of  $\exists \text{Damages}.\text{Joint}$

- Analogously for  $\forall x.(\text{JuvDisease}(x) \rightarrow \forall y.(\text{Affects}(x, y) \rightarrow \text{Child}(y) \vee \text{Teenager}(y)))$ .

In DL:

$\text{JuvDisease}$  is subset of  $\forall \text{Affects} . (\text{Child} \sqcup \text{Teenager})$

## DL basic building blocks and their FOL counterparts

Concepts (unary predicates):

$$\blacksquare C \mapsto C(x)$$

“Roles” (binary predicates):

$$\blacksquare R \mapsto R(x, y)$$

Complex concepts:

$$\blacksquare \neg C \mapsto \neg C(x)$$

$$\blacksquare C \sqcup D \mapsto C(x) \vee D(x)$$

$$\blacksquare C \sqcap D \mapsto C(x) \wedge D(x)$$

$$\blacksquare \exists R.C \mapsto \exists y.(R(x, y) \wedge C(y))$$

$$\blacksquare \forall R.C \mapsto \forall y.(R(x, y) \rightarrow C(y))$$

## A decidable fragment of FOL

This is a fragment of **FO2**, that is First Order Logic consisting of all first-order sentences with at most two distinct variables.

- FO2 has the finite-model property and the satisfiability problem is decidable [Mortimer 1975]<sup>1</sup>
- the satisfiability problem of FO2 is NEXPTIME-complete [Grädel, Kolaitis, Vardi 1997]<sup>2</sup>

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<sup>1</sup>Michael Mortimer. "On language with two variables". In: *Zeit. für Math. Logik und Grund. der Math.* 21 (1975), pp. 135–140.

<sup>2</sup>Erich Grädel, Phokion G. Kolaitis, and Moshe Y. Vardi. "On the Decision Problem for Two-Variable First-Order Logic". In: *The Bulletin of Symbolic Logic* 3.1 (1997), pp. 53–69.

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## Basic language

(Basic) DL languages are inductively defined from:

- **Unary** FOL predicate symbols
- **Binary** FOL predicate symbols
- FOL **constants**
- Symbol  $\top$ : truth
- Symbol  $\perp$ : falsum
- Propositional connectives:  $\neg$ ,  $\sqcup$ ,  $\sqcap$
- The existential and universal quantifiers:  $\exists$ ,  $\forall$
- Parentheses ( and )

No functions. No predicates of arity greater than 2.

ALC concepts are built inductively from a set of atomic concepts and a set of (atomic) roles, as follows:<sup>3</sup>

$$C ::= \top \mid \perp \mid A \mid \neg C \mid C \sqcup C \mid C \sqcap C \mid \exists R.C \mid \forall R.C$$

where  $A$  is an atomic concept and  $R$  is an atomic role.

A concept represents a set of object. E.g.:

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Child	the set of children
Child $\sqcup$ Teenager	the set of non-adult persons
Disease $\sqcap \forall \text{Affects}. \text{Child}$	the set of diseases affecting only children
Disease $\sqcap \exists \text{Affects}. \text{Child}$	the set of diseases affecting some children
Male $\sqcap \exists \text{hasChild}. (\exists \text{hasChild}. \top)$	the set of grandfathers
Female $\sqcap \forall \text{hasChild}. (\exists \text{hasChild}. \text{Male})$	the set of women whose children all have a boy

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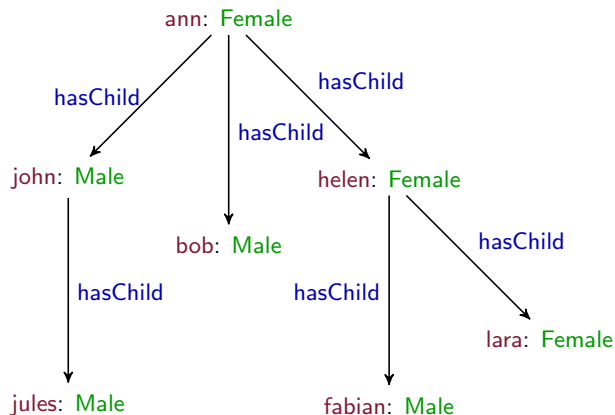
<sup>3</sup>ALC stands for Attributive Language with Concept negation.

## Graphical visualisation of objects, concepts, and roles

The set of women whose all children have no children or have a boy:

$$\text{Female} \sqcap \forall \text{hasChild}.((\forall \text{hasChild}.\perp) \sqcup (\exists \text{hasChild}.\text{Male}))$$

Ann (**ann**) is one of them.

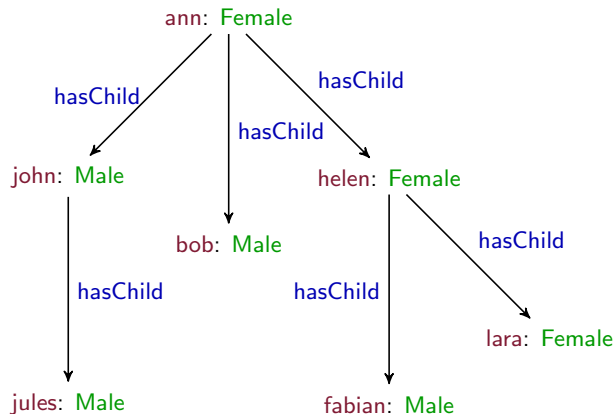


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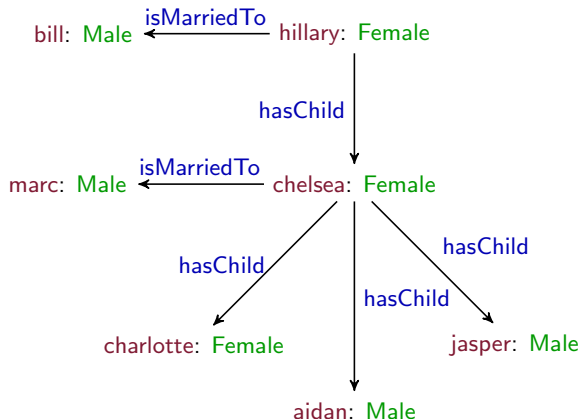
This is a knowledge graph!

## Graphical visualisation of objects, concepts, and roles


The set of women whose all children have no children or have a boy:

$$\text{Female} \sqcap \forall \text{hasChild}.((\forall \text{hasChild}.\perp) \sqcup (\exists \text{hasChild}.\text{Male}))$$

Hillary (hillary) is one of them.



**Hillary Clinton**  
Former United States Secretary of State



[hillaryclinton.com](http://hillaryclinton.com)

Hillary Diane Rodham Clinton is an American politician, diplomat, lawyer, writer, and public speaker who served as the 67th United States secretary of state from 2009 to 2013, as a United States senator ... [Wikipedia](#)

**Born:** October 26, 1947 (age 74 years), [Edgewater Hospital, Chicago](#)

**Spouse:** [Bill Clinton](#) (m. 1975)

**Party:** [Democratic Party](#)

**Children:** [Chelsea Clinton](#)

**Education:** [Yale Law School](#) (1969–1973), [MORE](#)

**Previous offices:** [United States Secretary of State](#) (2009–2013), [MORE](#)

**Chelsea Clinton**



American writer

Chelsea Victoria Clinton is an American writer and global health advocate. She is the only child of former U.S. President Bill Clinton and former U.S. Secretary of State and 2016 presidential candidate Hillary Clinton. [Wikipedia](#)

**Born:** February 27, 1980 (age 42 years), [Little Rock, Arkansas, United States](#)

**Full name:** [Chelsea Victoria Clinton](#)

**Spouse:** [Marc Mezvinsky](#) (m. 2010)

**Children:** [Charlotte Clinton Mezvinsky](#), [Jasper Clinton Mezvinsky](#), [Aidan Clinton Mezvinsky](#)

## General concept inclusions

Formalisation in FOL:

- $\forall x.(\text{JuvDisease}(x) \rightarrow \forall y.(\text{Affects}(x, y) \rightarrow \text{Child}(y) \vee \text{Teenager}(y)))$
- $\forall x.(\text{Child}(x) \vee \text{Teenager}(x) \rightarrow \neg \text{Adult}(x))$
- $\forall x.(\text{Person}(x) \rightarrow (\text{Child}(x) \vee \text{Teenager}(x) \vee \text{Adult}(x)))$
- $\forall x.(\text{JuvArthritis}(x) \rightarrow \text{Arthritis}(x) \wedge \text{JuvDisease}(x))$
- $\forall x.(\text{Arthritis}(x) \rightarrow \exists y.(\text{Damages}(x, y) \wedge \text{Joint}(y)))$

They all have the form:

$$\forall x.(\mathbf{C}(x) \rightarrow \mathbf{D}(x))$$

In DL:

$$\mathbf{C} \sqsubseteq \mathbf{D}$$

where  $\mathbf{C}$  and  $\mathbf{D}$  are  $\mathcal{ALC}$  concepts.

We call these statements **General concept inclusions** (GCIs).

## General concept inclusions

Formalisation in FOL:

- $\forall x.(\text{JuvDisease}(x) \rightarrow \forall y.(\text{Affects}(x, y) \rightarrow \text{Child}(y) \vee \text{Teenager}(y)))$
- $\forall x.(\text{Child}(x) \vee \text{Teenager}(x) \rightarrow \neg \text{Adult}(x))$
- $\forall x.(\text{Person}(x) \rightarrow (\text{Child}(x) \vee \text{Teenager}(x) \vee \text{Adult}(x)))$
- $\forall x.(\text{JuvArthritis}(x) \rightarrow \text{Arthritis}(x) \wedge \text{JuvDisease}(x))$
- $\forall x.(\text{Arthritis}(x) \rightarrow \exists y.(\text{Damages}(x, y) \wedge \text{Joint}(y)))$

Formalisation in DL:

- $\text{JuvDisease} \sqsubseteq \forall \text{Affects}.(\text{Child} \sqcup \text{Teenager})$
- $\text{Child} \sqcup \text{Teenager} \sqsubseteq \neg \text{Adult}$
- $\text{Person} \sqsubseteq \text{Child} \sqcup \text{Teenager} \sqcup \text{Adult}$
- $\text{JuvArthritis} \sqsubseteq \text{Arthritis} \sqcap \text{JuvDisease}$
- $\text{Arthritis} \sqsubseteq \exists \text{Damages}.\text{Joint}$

## Terminological statements

- Subtyping:

$\text{Arthritis} \sqsubseteq \text{Disease}$

- Definitions:

$\text{JuvArthritis} \equiv \text{JuvDisease} \sqcap \text{Arthritis}$

- Disjointness:

$\text{Child} \sqsubseteq \neg \text{Adult}$

- Covering:

$\text{Person} \sqsubseteq \text{Child} \sqcup \text{Teenager} \sqcup \text{Adult}$

- Domain restriction:

$\exists \text{Affects}.\top \sqsubseteq \text{Disease}$

- Range restriction:

$\top \sqsubseteq \forall \text{Affects}.\text{LivingThing}$



# Axioms

Domain knowledge is represented with a set of CGIs of the form:

$$C \sqsubseteq D$$

where  $C$  and  $D$  are concepts.

They form the terminological box  $TBox$ .

Data is represented with a set of statements of the form:

$$C(a)$$

where  $C$  is a concept and  $a$  is an object/individual; and of the form

$$R(a, b) \quad ; \quad \neg R(a, b)$$

where  $R$  is a role, and  $a$  and  $b$  are individuals.

They form the assertion box  $ABox$ .

## Description Logic knowledge bases

An  $\mathcal{ALC}$  knowledge base is composed of:

- a TBox
- an ABox

E.g.,

TBox:

- $\text{JuvDisease} \sqsubseteq \forall \text{Affects}.(\text{Child} \sqcup \text{Teenager})$
- $\text{Child} \sqcup \text{Teenager} \sqsubseteq \neg \text{Adult}$
- $\text{Person} \sqsubseteq \text{Child} \sqcup \text{Teenager} \sqcup \text{Adult}$
- $\text{JuvArthritis} \sqsubseteq \text{Arthritis} \sqcap \text{JuvDisease}$
- $\text{Arthritis} \sqsubseteq \exists \text{Damages}.\text{Joint}$

ABox:

- $\text{JuvArthritis}(\text{jra})$
- $\text{Child}(\text{johnSmith})$
- $(\text{Child} \sqcup \text{Teenager})(\text{maryJones})$
- $\text{Affects}(\text{jra}, \text{johnSmith})$

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An **interpretation** for  $\mathcal{ALC}$  is a tuple  $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ , where:

- $\Delta^{\mathcal{I}}$  is non-empty set; the **domain of interpretation**
- $\cdot^{\mathcal{I}}$  is the **interpretation function** that associates:
  - ▶ every concept name  $A$  to a subset  $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$
  - ▶ every role name  $R$  to a subset  $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
  - ▶ every individual name  $a$  an object  $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$ .
  - ▶  $\top^{\mathcal{I}} = \Delta^{\mathcal{I}}$
  - ▶  $\perp^{\mathcal{I}} = \emptyset$

Given an interpretation  $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ , the **meaning** of a concepts is given inductively, as follows:

- $(\neg C)^{\mathcal{I}} = \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
- $(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}}$
- $(C \sqcup D)^{\mathcal{I}} = C^{\mathcal{I}} \cup D^{\mathcal{I}}$
- $(\forall R.C)^{\mathcal{I}} = \{x \mid \forall y.(x, y) \in R^{\mathcal{I}} \rightarrow y \in C^{\mathcal{I}}\}$
- $(\exists R.C)^{\mathcal{I}} = \{x \mid \exists y.(x, y) \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}$

## Example

Let  $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$  be the interpretation defined as:

- $\Delta^{\mathcal{I}} = \{\text{jra}, \text{flu}, \text{johnSmith}\}$
- $\text{JuvDisease}^{\mathcal{I}} = \{\text{jra}\}$
- $\text{Child}^{\mathcal{I}} = \{\text{johnSmith}\}$
- $\text{Teenager}^{\mathcal{I}} = \emptyset$
- $\text{Affects}^{\mathcal{I}} = \{(\text{jra}, \text{johnSmith})\}$

We have:

- $(\text{JuvDisease} \sqcap \text{Child})^{\mathcal{I}} = \emptyset$
- $(\text{Child} \sqcup \text{Teenager})^{\mathcal{I}} = \{\text{johnSmith}\}$
- $(\exists \text{Affects}.\top)^{\mathcal{I}} = \{\text{jra}\}$
- $(\exists \text{Affects}.\text{Teenager})^{\mathcal{I}} = \emptyset$
- $(\exists \text{Affects}.\text{Child})^{\mathcal{I}} = \{\text{jra}\}$
- $(\neg \text{Child})^{\mathcal{I}} = \{\text{jra}, \text{flu}\}$
- $(\forall \text{Affects}.\text{Teenager})^{\mathcal{I}} = \{\text{flu}, \text{johnSmith}\}$

Given an interpretation  $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ , we define the **satisfiability** of axioms as:

- $\mathcal{I} \models C \sqsubseteq D$  iff  $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ .
- $\mathcal{I} \models C \equiv D$  iff  $C^{\mathcal{I}} = D^{\mathcal{I}}$ .
- $\mathcal{I} \models C(a)$  iff  $a^{\mathcal{I}} \in C^{\mathcal{I}}$ .
- $\mathcal{I} \models R(a, b)$  iff  $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$ .
- $\mathcal{I} \models \neg R(a, b)$  iff  $(a^{\mathcal{I}}, b^{\mathcal{I}}) \notin R^{\mathcal{I}}$ .

An  $\mathcal{ALC}$  knowledge base  $\mathcal{K}$  is composed of:

- a TBox  $\mathcal{T}$  ( $C \sqsubseteq D$  axioms)
- an ABox  $\mathcal{A}$  ( $C(a)$ ,  $R(a, b)$ ,  $\neg R(a, b)$  axioms)

An interpretation  $\mathcal{I}$  is a **model** of the knowledge base  $\mathcal{K}$  if  $\mathcal{I} \models ax$  for every axiom  $ax$  in  $\mathcal{T} \cup \mathcal{A}$ .

A knowledge base is **satisfiable** if it has a model.

## Example

Let  $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$  be the interpretation,

- $\Delta^{\mathcal{I}} = \{\text{jra}, \text{flu}, \text{johnSmith}\}$
- $\text{JuvDisease}^{\mathcal{I}} = \{\text{jra}\}$
- $\text{Child}^{\mathcal{I}} = \{\text{johnSmith}\}$
- $\text{Teenager}^{\mathcal{I}} = \emptyset$
- $\text{Affects}^{\mathcal{I}} = \{(\text{jra}, \text{johnSmith})\}$

We have:

- $(\forall \text{Affects}.(\text{Child} \sqcup \text{Teenager}))^{\mathcal{I}} = \Delta^{\mathcal{I}}$
- $(\forall \text{Affects}.\text{Teenager})^{\mathcal{I}} = \{\text{flu}, \text{johnSmith}\}$

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Consider the knowledge base  $\mathcal{K}$ ,

- $\text{JuvDisease}(\text{jra})$
- $\text{Affects}(\text{jra}, \text{johnSmith})$
- $\text{JuvDisease} \sqsubseteq \forall \text{Affects}.(\text{Child} \sqcup \text{Teenager})$

$\mathcal{I}$  is a model of  $\mathcal{K}$ .

Consider the knowledge base  $\mathcal{K}'$ ,

- $\text{JuvDisease}(\text{jra})$
- $\text{Affects}(\text{jra}, \text{johnSmith})$
- $\text{JuvDisease} \sqsubseteq \forall \text{Affects}.\text{Teenager}$

$\mathcal{I}$  is not a model of  $\mathcal{K}'$ .

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## Reasoning problems

Let  $\mathcal{K} = (\mathcal{T}, \mathcal{A})$  be an  $\mathcal{ALC}$  knowledge base, let  $C, D$  be  $\mathcal{ALC}$  concepts, and let  $b$  be a named individual.

- $C$  is **satisfiable** wrt. to  $\mathcal{T}$  if there exists a model  $\mathcal{I}$  of  $\mathcal{T}$  and some  $d \in \Delta^{\mathcal{I}}$  with  $d \in C^{\mathcal{I}}$ .
- $C$  is **subsumed by**  $D$  wrt. to  $\mathcal{T}$  if  $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$  for every model  $\mathcal{I}$  of  $\mathcal{T}$ .
- $C$  and  $D$  are **equivalent** wrt.  $\mathcal{T}$  if  $C^{\mathcal{I}} = D^{\mathcal{I}}$  for every model  $\mathcal{I}$  of  $\mathcal{T}$ .
- $\mathcal{K}$  is **consistent/satisfiable** if there exists a model of  $\mathcal{K}$ .
- $b$  is an **instance** of  $C$  wrt.  $\mathcal{K}$  if  $b^{\mathcal{I}} \in C^{\mathcal{I}}$  for every model  $\mathcal{I}$  of  $\mathcal{K}$ .

All reduceable from knowledge base consistency/satisfiability.

## Reasoning problems (reduceability to KB consistency/satisfiability)

Reducing **equivalence** to **subsumption**:

$$\mathcal{T} \models C \equiv D \text{ iff } \mathcal{T} \models C \sqsubseteq D \text{ and } \mathcal{T} \models D \sqsubseteq C$$

Reducing **subsumption** to **concept satisfiability**:

$$\mathcal{T} \models C \sqsubseteq D \text{ iff } C \sqcap \neg D \text{ is not satisfiable wrt. } \mathcal{T}$$

Reducing **concept satisfiability** to **knowledge base consistency**:

$$C \text{ is satisfiable wrt. } \mathcal{T} \text{ iff } (\mathcal{T}, \{C(b)\}) \text{ is consistent.}$$

Reducing **instance checking** to **knowledge base consistency**:

$$(\mathcal{T}, \mathcal{A}) \models C(b) \text{ iff } (\mathcal{T}, \mathcal{A} \cup \{\neg C(b)\}) \text{ is not consistent.}$$

The problem of deciding the satisfiability of  $\mathcal{ALC}$  knowledge bases is EXPTIME-complete.  
Efficient tableau method in [Donini and Massacci 2000]<sup>4</sup>.

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<sup>4</sup>Francesco M. Donini and Fabio Massacci. "EXTime tableaux for ALC". In: *Artificial Intelligence* 124.1 (2000), pp. 87–138.

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## Additional concepts: Qualified number restrictions

**Syntax:** For every integer  $n$ , role name  $R$ , and concept  $C$ , we can also have the concept  $(\leq n R.C)$ , that refers to things have less than  $n$   $R$ -successors that are  $C$ .

Similarly, we can have the concept  $(\geq n R.C)$ .

Let  $\mathcal{DL}$  be a description logic. The set of  $\mathcal{DLQ}$  concepts is the smallest set of concepts that contains all  $\mathcal{DL}$  concepts and  $(\leq n R.C)$  and  $(\geq n R.C)$  for every  $n \in \mathbb{N}$ , role  $R$ , and concept  $C$ .

**Semantics:** Let  $\#RC = \#(\{y \mid (x, y) \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\})$

$$(\leq n R.C)^{\mathcal{I}} = \{x \mid \#RC \leq n\} \quad ; \quad (\geq n R.C)^{\mathcal{I}} = \{x \mid \#RC \geq n\}$$

**Example:**

$\text{MildArthritis} \equiv \text{Arthritis} \sqcap (\leq 2 \text{ Damages.Joint}) \quad ; \quad \text{SevereArthritis} \equiv \text{Arthritis} \sqcap (\geq 5 \text{ Damages.Joint})$

**Remarks:**

- $\exists R.C$  and  $(\geq 1 R.C)$  are equivalent.
- One can define
  - ▶  $(= n R.C) = (\geq n R.C) \sqcap (\leq n R.C)$
  - ▶  $(> n R.C) = \neg(\leq n R.C)$
  - ▶  $(< n R.C) = \neg(\geq n R.C)$
- Functionality ( $\mathcal{F}$ ) can be expressed with:  $\top \sqsubseteq (\leq 1 \text{ hasSSN.}\top)$

## Additional concepts: Nominals

**Syntax:** For every individual name  $a$ , we can also have the concept  $\{a\}$ , to represent the singleton containing  $a$ .

Let  $\mathcal{DL}$  be a description logic. The set of  $\mathcal{DLO}$  concepts is the smallest set of concepts that contains all  $\mathcal{DL}$  concepts and  $\{a\}$  for every individual name  $a$ .

**Semantics:**

$$(\{a\})^{\mathcal{I}} = \{a^{\mathcal{I}}\}$$

**Example:** Individuals affected by a disease that also affects  $\text{johnSmith}$

$$\exists \text{Affects}^-. (\text{Disease} \sqcap \exists \text{Affect}. \{\text{johnSmith}\})$$

Simulating ABox axioms:

$$\{\text{johnSmith}\} \sqsubseteq \text{Child} \quad ; \quad \{\text{jra}\} \sqsubseteq \exists \text{Affects}. \{\text{johnSmith}\} \quad ; \quad \{\text{jra}\} \sqsubseteq \neg \exists \text{Affects}. \{\text{johnSmith}\}$$

## Additional axioms: Role hierarchy

**Syntax:** We can add an RBox  $\mathcal{R}$ , and for every  $R$  and  $S$ , we can add an axiom  $R \sqsubseteq S$  to represent the fact that  $R$  is subsumed by  $S$ .

Let  $\mathcal{DL}$  be a description logic. A  $\mathcal{DLH}$  ontology is a  $\mathcal{DL}$  ontology that may contain an RBox with axioms of the form  $R \sqsubseteq S$ , where  $S$  and  $R$  roles in  $\mathcal{DL}$ .

**Semantics:**

$$\mathcal{I} \models R \sqsubseteq S \text{ iff } R^{\mathcal{I}} \subseteq S^{\mathcal{I}}$$

**Example:**

$$\text{hasSister} \sqsubseteq \text{hasSibling} \quad ; \quad \text{hasParent} \sqsubseteq \text{hasAncestor}$$

## Additional roles: Inverse roles

**Syntax:** For every role name  $R$ , we can also have the role  $R^-$  that represents the inverse of role  $R$ .

Let  $\mathcal{DL}$  be a description logic. The set of  $\mathcal{DLI}$  concepts is the smallest set of concepts that contains all  $\mathcal{DL}$  concepts and where inverse roles can occur in all places of role names.

**Semantics:**

$$(R^-)^{\mathcal{I}} = \{(x, y) \mid (y, x) \in R^{\mathcal{I}}\}$$

**Example** (an RBox axiom):

$$\text{Affects}^- \equiv \text{AffectedBy}$$

Another way to capture range restrictions:

$$\exists \text{Affects}^-. \top \sqsubseteq \text{Person}$$

Expressing symmetry:

$$\text{hasSibling} \sqsubseteq \text{hasSibling}^-$$



## Additional roles: Chain roles

**Syntax:** For every two roles  $R$  and  $S$ , we can also have the role  $R \circ S$  that represents the composition of role  $R$  with role  $S$ .

**Semantics:**

$$(R \circ S)^{\mathcal{I}} = \{(x, z) \mid (x, y) \in R^{\mathcal{I}} \wedge (y, z) \in S^{\mathcal{I}}\}$$

**Example:**

$$\text{hasUncle} \sqsubseteq \text{hasMother} \circ \text{hasBrother}$$

Capturing transitive ( $\mathcal{S}$ ) roles:

$$\text{hasAncestor} \circ \text{hasAncestor} \sqsubseteq \text{hasAncestor}$$

'Self' concepts:

$$(\exists R.Self)^{\mathcal{I}} = \{x \mid (x, x) \in R^{\mathcal{I}}\}$$

Universal role:

$$U^{\mathcal{I}} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$$

Identity ABox axioms:

$$\mathcal{I} \models a = b \text{ iff } a^{\mathcal{I}} = b^{\mathcal{I}}$$

$$\mathcal{I} \models a \neq b \text{ iff } a^{\mathcal{I}} \neq b^{\mathcal{I}}$$

## Partial summary

$\mathcal{DL}$	$C$	$R$	axioms
$\mathcal{ALC}$	$\neg C, C \sqcap C, \exists R.C$	$R$	$C \sqsubseteq C, C(a), R(a, b), \neg R(a, b)$
$\mathcal{ALCI}$	$\neg C, C \sqcap C, \exists R.C$	$R, R^-$	$C \sqsubseteq C, C(a), R(a, b), \neg R(a, b)$
$\mathcal{ALCQ}$	$\neg C, C \sqcap C, \exists R.C, (\bowtie_n R.C)$	$R$	$C \sqsubseteq C, C(a), R(a, b), \neg R(a, b)$
$\mathcal{ALCO}$	$\neg C, C \sqcap C, \exists R.C, \{a\}$	$R$	$C \sqsubseteq C, C(a), R(a, b), \neg R(a, b)$
$\mathcal{ALCH}$	$\neg C, C \sqcap C, \exists R.C$	$R$	$C \sqsubseteq C, C(a), R(a, b), \neg R(a, b), R \sqsubseteq R$
$\mathcal{ALCF}$	$\neg C, C \sqcap C, \exists R.C$	$R$	$C \sqsubseteq C, C(a), R(a, b), \neg R(a, b), \top \sqsubseteq (\leq 1 R. \top)$
$\mathcal{ALCS}$	$\neg C, C \sqcap C, \exists R.C$	$R$	$C \sqsubseteq C, C(a), R(a, b), \neg R(a, b), R \circ R \sqsubseteq R$
...			
$\mathcal{SROIQ}^5$	$\neg C, C \sqcap C, \exists R.C, (\bowtie_n R.C), \{a\}, \exists R.Self$	$R, R^-, R \circ \dots \circ R, U$	$C \sqsubseteq C, C(a), R(a, b), \neg R(a, b), a = b, a \neq b, R \sqsubseteq R$
... and more			

<sup>5</sup>Ian Horrocks, Oliver Kutz, and Ulrike Sattler. "The Even More Irresistible SROIQ". In: *KR 2006*. 2006.

## Some complexity results

See <http://www.cs.man.ac.uk/~ezolin/dl/>.

Complexity of reasoning problems:

$\mathcal{DL}$	complexity
$\mathcal{ALC}$	EXPTIME-complete
$\mathcal{ALCI}$	EXPTIME-complete
$\mathcal{ALCQ}$	EXPTIME-complete
$\mathcal{ALCO}$	EXPTIME-complete
$\mathcal{ALCH}$	EXPTIME-complete
$\mathcal{ALCF}$	EXPTIME-complete
$\mathcal{ALCS}$	EXPTIME-complete
$\mathcal{ALCHOQ}$	EXPTIME-complete
$\mathcal{ALCOIQ}$	NEXPTIME-complete
$\mathcal{ALCHOIQ}$	NEXPTIME-complete
$\mathcal{ALCSHOIQ} = \mathcal{SROIQ}$	N2EXPTIME-complete
...	

# Outline

- 1 A fragment of FOL
- 2 Syntax of basic Description Logics
- 3 Semantics of basic Description Logics
- 4 Reasoning
- 5 More expressive DLs
- 6 Less expressive DLs

## The need for light-weight Description Logics

The complexity of the extensions of  $\mathcal{ALC}$  are still very hard computationally to reason with.

Some ontologies have a very large TBox (e.g., medical domain, SNOWMED)

Often we want to reason with a very large amount of data (e.g., diagnostic or monitoring data).

$\mathcal{EL}$  is the fragment of  $\mathcal{ALC}$  without  $\sqcup$  and  $\neg$ :

$$C ::= A \mid C \sqcap C \mid \exists R.C$$

$\mathcal{EL}^{++}$  is the minimal  $\mathcal{DL}$  obtained by adding to  $\mathcal{EL}$ :

- $\top$  and  $\perp$  concepts;
- a concept  $\{a\}$  for every named individual  $a$ ;
- an RBox, that can contain any  $R_1 \circ \dots \circ R_n \sqsubseteq S$ .

Reasoning with  $\mathcal{EL}^{++}$  is PTIME-complete [Baader, Brandt and Lutz 2005]<sup>6</sup>.

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<sup>6</sup>Franz Baader, Sebastian Brandt, and Carsten Lutz. "Pushing the EL Envelope". In: *IJCAI-05*.

Let  $R$  be a role name,  $A$  be a concept name,  $a$  and  $b$  be two named individuals.

Subconcepts:

$$B ::= A \mid B \sqcap B \mid B \sqcup B \mid \exists R.B$$

Superconcepts:

$$C ::= A \mid \perp \mid C \sqcap C \mid \forall R.C$$

TBox:

$$B \sqsubseteq C$$

ABox:

$$A(a) \quad R(a, b)$$

$\mathcal{RL}$  reasoning can be captured by rule-based reasoning in Datalog (forward and backward chaining techniques), a subset of Prolog.

Knowledge base satisfiability with  $\mathcal{RL}$  is in PTIME.



## Allowed in $\mathcal{RL}$

- Subtyping:

$$\text{Arthritis} \sqsubseteq \text{Disease}$$

- Definitions:

$$\text{JuvArthritis} \equiv \text{JuvDisease} \sqcap \text{Arthritis}$$

- Disjointness:

$$\text{Child} \sqcap \text{Adult} \sqsubseteq \perp$$

- Covering:

$$\text{Person} \sqsubseteq \text{Child} \sqcup \text{Teenager} \sqcup \text{Adult}$$

- Domain restriction:

$$\exists \text{Affects}.\top \sqsubseteq \text{Disease}$$

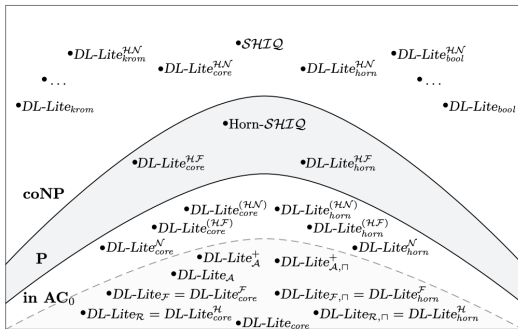
- Range restriction:

$$\top \sqsubseteq \forall \text{Affects}.\text{LivingThing}$$

## $\mathcal{DL}$ -Lite family

[Calvanese et al. 2007]<sup>7</sup>

The  $\mathcal{DL}$ -lite extended family (grouped according to the data complexity of “positive existential query answering under the unique name assumption”) [Artale et al. 2014]<sup>8</sup>:



<sup>7</sup>Diego Calvanese et al. “Tractable Reasoning and Efficient Query Answering in Description Logics: The  $DL$ -Lite Family”. In: *J. Autom. Reason.* 39.3 (2007), pp. 385–429.

<sup>8</sup>Alessandro Artale et al. “The  $DL$ -Lite Family and Relations”. In: *CoRR* abs/1401.3487 (2014).

Let  $R$  be a role name,  $A$  be a concept name,  $a$  and  $b$  be two named individuals.

Subconcepts:

$$B ::= A \mid \exists P.\top$$

Superconcepts:

$$C ::= A \mid \neg A \mid A \sqcap A \mid \exists P.\top \mid \exists P.C$$

Roles:

$$P ::= R \mid R^{-}$$

TBox:

$$B \sqsubseteq C$$

RBox:

$$P \sqsubseteq P \quad Dis(P, P)$$

ABox:

$$A(a) \quad R(a, b)$$

## Disallowed in $\mathcal{DL}\text{-Lite}_{\mathcal{R}}$

- Definitions:

$$\text{JuvArthritis} \equiv \text{JuvDisease} \sqcap \text{Arthritis}$$

- Covering:

$$\text{Person} \sqsubseteq \text{Child} \sqcup \text{Teenager} \sqcup \text{Adult}$$

- Transitive role:

$$\text{hasAncestor} \circ \text{hasAncestor} \sqsubseteq \text{hasAncestor}$$

- Cardinality restrictions:

$$\text{SevereArthritis} \sqsubseteq \text{Arthritis} \sqcap (\geq 5 \text{ Damages.Joint})$$

- Functional role:

$$\top \sqsubseteq (\leq 1 \text{ hasSSN}.\top)$$

- ...

## Allowed in $\mathcal{DL}\text{-Lite}_{\mathcal{R}}$

- Subtyping:

$\text{Arthritis} \sqsubseteq \text{Disease}$

- Concept disjointness:

$\text{Child} \sqsubseteq \neg \text{Adult}$

- Role disjointness:

$\text{Dis}(\text{hasChild}, \text{hasMother})$

- Domain restriction:

$\exists \text{Affects}.\top \sqsubseteq \text{Disease}$

- Range restriction:

$\exists \text{Affects}^{\neg}.\top \sqsubseteq \text{LivingThing}$

- Symmetric role:

$\text{hasSibling} \sqsubseteq \text{hasSibling}^{\neg}$

- ...

Problem	Parameter	Computational complexity
satisfiability	size of knowledge base	NLOGSPACE
instance checking	size of data	$AC^0$
query answering	size of data	$AC^0$

## Summary

- $\mathcal{EL}^{++}$  is good when focusing on TBox reasoning; tractable satisfiability.
- $\mathcal{RL}$  is good when focusing on ABox rule-based reasoning.
- $\mathcal{DL}\text{-Lite}_{\mathcal{R}}$  is good for query answering with limited TBox/RBox expressivity and large ABox.

$\mathcal{EL}^{++}$ ,  $\mathcal{RL}$ , and  $\mathcal{DL}\text{-Lite}_{\mathcal{R}}$  correspond to OWL 2 profiles, as defined by the World Wide Web Consortium (W3C).

<https://www.w3.org/TR/owl2-profiles/>

They are (part of) of what we look at next.

## Credits

Many slides and examples based on Ian Horrocks's KRR lectures

<https://www.cs.ox.ac.uk/people/ian.horrocks/>.

<https://www.cs.ox.ac.uk/teaching/courses/2020-2021/KRR/>