Description Logics in one example

Σ:

TBox

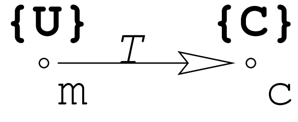
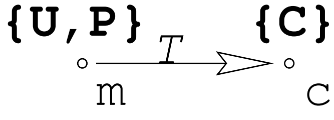
$\exists\texttt{TEACHES}.\texttt{Course} \stackrel{.}{\sqsubseteq} \neg\texttt{Undergrad} \sqcup \texttt{Prof}$

ABox

$\texttt{TEACHES}(\texttt{mary}, \texttt{cs415}), \texttt{Course}(\texttt{cs415}), \texttt{Undergrad}(\texttt{mary})$

$\Sigma \models \texttt{Prof}(\texttt{mary})$

Correspondence with Modal Logics

$\mathcal{ALC}$	$\mathbf{K}_{(\mathbf{m})}$
$C^{\mathcal{I}}$ is a set of individuals	$\alpha_C^{\mathcal{I}}$ is a set of worlds
$R^{\mathcal{I}}$ is a set of pairs of individuals	R is an accessibility relation
A	P_A
$C \sqcap D$	$\alpha_C \wedge \alpha_D$
$C \sqcup D$	$\alpha_C \vee \alpha_D$
$\neg C$	$\neg \alpha_C$
$\forall R.C$	$\Box_R\ \alpha_C$
$\exists R.C$	$\Diamond_R\ \alpha_C$
$o \in C^{\mathcal{I}}$	$\mathcal{I}, o \models \alpha_C$
$\exists \mathtt{T}.\mathtt{C} \sqsubseteq \neg \mathtt{U} \sqcup \mathtt{P}$	$\Diamond_{\mathtt{T}}\mathtt{C} \rightarrow \neg \mathtt{U} \vee \mathtt{P}$
$\mathtt{U}(\mathtt{m}), \mathtt{T}(\mathtt{m}, \mathtt{c}), \mathtt{C}(\mathtt{c})$	
$\Sigma \models \mathtt{P}(\mathtt{m})$	

Combining Modal Logics
in the Description Logics perspective

- DLs extend modal logic in interesting ways:
  - Reasoning in DLs is always reasoning with theories.
  - Nominals.
- Studying the effects of augmenting the expressivity is central:
  - adding operators cab be seen as **combining** modal logics with the basic $\mathbf{K}_{(\mathbf{m})}$;
  - if the basic logic is expressive enough (e.g., $\mathcal{PDL}$), possible reductions are studied;
  - more *typical* combinations are also important, such as combinations with tense logic or modal logic with concrete domains.

Combining Modal Logics vs Reductions

- Combining the basic $\mathbf{K}_{(\mathbf{m})}$ with a modal logic having:
  - inverse, or graded, or deterministic modalities, ...
- Reducing a complex combination to the basic $\mathcal{PDL}$:
  - $\mathcal{DCPDL}$ to $\mathcal{PDL}$, $\mathcal{CPDL}$ + nominals to $\mathcal{PDL}$, ...
- The combination approach may be more interesting than the reduction approach; example:
  - $\mathcal{PDL}$ versus $\mathbf{K}^{\mathcal{H}}_{(\mathbf{m})} \cup \mathbf{S4}$,
  - the combination can be seen as the FOL fragment of $\mathcal{PDL}$,
  - same complexity class,
  - different algorithmic properties (*cut rule*).

Combining Modal Logics in the Description Logics perspective

- Decidability,
- complexity class,
- (how to extend) algorithms,
- (how to re-adapt) strategies and optimisations.

Some examples of how combining modalities with different properties can affect:

- Complexity class
- Algorithmic complexity
- Decidability

- Decision problems for $\mathbf{K}^{\mathcal{H}}_{(\mathbf{m})}$ and $\mathbf{S4}_{(\mathbf{m})}$ known to be in $\mathrm{PS}\textsc{pace}$
- Combination allows use of *universal modality* to *internalise* arbitrary set of axioms:
  - Define new transitive modality u that includes all other modalities
  - Satisfiability of ϕ w.r.t. $\psi_1 \rightarrow \varphi_1, \ldots, \psi_n \rightarrow \varphi_n$ equivalent to satisfiability of $\phi \wedge \Box_u((\psi_1 \rightarrow \varphi_1) \wedge \ldots \wedge (\psi_n \rightarrow \varphi_n))$
- Decision problem w.r.t. arbitrary set of axioms known to be in $\mathrm{E}\textsc{xp}\mathrm{T}\textsc{ime}$ even for $\mathbf{K}_{(\mathbf{m})}$

- Decision problem for $\mathbf{K}^{\mathcal{H}}_{\mathbf{(m)}} \cup \mathbf{S4}_{\mathbf{(m)}}$ in $\textsc{ExpTime}$, but tableaux algorithm presents no special problems:
  - For transitive modalities, propogate $\Box_i \phi$ terms along i modalities
  - Use simple *blocking* technique to check for cycles caused by e.g. $\Box_i \Diamond_i \phi$
  - Cycle in algorithm $\Rightarrow$ valid cyclical model

- Decision problem for $\mathbf{K}^{\mathcal{H}}_{(\mathbf{m})}$ combined with deterministic and converse modalities still in $\mathrm{PS{\scriptsize PACE}}$ and solvable with simple tableaux algorithm (no blocking)
- Combination no longer has finite model property — requires new blocking technique to detect cycles implying valid but non-finite models, e.g. for:

$$\neg\phi \wedge [R^{\smile}]\langle S^{\smile}\rangle\phi$$

  where R is transitive, S is deterministic and R includes S

- Decision problem for $\mathbf{K}^{\mathcal{H}}_{(\mathbf{m})} \cup \mathbf{S4}_{(\mathbf{m})}$ in ExpTime
- Decision problem for $\mathbf{K}^{\mathcal{H}}_{(\mathbf{m})}$ combined with graded modalities in PSpace
- Decision problem for $\mathbf{K}^{\mathcal{H}}_{(\mathbf{m})} \cup \mathbf{S4}_{(\mathbf{m})}$ combined with graded modalities is undecidable — shown by reduction of domino problem
- Representing $\mathbb{N} \times \mathbb{N}$ grid (the tricky bit) uses combination of $\mathcal{H}$, transitive and non-transitive modalities and graded modalities
- Decidability restored by restricting the way transitive and graded modalities can be combined