6. Reasoning in Description Logics

Exercise 6.1 Let $\mathcal{T}$ be a TBox consisting of concept inclusions of the form $A_1 \sqsubseteq A_2$ and concept disjointness assertion of the form $A_1 \sqsubseteq \neg A_2$, for atomic concepts A_1 and A_2.

Describe an algorithm for checking concept satisfiability with respect to $\mathcal{T}$, i.e., whether for some concept A it holds that A is satisfiable with respect to $\mathcal{T}$.

What is the complexity of the algorithm?

Solution: Let $\mathcal{C}$ be the set of atomic concepts appearing in $\mathcal{T}$. Construct a directed graph $G_\mathcal{T} = (N, E)$ as follows:

- the set of nodes is $N = \mathcal{C} \cup \{\neg A \mid A \in \mathcal{C}\}$;
- the set of directed edges is $R = \{A_1 \to A_2,\ \neg A_2 \to \neg A_1 \mid A_1 \sqsubseteq A_2 \in \mathcal{T}\} \cup \{A_1 \to \neg A_2,\ A_2 \to \neg A_1 \mid A_1 \sqsubseteq \neg A_2 \in \mathcal{T}\}$.

Then one can show that an atomic concept A is unsatisfiable with respect to $\mathcal{T}$ if and only if there is a path from A to $\neg A$. The algorithm for reachability checking can be done in linear time.

NOTE: the reachability checking problem is in NLOGSPACE.

Exercise 6.2 Consider TBoxes $\mathcal{T}$ consisting of axioms of the forms

$$\begin{array}{lll} B_1 \sqsubseteq B_2, & \text{where} & B_1, B_2 ::= A \mid \exists P \mid \exists P^-, \\ R_1 \sqsubseteq R_2, & \text{where} & R_1, R_2 ::= P \mid P^-, \end{array}$$

where A denotes an atomic concept, and P an atomic role.

- Describe an algorithm for checking concept subsumption with respect to a given $\mathcal{T}$, i.e., whether for two concepts B_1 and B_2 it holds that $\mathcal{T} \models B_1 \sqsubseteq B_2$.
- Let $\mathcal{A}_0 = \{A_0(a)\}$, for some atomic concept A_0 and individual a, and let $\mathcal{T}$ be a(n arbitrary) TBox of the above form. Can we determine whether $\langle \mathcal{T}, \mathcal{A}_0 \rangle$ is satisfiable?

Solution: Let $\mathcal{C}$ be the set of atomic concepts and $\mathcal{R}$ the set of atomic roles appearing in $\mathcal{T}$. For an atomic or inverse role R, we use R^- to denote P^- if R is an atomic role P, and to denote P if R is an inverse role P^-.

Construct a directed graph $G_\mathcal{T} = (N, E)$ as follows:

- the set of nodes is $N = \mathcal{C} \cup \{\exists P \mid P \in \mathcal{R}\} \cup \{\exists P^- \mid P \in \mathcal{R}\}$;
- the set of directed edges is $R = \{B_1 \to B_2 \mid B_1 \sqsubseteq B_2 \in \mathcal{T}\} \cup \{\exists R_1 \to \exists R_2 \mid R_1 \sqsubseteq R_2 \in \mathcal{T}\} \cup \{\exists R_1^- \to \exists R_2^- \mid R_1 \sqsubseteq R_2 \in \mathcal{T}\}$.

Then one can show that $\mathcal{T} \models B_1 \sqsubseteq B_2$ if and only if there is a path from B_1 to B_2 in $G_\mathcal{T}$.

The TBox $\mathcal{T}$ does not contain assertions involving negation. Hence, every knowledge base having $\mathcal{T}$ as TBox and an arbitrary ABox (including $\mathcal{A}_0$) is satisfiable.

Exercise 6.3 Show that concept satisfiability in $\mathcal{ALC}$ is NP-hard.

Hint: show the claim by reduction from SAT.

Solution: We provide a (straightforward) reduction φ from SAT to concept satisfiability in $\mathcal{ALC}$. Given a propositional formula f, we obtain the $\mathcal{ALC}$ concept $\varphi(f)$ by simply viewing every propositional variable in f as an atomic concept, and replacing in f every occurrence of '$\sqcap$' with '$\wedge$', and every occurrence of '$\sqcup$' with '$\vee$'. Notice that $\varphi(f)$ is an $\mathcal{ALC}$ concept not containing roles.

We now show that $\varphi(f)$ is satisfiable if and only if f is so.

For the "if" direction, let f be satisfiable, and τ a truth value assignment such that $f\tau$ evaluates to true. We construct an interpretation $(\Delta^{\mathcal{I}_\tau}, \cdot^{\mathcal{I}_\tau})$ of $\varphi(f)$ as follows: $\Delta^{\mathcal{I}_\tau} = \{o\}$, and for an atomic concept A, we set

$A^{\mathcal{I}_\tau} = \{o\}$ if $A\tau =$ *true*, and $A^{\mathcal{I}_\tau} = \{\}$ if $A\tau =$ *false*. It is easy to show, by induction on the structure of f, that $\varphi(f)^{\mathcal{I}_\tau} = \{o\}$, hence $\varphi(f)$ is satisfiable.

For the "only-if" direction, let $\varphi(f)$ be satisfiable, $\mathcal{I}$ an interpretation such that $(\varphi(f))^{\mathcal{I}} \neq \emptyset$, and $o \in (\varphi(f))^{\mathcal{I}}$. We construct a truth value assignment $\tau_{\mathcal{I}}$ for f as follows: for a propositional variable A in f, we set $A\tau_{\mathcal{I}} =$ *true* if $o \in A^{\mathcal{I}}$, and $A\tau_{\mathcal{I}} =$ *false* if $o \notin A^{\mathcal{I}}$. It is easy to show, by induction on the structure of f, that $f\tau_{\mathcal{I}} =$ *true*, hence f is satisfiable. This concludes the proof.

Exercise 6.4 Let q_n, for $n \geq 1$, be a Boolean conjunctive query with $n+1$ existential variables of the form $\exists x_0, \ldots, x_n.\ P(x_0, x_1) \land P(x_1, x_2) \land \cdots \land P(x_{n-1}, x_n)$. Given $n \geq 1$:

1. construct an $\mathcal{ALC}$ KB $\mathcal{K}_n$ such that $\mathcal{K}_n \models q_n$.
2. construct an $\mathcal{ALC}$ KB $\mathcal{K}'_{2^n}$ of size polynomial in n such that $\mathcal{K}'_{2^n} \models q_{2^n}$ and $\mathcal{K}'_{2^n} \not\models q_{2^n+1}$.

   Hint: $\mathcal{K}'_{2^n}$ "implements" a binary counter by means of n atomic concepts representing the bits of the counter, and such that the models of $\mathcal{K}'_{2^n}$ contain a P-chain of objects of length 2^n.

Solution:

1. There are many possible ways to construct $\mathcal{K}_n = \langle \mathcal{T}_n, \mathcal{A}_n \rangle$. We provide a few alternatives:
   (a) $\mathcal{T}_n = \emptyset$ and $\mathcal{A}_n = \{P(a, a)\}$;
   (b) $\mathcal{T}_n = \{A \sqsubseteq \exists P.A\}$ and $\mathcal{A}_n = \{A(c)\}$;
   (c) $\mathcal{T}_n = \emptyset$ and $\mathcal{A}_n = \{P(c_0, c_1), P(c_1, c_2), \ldots, P(c_{n-1}, c_n)\}$;
   (d) $\mathcal{T}_n = \{A \sqsubseteq \exists P.\exists P.\cdots \exists P.\exists P\}$ and $\mathcal{A}_n = \{A(c)\}$, where the number of (nested) existential restrictions in the right-hand side of the concept inclusion in $\mathcal{T}_n$ is equal to n.
   (e) $\mathcal{T}_n = \{A \sqsubseteq \exists P.A_1,\ A_1 \sqsubseteq \exists P.A_2, \ldots, A_{n-2} \sqsubseteq \exists P.A_{n-1},\ A_{n-1} \sqsubseteq \exists P\}$ and $\mathcal{A}_n = \{A(c)\}$.

   Notice that in alternatives (a) and (b), $\mathcal{T}_n$ and $\mathcal{A}_n$ do not depend on n, and work for every possible value $n \geq 1$.

2. We introduce $2n$ concepts $B_i, \overline{B}_i$, $1 \leq i \leq n$. Intuitively, $B_i(a)$ (resp. $\overline{B}(a)$) says that the i-th bit of the number a is 1 (resp. 0). $\mathcal{K}'_{2^n} = \langle \mathcal{T}_n, \mathcal{A}_n \rangle$, where $\mathcal{T}_n$ consists of the following axioms:

$$\begin{array}{rll} \overline{B}_i \sqsubseteq \exists P.\top, & & 1 \leq i \leq n \\ \overline{B}_1 \sqsubseteq \forall P.B_1 & & \\ B_1 \sqcap \cdots \sqcap B_i \sqcap \overline{B}_{i+1} \sqsubseteq \forall P.(\overline{B}_1 \sqcap \cdots \sqcap \overline{B}_i \sqcap B_{i+1}) & & 1 \leq i \leq n-1 \\ \overline{B}_i \sqcap \overline{B}_j \sqsubseteq \forall P.\overline{B}_j & & 1 \leq i < j \leq n \\ \overline{B}_i \sqcap B_j \sqsubseteq \forall P.B_j & & 1 \leq i < j \leq n \end{array}$$

   and $\mathcal{A}_n = \{\overline{B}_1(a), \ldots, \overline{B}_n(a)\}$