Exercises on space complexity

Exercise 1: Let A be an algorithm of space complexity $s(n)$.

Show that there is an algorithm A' such that

- $\mathcal{L}(A) = \mathcal{L}(A')$
- A' has space complexity $s'(n) = O(s(n) + \log n)$
- A' does not scan the input-tape beyond the boundaries of the input

Proof: we proceed in two steps

1) We prove that on input x, there is an algorithm A_1 such tha

   - $\mathcal{L}(A_1) = \mathcal{L}(A)$
   - A_1 does not scan the input-tape beyond location $2^{O(s(n) + \log_2 n)}$ from the input
   - A_1 has space complexity $s_1(n) = O(s(n) + \log n)$

This proof is analogous to the one that we did in class to show that a poly-space bounded (N)TM is equivalent to one that has running time $t(n) \leq c^{q(n)}$ with $q(n) = O(s(n))$ (where $s(n)$ is a polynomial space bound)

We showed $q(n) = 2 \cdot s(n) + d$, where $c = |\Gamma| + |Q|$

In our case:

- $c^{q(n)} = 2^{\log_2 c \cdot q(n)} = 2^{O(s(n))}$
- we have also the position on the input tape that contributes to the configuration:

$$\Rightarrow n \cdot 2^{O(s(n))} = 2^{\log_2 n} \cdot 2^{O(s(n))} = 2^{O(s(n) + \log_2 n)}$$

different configurations at most

Note: a TM with running time $t(n) \leq 2^{O(s(n) + \log n)}$ can scan at most $2^{O(s(n) + \log n)}$ cells of the input tape

E 11.2

2) We modify the algorithm A_1 in such a way that it does not move beyond the input.

The resulting algorithm A' works as follows:

- whenever A_1 would move right past the end of the input, A' instead:

  - does not move past the end of the input, but maintains a counter on the work tape
  - whenever A_1 moves right, the counter is incremented
    — " — left, — " — decremented

  In this way, A' can keep track of the position of the input head of A_1

  Whenever A_1 moves back again over the input symbol, A' does not update the counter (leaving it to 0)

- A' operates similarly whenever A_1 moves left past the beginning of the input

How much space does the counter use:

Since A_1 does not scan the input tape beyond $s_I(n) = 2^{O(s(n)+\log n)}$

the counter takes $\log_2 s_I(n) = O(s(n) + \log n)$

Hence, the total space used by A' is

$$s(n) + O(s(n) + \log 2) = O(s(n) + \log n)$$

Exercise 2: Let A be an algorithm of space complexity s.

Show that there is an algorithm A' such that

- A' computes the same function as A, i.e. $A'(w) = A(w)\ \forall w \in \{0,1\}^*$
- A' has space complexity $s'(n) = s(n) + \mathcal{O}(\log l(n))$ where $l(n) = \max_{w \in \{0,1\}^n} |A(w)|$ is the size of the maximum output for input x of length n
- A' never rewrites on the same location of its output tape

Proof:

A' proceeds in successive iterations each time simulating the whole computation of A:

in the i-th iteration, A' outputs the i-th bit of $A(x)$

When emulating A, in its i-th iteration A' proceeds as follows:

- it does not directly (re)write on the output tape
- instead, it maintains on the work tape:
  - the counter i of the next output bit that will be written
  - a counter c of the bit that A is currently writing
  - the value of the bit written by A in position i
- when A would write an output bit, A' operates depending on the values of i and c:
  - if $i \neq c$, then A' does not output anything
  - if $i = c_e$, then A' stores the written bit on its worktape
- At the end of its simulation, A' outputs the stored bit to the i-th position of the output tape

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How much space $s'(n)$ does A' use on the working tape for inputs of length n?

- $s(n)$ cells, since it performs the computation of A
- the space for the counters i, and c.

c_l and c_o here to count positions on the output tape, and hence will use $\log_2 l(n)$ bits each.

We get that $s'(n) = s(n) + O(\log_2 l(n))$