E5.1

Turing Machines computing functions:

We consider number-theoretic functions $f : \mathbb{N}^k \to \mathbb{N}$

- natural numbers are represented in unary

  n is represented by $\bar{n} = 1^{n+1}$

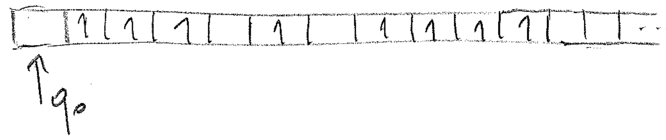
  e.g.

n	$\bar{n}$
0	1
1	11
2	111
⋮	⋮

Hence, the input alphabet of a TM is $\Sigma = \{1\}$

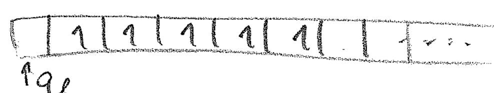
- consecutive numbers on the input are separated by a $\not{b}$
- the TM starts its computation with the head placed on a $\not{b}$ preceding the first number

  e.g. for $f(2,0,3)$



- when the computation terminates, the input has been replaced on the tape with the result of the function

  e.g. if $f(2,0,3) = 4$



We make some assumptions to ease composition of TMs:

1) there is a single final state q_f
2) the only transition from q_0 is $\delta(q_0, \not{b}) = (q_i, \not{b}, R)$
3) there are no transitions entering q_0 or of the form $\delta(q_f, \not{b})$
4) the computation loops whenever $f(n)\uparrow$, i.e. $f(n)$ is undefined

This allows us to sequentially compose TMs:
represented by the diagram $\circledast_1 \longrightarrow \boxed{M_1} \longrightarrow \circledast_2 \longrightarrow \boxed{M_2} \longrightarrow \circledast_3$

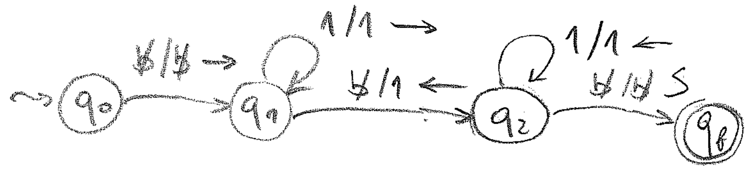
1 ... initial state of M_1 and of the combination
3 ... final — " — M_2 — " —
2 ... final state of M_1 is also the initial state of M_2

E5.2

1) Construct a TM computing the successor function

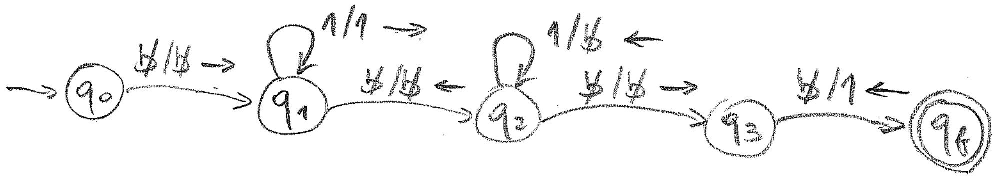
$s(n) = n+1$

S:



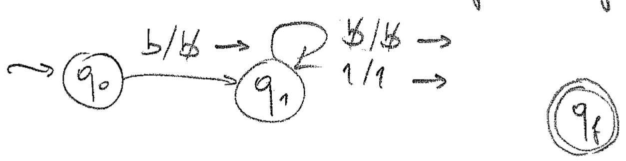
2) Construct a TM computing the zero function $z(n)=0$

Z :



3) Construct a TM computing the empty function $e(n)\uparrow$ i.e., the function that is undefined for every $n \in \mathbb{N}$

E :

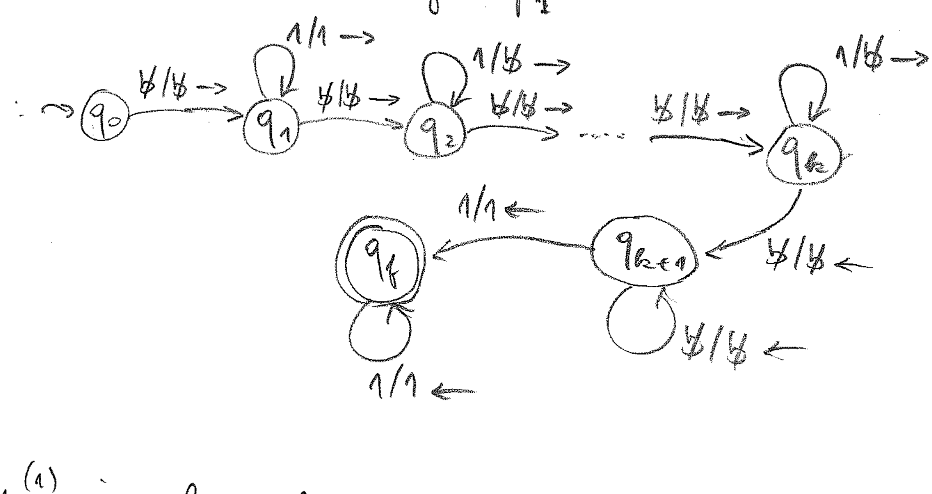


The k-variable projection function $\pi_i^{(k)}$ is defined as

$\pi_i^{(k)}(n_1, \dots, n_k) = n_i$ (for $1 \le i \le k$)

4) Construct a TM computing $\pi_1^{(k)}$

$P_1^{(k)}$:



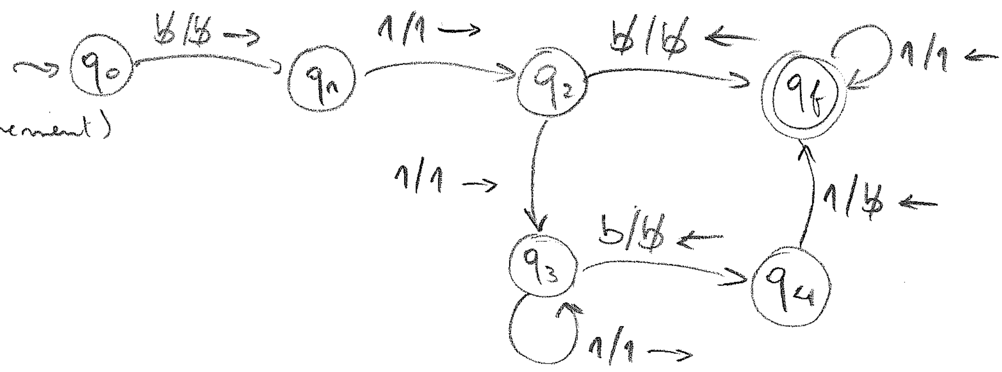
Note $\pi_1^{(1)}$ is also called the identity function $id(n) = n$

E5.3

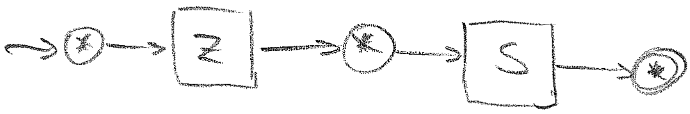
5) Construct a TM computing the predecessor function

$$pred(n) = \begin{cases} 0 & \text{if } n = 0 \\ n-1 & \text{if } n > 0 \end{cases}$$

D :
(for decrement)



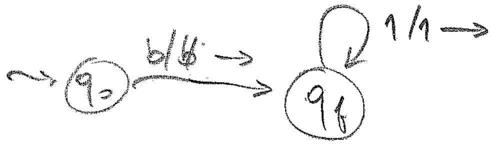
6) Using sequential composition, construct a TM computing the constant function $c(n) = 1$

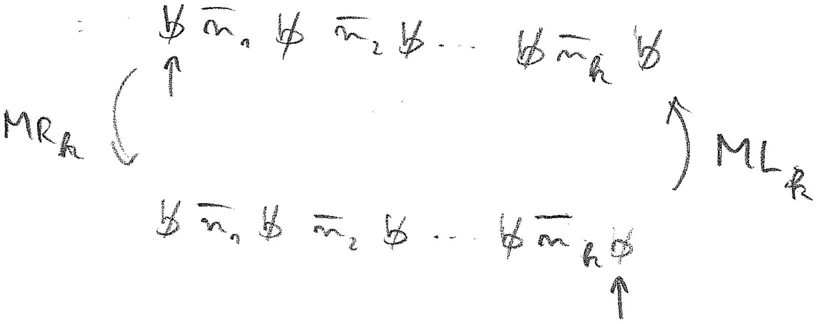


We now define some TMs that can be used as macros:

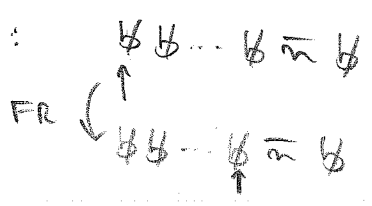
MR_k: move the tape head to the right through k consecutive natural numbers

e.g. MR_1





FR (find right) :



E5.4

FL (find left):

$$\not b\, \bar n\, \not b \cdots \underset{\uparrow}{\not b} \quad \longrightarrow \quad \underset{\uparrow}{\not b}\, \bar n\, \not b \cdots \not b$$

E_k (erase):

$$\underset{\uparrow}{\not b}\, \bar n_1\, \not b\, \bar n_2\, \not b \cdots \not b\, \bar n_k\, \not b \quad \longrightarrow \quad \underset{\uparrow}{\not b}\, \not b \cdots \not b\, \not b$$

CPY_k (copy):

$$\underset{\uparrow}{\not b}\, \bar n_1\, \not b\, \bar n_2\, \not b \cdots \not b\, \bar n_k\, \not b\, \not b \cdots \not b\, \not b$$
$$\longrightarrow \quad \underset{\uparrow}{\not b}\, \bar n_1\, \not b\, \bar n_2\, \not b \cdots \not b\, \bar n_k\, \not b\, \bar n_1\, \not b\, \bar n_2\, \not b \cdots \not b\, \bar n_k\, \not b$$

$CPY_{k,i}$ (copy through i numbers)

$$\underset{\uparrow}{\not b}\, \bar n_1\, \not b \cdots \not b\, \bar n_k\, \not b\, \bar n_{k+1}\, \not b \cdots \not b\, \bar n_{k+i}\, \not b\, \not b \cdots \not b\, \not b$$
$$\longrightarrow \quad \underset{\uparrow}{\not b}\, \bar n_1\, \not b \cdots \not b\, \bar n_k\, \not b\, \bar n_{k+1}\, \not b \cdots \not b\, \bar n_{k+i}\, \not b\, \bar n_1\, \not b \cdots \not b\, \bar n_k\, \not b$$

for CPY_k and $CPY_{k,i}$ the blank portion following $\bar n_1 \not b \cdots \not b \bar n_k$ is assumed to be long enough to contain the copy.

T (translate)

$$\not b\, \not b \cdots \not b\, \bar n\, \not b \ \text{(head at first } \not b\text{)} \quad \longrightarrow \quad \underset{\uparrow}{\not b}\, \bar n\, \not b \cdots \not b$$

BRN (branch on zero)

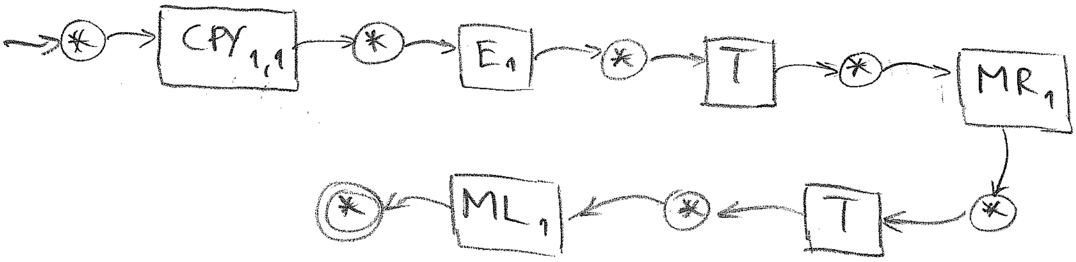
$$\begin{tikzcd} \arrow[r, squiggly] & \circledast \arrow[r] & \boxed{BRN} \arrow[r, "n=0"] \arrow[d, "n>0"] & \circledast \\ & & \circledast & \end{tikzcd}$$

it does not alter the tape or change the head position

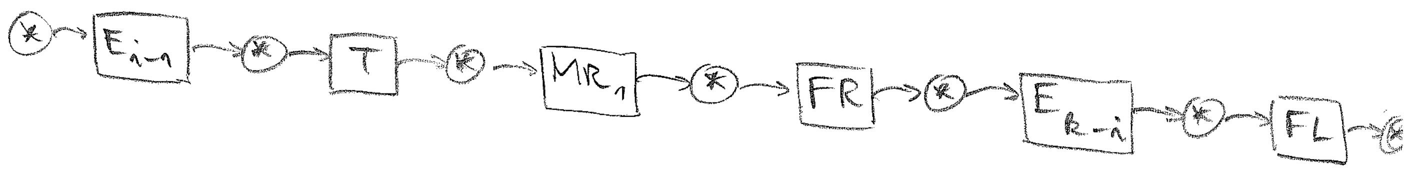
E5.5

7) Using composition and the defined macros, construct a TM INT (interchange):

$$\not{b}\,\bar{n}\,\not{b}\,\bar{m}\,\not{b}\,\not{b}^{n+1}\,\not{b}$$
$$\uparrow$$
$$\not{b}\,\bar{m}\,\not{b}\,\bar{n}\,\not{b}\,\not{b}^{n+1}\,\not{b}$$



8) Using composition and the defined macros, construct a TM for $p_i^{(k)}$



Function composition:

Let $g_i : \mathbb{N}^k \to \mathbb{N}$ for $1 \leq i \leq n$

$h : \mathbb{N}^n \to \mathbb{N}$

The composition of h with $g_1, \dots, g_n$, written

$f = h \circ (g_1, \dots, g_n)$

is the function $f : \mathbb{N}^k \to \mathbb{N}$ defined by

$f(x_1, \dots, x_k) = h(g_1(x_1, \dots, x_k), \dots, g_n(x_1, \dots, x_k))$

The function $f(x_1, \dots, x_k)$ is undefined if either

1) $g_i(x_1, \dots, x_k) \uparrow$ for some $i \in \{1, \dots, n\}$
2) $g_i(x_1, \dots, x_k) = y_i$ for $i \in \{1, \dots, n\}$ and $h(y_1, \dots, y_n) \uparrow$

We show that the composition of Turing-computable functions is also Turing-computable.

Exercise: given $g_1 : \mathbb{N}^3 \to \mathbb{N}$, $g_2 : \mathbb{N}^3 \to \mathbb{N}$, $h : \mathbb{N}^2 \to \mathbb{N}$ and TMs G_1, G_2, H computing respectively g_1, g_2, h, construct a TM computing $h \circ (g_1, g_2)$.

Use macros and TM composition, and construct the configurations obtained after every macro application.

E5.7

macro	configuration	
	$\underline{\not b}\,\overline{x_1}\,\not b\,\overline{x_2}\,\not b\,\overline{x_3}\,\not b$	
CPY_3	$\underline{\not b}\,\overline{x_1}\,\not b\,\overline{x_2}\,\not b\,\overline{x_3}\,\not b\,\overline{x_1}\,\not b\,\overline{x_2}\,\not b\,\overline{x_3}\,\not b$	
MR_3	$\not b$ —"— $\underline{\not b}$ —"— $\not b$	
G_1	$\not b$ —"— $\underline{\not b}\,\overline{y_1}\,\not b$	(where $y_1 = g_1(x_1, x_2, x_3)$)
ML_3	$\underline{\not b}\,\overline{x_1}\,\not b\,x_2\,\not b\,x_3\,\not b\,\overline{y_1}\,\not b$	
$CPY_{3,1}$	$\underline{\not b}$ —"— $\not b\,\overline{x_1}\,\not b\,\overline{x_2}\,\not b\,\overline{x_3}\,\not b$	
MR_4	$\not b$ —"— $\underline{\not b}$ —"— $\not b$	
G_2	$\not b$ —"— $\not b\,\overline{y_1}\,\underline{\not b}\,\overline{y_2}\,\not b$	(where $y_2 = g_2(x_1, x_2, x_3)$)
ML_1	$\not b$ —"— $\underline{\not b}\,\overline{y_1}\,\not b\,\overline{y_2}\,\not b$	
H	$\not b$ —"— $\underline{\not b}\,\overline{z}\,\not b$	(where $z = h(y_1, y_2)$)
ML_3	$\underline{\not b}$ —"— $\not b\,\overline{z}\,\not b$	
E_3	$\underline{\not b}\,\not b \ldots\ \not b\,\overline{z}\,\not b$	
T	$\not b\,\overline{z}\,\not b$	