# Regular expressions



Are a formalism for describing a certain class of languages: declarative rather than computational view

Definition: given an alphabet $\Sigma$, regular expressions are strings over the alphabet $\Sigma \cup \{+, *, (, ), \cdot, \varepsilon, \emptyset\}$ defined inductively as follows:

- basis: $\varepsilon$, $\emptyset$, and each $a \in \Sigma$ is a R.E.
- inductive step: if E and F are R.E., then so are:
  - $E + F$ (union)
  - $E \cdot F$ (concatenation)
  - $E^*$ (closure)
  - $(E)$ (parentheses)

Example: $a \cdot (a+b)^* \cdot b^* a$

Definition: language $\mathcal{L}(E)$ defined by a R.E. E is also defined inductively:

- $\mathcal{L}(\varepsilon) = \{\varepsilon\}$ empty word
- $\mathcal{L}(\emptyset) = \emptyset$ empty language
- for each $a \in \Sigma$ $\mathcal{L}(a) = \{a\}$
- $\mathcal{L}(E+F) = \mathcal{L}(E) \cup \mathcal{L}(f)$
- $\mathcal{L}((E)) = \mathcal{L}(E)$
- $\mathcal{L}(E \cdot F) = \mathcal{L}(E) \cdot \mathcal{L}(F)$ ... concatenation

<u>concatenation</u> of two languages $L_1$ and $L_2$:

$$L_1 \cdot L_2 = \{w \mid w = x \cdot y, x \in L_1, y \in L_2\}$$

Example: $E = \varepsilon + 1 \Rightarrow \mathcal{L}(E) = \{\varepsilon, 1\}$

$F = \varepsilon + 0 + 1 \Rightarrow \mathcal{L}(F) = \{\varepsilon, 0, 1\}$

$G = E \cdot F = (\varepsilon+1) \cdot (\varepsilon+0+1) \Rightarrow \mathcal{L}(G) = \{\varepsilon, 0, 1, 10, 11\}$

- $\mathcal{L}(E^*) = (\mathcal{L}(E))^*$ ... closure

closure of a language L ?

we first define the powers of a language L :

- $L^0 = \{\varepsilon\}$ .
- $L^k = L^{k-1} . L$

Hence $L^k = \{ w \mid w = x_1 \cdots x_k \text{, with } \forall i, x_i \in L \}$

Closure of L : $L^* = L^0 \cup L^1 \cup L^2 \cup \cdots = \bigcup_{k \geq 0} L^k$

Example: $\begin{cases} E = 0 + 1 & \Rightarrow \mathcal{L}(E) = \{0, 1\} \\ F = E^* & \Rightarrow \mathcal{L}(F) = \text{set of all binary strings} \end{cases}$

$\{ E = 0.0 \Rightarrow \mathcal{L}(E^*) = \{\varepsilon, 00, 0000, 000000, \ldots\}$

= all even-length strings of 0's

Positive closure of a language L

$L^+ = L^1 \cup L^2 \cup \ldots$

We can introduce a positive closure operator in R.E.

$\mathcal{L}(E^+) = (\mathcal{L}(E))^+$

Note: we have to distinguish between an expression E and the language $\mathcal{L}(E)$ defined by E

When we write E = F, we usually mean not syntactic equality but equality of the corresponding languages, i.e. $\mathcal{L}(E) = \mathcal{L}(F)$. In other words, equality is in the algebra of R.E.

Precedence of operators:

high $*$

$\cdot$

low $+$

example: $E + F \cdot G^* = E + (F \cdot (G^*))$

## Algebraic laws for R.E.

similar to the laws for arithmetic expressions, we can express laws for R.E.: treat $+$ as sum
$\cdot$ as product

(Note! the laws are not the same as for arithmetic expressions)

- associativity of $\cdot$ and $+$

$$(E \cdot F) \cdot G = E \cdot (F \cdot G) = E \cdot F \cdot G$$

$$(E + F) + G = E + (F + G) = E + F + G$$

- commutativity of $+$

$$E + F = F + E$$

Note: $\cdot$ is not commutative: $E \cdot F \neq F \cdot E$

- distributivity:

1) left distributive law of $\cdot$ over $+$ : $E \cdot (F + G) = E \cdot F + E \cdot G$

2) right — " — : $(F + G) \cdot E = F \cdot E + G \cdot E$

Proof of 1): the law actually holds for arbitrary languages, and does not require $E, F, G$ to be R.E.
(OPTIONAL)

Hence, we prove: for arbitrary languages $L, M, N$:

$$L \cdot (M \cup N) = L \cdot M \cup L \cdot N$$

We show that for a string $w$ we have $w \in L \cdot (M \cup N)$ iff $w \in L \cdot M \cup L \cdot N$

"Only-if": $w \in L \cdot (M \cup N) \Rightarrow w = x \cdot y$ with $x \in L$, $y \in M \cup N$

Since $y \in M \cup N$, either $y \in M$ or $y \in N$ (or both).

If $y \in M$, then $w = x \cdot y \in L \cdot M$, hence $w \in L \cdot M \cup L \cdot N$

(similarly for $y \in N$)

"If": $w \in L \cdot M \cup L \cdot N$, hence either $w \in L \cdot M$ or $w \in L \cdot N$.

If $w \in L \cdot M$, then $w = x \cdot y$, with $x \in L$, $y \in M$. (Similarly for $w \in L \cdot N$)

Hence $y \in M \cup N$, and $w = x \cdot y \in L \cdot (M \cup N)$.

Example: $0\cdot 0 + 0\cdot 1^* = 0\cdot(0+1^*)$

we can factor out a 0 from the union

What about $0 + 0\cdot 1^*$ ?

if we factor out a 0, what remains for the summand on the left?

$$0 + 0\cdot 1^* \underset{\text{identity}}{=} 0\cdot\varepsilon + 0\cdot 1^* = 0\cdot(\varepsilon + 1^*) = 0\cdot 1^*$$

since $\varepsilon \in \mathcal{L}(1^*)$

- identities and annihilators (hold for arbitrary languages)
  - $\emptyset + E = E + \emptyset = E$
  - $\varepsilon \cdot E = E \cdot \varepsilon = E$
  - $\emptyset \cdot E = E \cdot \emptyset = \emptyset$

- idempotency
  - $E + E = E$
  - $(E^*)^* = E^*$ Proof: $\boxed{\text{Exercise 3.4.1 f}}$

- other laws for closure (already seen)
  - $\emptyset^* = \varepsilon$
  - $\varepsilon^* = \varepsilon$
  - $E^+ = E\cdot E^* = E^*\cdot E$
  - $E^* = E^+ + \varepsilon$

    Note: if $\varepsilon \in \mathcal{L}(E)$, then $E^* = E^+$

$\boxed{\text{Exercise 3.4.4.}}$ Prove that $(E^*\cdot F^*)^* = (E+F)^*$

$\boxed{\text{Exercise 3.1.1}}$ Write R.E.s for the following languages

a) $\{ w \in \{a, b, c\}^* \mid w$ contains at least one a and at least one b $\}$

b) $\{ w \in \{0, 1\}^* \mid w$'s tenth symbol from the right is a 1 $\}$

c) $\{ w \in \{0, 1\}^* \mid w$ contains at most one pair of consecutive 1's $\}$

$\boxed{\text{Exercise 3.1.2}}$ Write R.E.s for the following languages

a) The set of all strings over $\{0, 1\}$ s.t. every pair of adjacent 0's appears before any pair of adjacent 1's

b) The set of strings of 0's and 1's whose number of 0's is divisible by 5

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Solutions:

3.1.1 a) $c^* \cdot a \cdot (a + c)^* \cdot b \cdot (a + b + c)^* +$
$c^* \cdot b \cdot (b + c)^* \cdot a \cdot (a + b + c)^*$

b) $(0+1)^* \, 1 \cdot \underbrace{(0+1) \cdot \ldots \cdot (0+1)}_{9 \text{ times}}$

c) $0^* \cdot (1 \cdot 0^+)^* \cdot 1 \cdot 1 \cdot (0^+ \cdot 1) \cdot 0^* + 0^* \cdot (1 \cdot 0^+)^* \cdot (\varepsilon + 1)$
or, simpler: $(0 + 1 \cdot 0)^* (\varepsilon + 11) \cdot (0 + 01)^*$

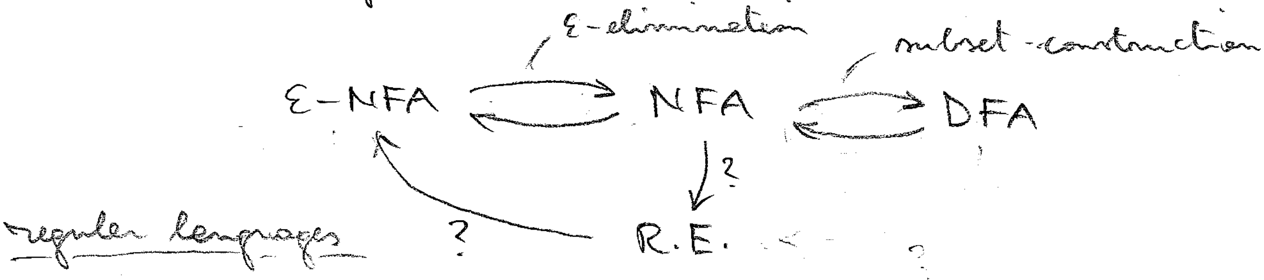
3.1.2 a) $\underbrace{(0 + 10)^*}_{\text{no pair of adjacent 1's}} \cdot \underbrace{(1 + 10)^*}_{\text{no pair of adjacent 0's}}$

b) $(1^* \cdot 0 \cdot 1^* \cdot 0 \cdot 1^* \cdot 0 \cdot 1^* \cdot 0 \cdot 1^* \cdot 0 \cdot 1^*)^*$

# Regular languages

20/11/2008

What is the relationship between the classes of languages studied so far?




**Theorem:** (R.E. $\rightarrow$ ε-NFA)

For every R.E. $E$ there is an ε-NFA $A_E$ s.t. $\mathcal{L}(A_E) = \mathcal{L}(E)$.

**Proof:** let us call an ε-NFA *simple* if

- it has only one final state
- the initial state has no incoming arcs
- the final — " — outgoing — " —

We show by structural induction that for each R.E. $E$ there is a simple ε-NFA $A_E$ s.t. $\mathcal{L}(E) = \mathcal{L}(A_E)$

Basis: $E = \varepsilon$, $E = \emptyset$, $E = a$ for some $a \in \Sigma$

$A_\varepsilon$


$A_\emptyset$


$A_a$

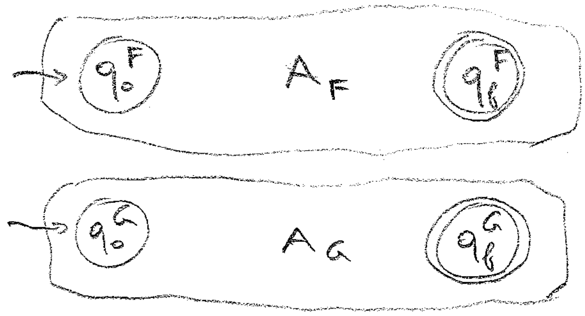

Inductive case: 1) $E = F + G$

2) $E = F \cdot G$

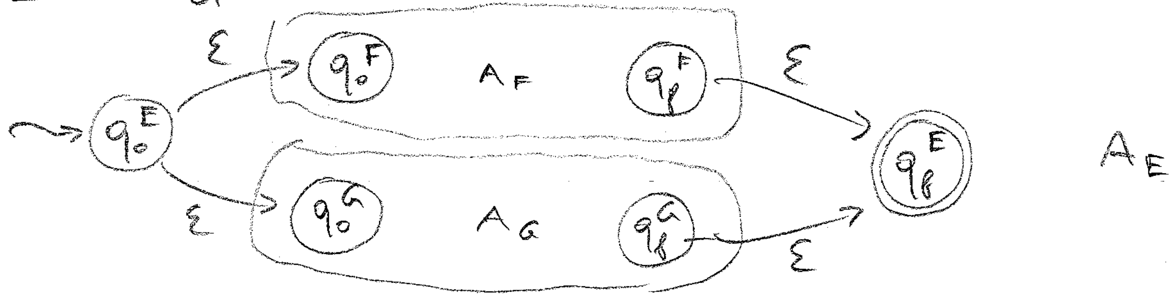
3) $E = F^*$

4) $E = (F)$

By I.H., there are simple $\varepsilon$-NFAs for F and G




1) $E = F + G$




$$\mathcal{L}(A_E) = \mathcal{L}(A_F) \cup \mathcal{L}(A_G) \underset{\text{by I.H.}}{=} \mathcal{L}(F) \cup \mathcal{L}(G) = \mathcal{L}(F+G) = \mathcal{L}(E)$$

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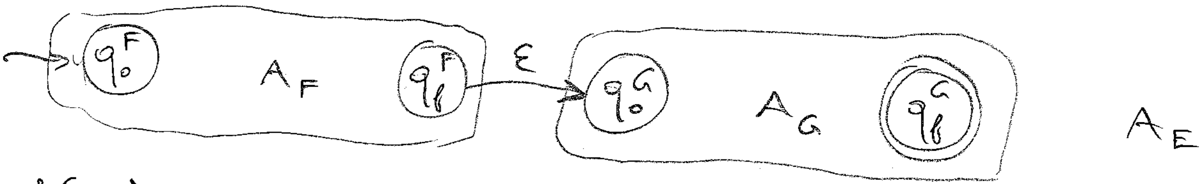
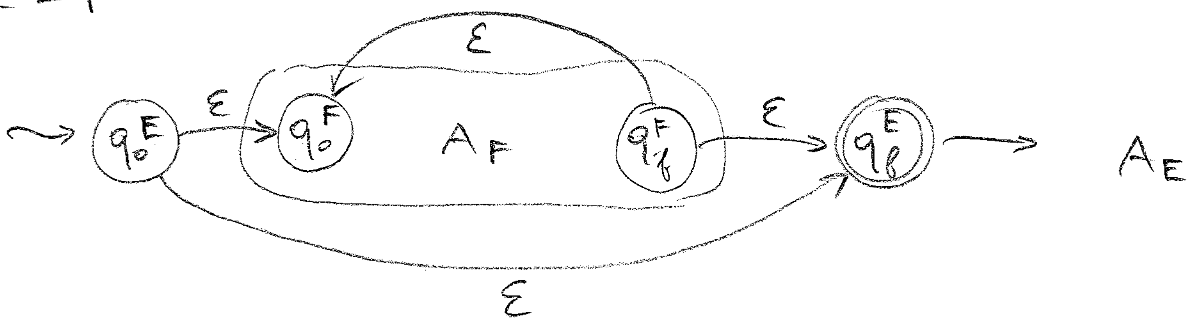
2) $E = F \cdot G$




$$\mathcal{L}(A_E) = \mathcal{L}(A_F) \cdot \mathcal{L}(A_G) \underset{\text{by I.H.}}{=} \mathcal{L}(F) \cdot \mathcal{L}(G) = \mathcal{L}(F \cdot G) = \mathcal{L}(E)$$

3) $E = F^*$

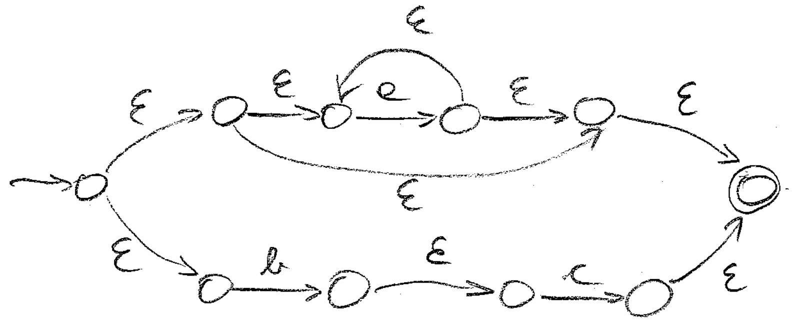



3) $E = (F)$

$A_E = A_F$

q.e.d.

Example: $E = a^* + b \cdot c$




OPTIONAL

Theorem (DFA $\to$ R.E.)

For every DFA $A$ there is a R.E. $E_A$ s.t. $\mathcal{L}(E_A) = \mathcal{L}(A)$

Proof: Let $A = (Q, \Sigma, \delta, q_1, F)$

We assume without loss of generality (w.l.o.g) that

$Q = \{q_1, q_2, \ldots, q_n\}$

Let us define $L_{ij} = \{w \mid \hat{\delta}(q_i, w) = q_j\} =$
$\forall i,j \in \{1, \ldots, n\}$
$= \{w \mid w \text{ takes } A \text{ from } q_i \text{ to } q_j\}$

note that $L_{ij} = \mathcal{L}(A_{ij})$ with $A_{ij} = (Q, \Sigma, \delta, q_i, \{q_j\})$

We aim at constructing R.E.s $E_{ij}$ for $L_{ij}$.

Then we can take $E_A = \sum_{q_j \in F} E_{1j}$, since

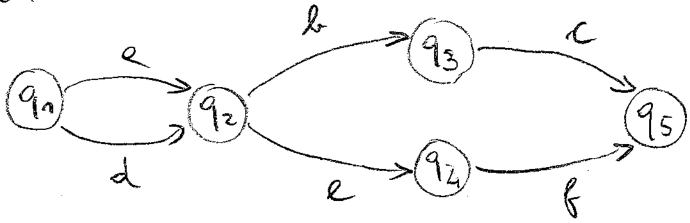
$$\mathcal{L}(E_A) = \bigcup_{q_j \in F} \mathcal{L}(E_{ij}) = \bigcup_{q_j \in F} L_{1j} = \{w \mid \hat{\delta}(q_1, w) \in F\} = L(A)$$

How can we compute $E_{ij}$?

Let us define $\forall i,j \in \{1, \ldots, n\}$, $\forall k \in \{0, \ldots, n\}$

$L_{ij}^{k} = \{w \mid A$ goes from $q_i$ to $q_j$ on input $w$, passing only through $q_1, \ldots, q_k$ as intermediate states$\}$

Example:



$abc \in L^3_{15}$

$def \notin L^3_{15}$ but $def \in L^4_{15}$

$def \in L_{15}$

$L^0_{12} = \{a, d\}$

$L^3_{15} = \{abc, dbc\}$

$L^4_{15} = L^5_{15} = L_{15} = \{abc, dbc, aef, def\}$

Note: $L^n_{ij} = L_{ij}$

Hence, we are done if we can construct R.E.s $E^k_{ij}$ for $L^k_{ij}$.

We can simply take $E_{ij} = E^n_{ij}$, and hence $E_A = \sum_{q_j \in F} E^n_{1j}$.

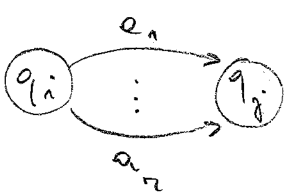
We construct $E^k_{ij}$ by induction on $k$:

Basis: we construct $E^0_{ij}$, for all $i, j \in \{1, \dots, n\}$

since $k=0$, we cannot go through any intermediate state.

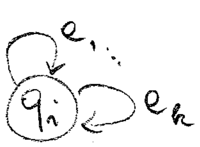
2 cases, each with 2 subcases:

$i \neq j$

-  $E^0_{ij} = a_1 + \dots + a_r$

-  $E^0_{ij} = \emptyset$

$i = j$

-  $E^0_{ii} = \varepsilon + a_1 + \dots + a_k$

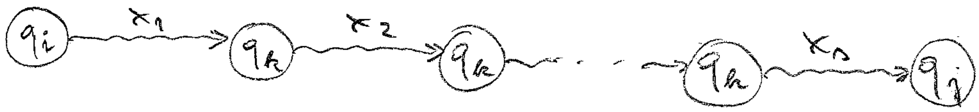
-  $E^0_{ii} = \varepsilon$

Induction: assume we have constructed $E_{ij}^{k-1}$ $\forall i, j \in \{1, \dots, n\}$ (3.10)

we show how to construct $E_{ij}^{k}$

Observe:

- $L_{ij}^{k}$ will include $L_{ij}^{k-1}$
- it additionally will contain those words that lead through $q_k$ at least once, when going from $q_i$ to $q_j$




$w = x_1 x_2 \cdots x_s$ where: $\rightsquigarrow$ represents transitions going at most through $\{x_1, \dots, x_{k-1}\}$

then $x_1 \in L_{ik}^{k-1}$

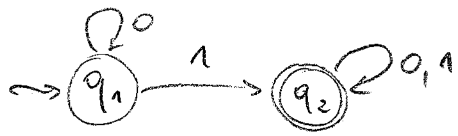
$x_2, \dots, x_{s-1} \in L_{kk}^{k-1}$

$x_s \in L_{kj}^{k-1}$

$$\Rightarrow w \in L_{ik}^{k-1} \cdot \left(L_{kk}^{k-1}\right)^* \cdot L_{kj}^{k-1}$$

$$\Rightarrow E_{ij}^{k} = E_{ij}^{k-1} + E_{ik}^{k-1} \cdot \left(E_{kk}^{k-1}\right)^* \cdot E_{kj}^{k-1}$$

Example:




A

| $k$ | $E_{11}^{k}$ | $E_{12}^{k}$ | $E_{21}^{k}$ | $E_{22}^{k}$ |
|---|---|---|---|---|
| 0 | $\varepsilon + 0$ | $1$ | $\emptyset$ | $\varepsilon + 0 + 1$ |
| 1 | $(\varepsilon + 0) + (\varepsilon + 0)(\varepsilon + 0)^* \cdot (\varepsilon + 0) = 0^*$ | $1 + (\varepsilon + 0) \cdot (\varepsilon + 0)^* \cdot 1 = 0^* \cdot 1$ | $\emptyset$ | $\varepsilon + 0 + 1$ |
| 2 | not needed | $0^* \cdot 1 + (0^* \cdot 1) \cdot (\varepsilon + 0 + 1)^* \cdot (\varepsilon + 0 + 1) = 0^* \cdot 1 \cdot (0 + 1)^*$ | not needed | not needed |

$\mathcal{L}(A) = \mathcal{L}(E_{12}) = \mathcal{L}(E_{12}^{2}) =$

$E_A = E_{12} = E_{12}^{2} = 0^* \cdot 1 \cdot (0 + 1)^*$

↑ OPTIONAL

## Theorem (NFA $\to$ R.E.)

For every NFA $A$ there is a R.E. $E_A$ s.t. $\mathcal{L}(E_A) = \mathcal{L}(A)$

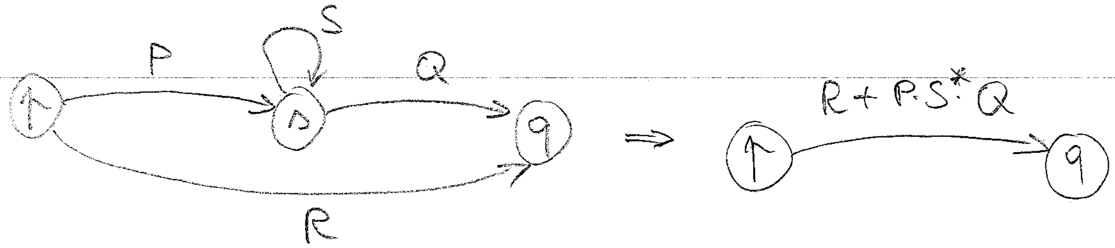
<u>Proof sketch</u>: We show how to construct $E_A$ by eliminating states of $A$.

Consider the elimination of a state $s$:

If there was a path from state $p$ to state $q$ over $s$, after eliminating $s$ the path does no longer exist

$\Rightarrow$ we have to compensate for that

We add a regular expression "connecting" $p$ and $q$ and capturing the missing path.

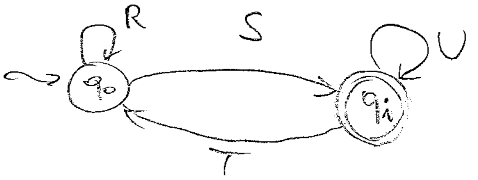



We can eliminate in this way all states except initial and final states:

Strategy: Assume $A$ has $k$ final states $q_1, \dots, q_k$.

For each final state $q_i$, eliminate all states except $q_i, q_0$

a) If $q_i \neq q_0$, we are left with:




The corresponding R.E. is

$$E_i = (R + S \cdot U^* \cdot T)^* \cdot S \cdot U^*$$

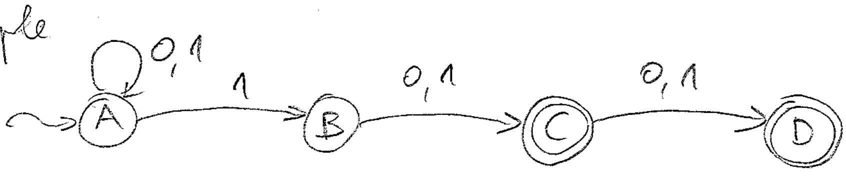
b) If $q_i = q_0 \in F$, we must eliminate all states except $q_0$.

We are left with

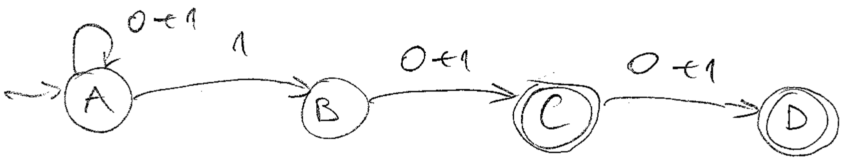

The corresponding R.E. is $E_i = R^*$

Then $E_A$ is the sum of all $E_i$: $E_A = E_1 + E_2 + \dots + E_k$

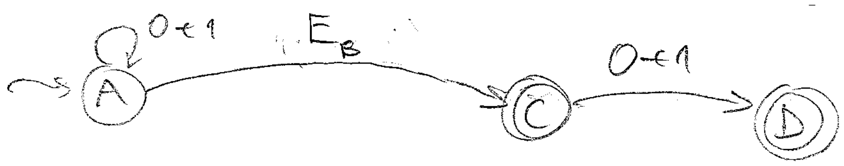
Example



We view all edge labels as R.E.s (missing labels mean $\emptyset$)



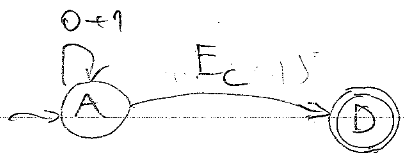
Eliminate B:



$$E_B = \emptyset + 1\cdot\emptyset^*\cdot(0+1) = 1\cdot\emptyset^*\cdot(0+1) = 1\cdot(0+1)$$

Eliminate

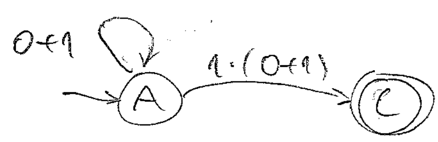
1) C:



$$E_C = \emptyset + 1\cdot(0+1)\cdot\emptyset^*\cdot(0+1) = 1\cdot(0+1)\cdot(0+1)$$

$$E_1 = (0+1)^*\cdot E_C = (0+1)^*\cdot 1\cdot(0+1)\cdot(0+1)$$

2) D:



$$E_2 = (0+1)^*\cdot 1\cdot(0+1)$$

$$E = E_1 + E_2 = (0+1)^*\cdot 1\cdot(0+1)\cdot(0+1) + (0+1)^*\cdot 1\cdot(0+1)$$

$$= (0+1)^*\cdot 1\,(0+1)\,(\varepsilon + 0 + 1)$$