PROPERTIES OF REGULAR LANGUAGES

12/12/2008

E8.1

## EXERCISE 1

Decide which of the following statements is true and which is false. Give a brief explanation of your answer.

a) For all languages $L_1$ and $L_2$, it holds that $(L_1^* \cdot L_2^*)^+ = (L_1^+ \cdot L_2^+)^*$.

b) If $L_1$ and $L_2$ are both not regular then $L_1 \cup L_2$ could be regular.

c) For all languages $L_1$ and $L_2$, if $L_1 \subseteq L_2$ then $L_1^* \subseteq L_2^*$.

## EXERCISE 2

Show that the following languages are not regular.

a) $\{0^n 1^m 0^{n+m} \mid m,n \geq 0\}$  b) $\{w \in \{0,1\}^* \mid w \text{ is a palindrome}\}$

## EXERCISE 3

Give algorithms to tell whether:

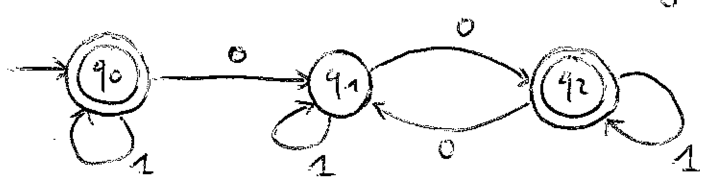
a) a regular language $L$ is universal (i.e. $L = \Sigma^*$);

b) two regular languages have at least one string in common.

## EXERCISE 4

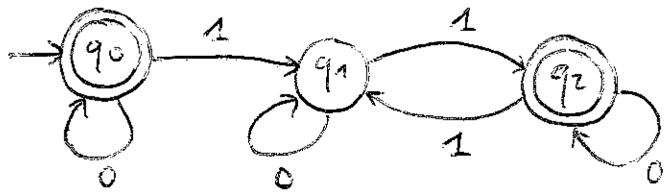
Show that if $L$ and $M$ are regular languages then so is $L \cap M$ (without using the De Morgan law $L \cap M = \overline{\overline{L} \cup \overline{M}}$). Apply the construction to the following automata:

$A_L^* =$





$A_M^* =$



1)a) False. Consider the languages $L_1 = \{a\}$ and $L_2 = \{b\}$. Then $b \in (L_1^* \cdot L_2^*)^+$ but $b \notin (L_1^+ \cdot L_2^+)^*$.

1)b) True. Assume that $L_1 = \overline{L_2}$, i.e. $L_2 = \overline{L_1}$. If $L_1$ is not regular then so is $L_2$ (because, if $L_2$ would be regular then, by the closure properties of regular languages, $L_1$ would be regular too, thus leading to a contradiction). Since $L_1 \cup L_2 = \Sigma^*$ we have that the union of two non-regular languages can be regular.

1)c) True. Given that, for all $w \in L_1$, we also have that $w \in L_2$, the argument goes as follows. If $w' \in L_1^*$ then $w' = w_1 \cdots w_n$ for some $n \in \mathbb{N}$ and $w_i \in L_1$ $(1 \leq i \leq n)$. But then each $w_i$ is also in $L_2$ and therefore $w' \in L_2^*$.

2)a) Assume that the language is regular.
Then, by the pumping lemma, we would have that:

there exists $n$ such that
for all $w \in L$ such that $|w| \geq n$
there are three strings $x, y, z$ such that $w = xyz$, $|xy| \leq n$, $|y| \geq 1$, and for all $k \geq 0$, $xy^kz \in L$.

Now, given some $n$, let $w = \overbrace{0 \cdots 0}^{n}\overbrace{1 \cdots 1}^{n}\overbrace{0 \cdots 0}^{2n} = 0^n 1^n 0^{2n}$. Since $|w| = 4n$ we have that $|w| \geq n$. In order to apply the pumping lemma we need to find strings $x$ and $y$ such that $|xy| \leq n$. The only possible choices are $x = 0^a$ and $y = 0^b$ where $b \geq 1$. But then we have that $xz = 0^{n-b}1^n 0^{2n}$ and thus that $n - b + n \neq 2n$. Therefore, for $k = 0$, $xy^kz \notin L$. Since we assumed that the language is regular this is a contradiction. Hence the language cannot be regular.

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2) b) Again, we use the pumping lemma.
Given some $n$, let $w = 0^n 1 0^n$.
If we consider $x, y, z$ such that
a) $w = xyz$, b) $|xy| \leq n$, c) $|y| \geq 1$
then $y$ can only be a non-empty string of 0's. Thus, for each $k > 1$, the string $xy^kz$ has more 0's on the left-hand side of 1 than on right-hand side. We can conclude that, for $k > 1$, $xy^kz \notin L$. Therefore we have that the language is not regular.

3) a) Note that if $L$ is universal then $\bar{L} = \Sigma^* - L = \phi$. Therefore we only need to check whether $\bar{L}$ is empty.

3) b) We can check whether the intersection $L$ of the two languages that we denote with $L_1$ and $L_2$ ~~is~~ non-empty, i.e. we can check whether $L = L_1 \cap L_2 = \overline{\bar{L}_1 \cup \bar{L}_2}$ is non-empty. Note that $L$ is regular because of the closure properties of regular languages.

4) Let $L$ and $M$ be the regular languages accepted by the automata $A_L = (Q_L, \Sigma_L, \delta_L, q_L, F_L)$ and $A_M = (Q_M, \Sigma_M, \delta_M, q_M, F_M)$.
We assume: a) $\Sigma_L = \Sigma_M = \Sigma$, b) $A_L$ and $A_M$ are deterministic.
We construct an automaton $A$ that simulates $A_L$ and $A_M$.
The states of $A$ are pairs of states $(p, q)$ where $p \in Q_L$ and $q \in Q_M$. If $a$ is an input symbol and $A$ is in state $(p, q)$ then $A$ goes in state $(p', q')$ where $p' = \delta_L(p, a)$ and $q' = \delta_M(q, a)$. The start state of $A$ is $(q_L, q_M)$ and the accepting states of $A$ are those pairs $(p, q)$ where both $p \in F_L$ and $q \in F_M$.

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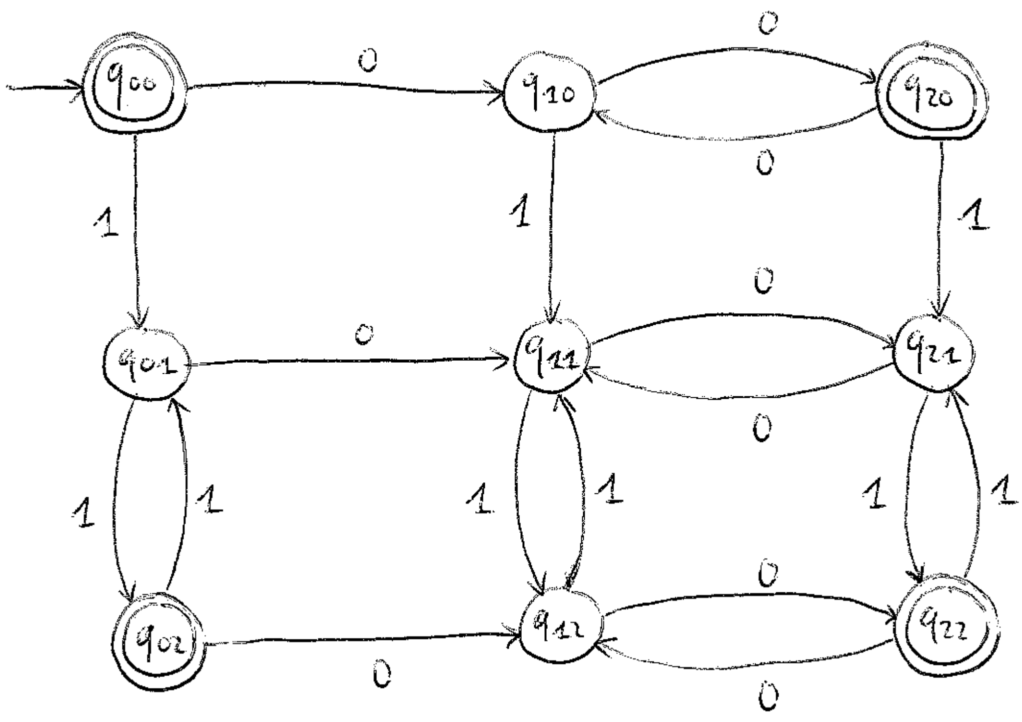
To sum up, we have that

$A = (Q_L \times Q_M, \Sigma, \delta, (q_L, q_M), F_L \times F_M)$

where $\delta((p,q),a) = (\delta_L(p,a), \delta_M(q,a))$.

Note that $A$ is constructed in such a way that $w$ is accepted by $A$ (i.e. $w \in L(A)$) if and only if $w$ is accepted by $A_L$ and $A_M$ (i.e. $w \in L(A_L)$ and $w \in L(A_M)$), i.e. iff $w \in L(A_L) \cap L(A_M)$ or $w \in L \cap M$.

By applying this construction to the automata $A_L^*$ and $A_M^*$ we get:




Note that $q_{ij}$ is shorthand for $(q_i, q_j)$

$q_j$ is in $Q_M^*$

$q_i$ is in $Q_L^*$