The polynomial hierarchy and PSPACE

There are many classes of problems that are more complex than problems in NP or coNP, but not "arbitrarily" complex.

- problems related to regular expressions / languages
  - e.g. – containment between the languages denoted by two regular expressions
    - universality of the language denoted by a regular expression / finite state automaton

- games in which players alternate moves on a board
  (note: we have to consider an $n \times n$ board, where n determines the size of the input)
  - existence of a winning strategy for one of the players; i.e.

    does there exist a move of P_1 s.t.
    for all moves of P_2
    there exists a move of P_1 s.t.
    ⋱
    P_1 wins

- problems related to special kinds of logics (that are more expressive than propositional logic, but less expressive than first-order logic)
  - modal logics
  - temporal logics (LTL, CTL, PDL, μ-calculus)
  - description logics, UML class diagrams, ER diagrams

We want to characterize the computational complexity of such problems

A first step is to resort to oracle TMs. (OTMs)

We define OTMs informally:

- let g be a function $\Sigma^* \to \Sigma^*$ (which we use as an oracle)
- an OTM M_g that uses oracle g is a TM with two tapes and a special oracle state o:
  - an ordinary tape
  - an oracle tape on which the TM can read and write normally, but also consult the oracle g at the cost of a single transition
- to consult the oracle, M_g:
  - writes the input string x for g on the oracle tape
  - enters the oracle state o
  - this activates the oracle, which replaces x with $g(x)$ on the oracle tape and places the head at the beginning of $g(x)$ (all in one step)
  - after consulting the oracle, M_g leaves the oracle state, but can use $g(x)$ on the oracle tape
- M_g accepts as usual, by entering a final state

Oracles can give TMs a lot of power.

Let us consider a class C of TMs computing functions:

<u>Definition</u>: $P^C = \{L \mid L$ is accepted by a (deterministic) poly-time OTM with an oracle in $C\}$

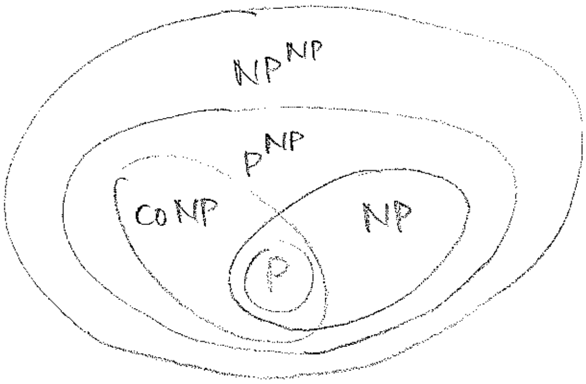
$NP^C = \{L \mid L$ is accepted by a non-deterministic poly-time OTM with an oracle in $C\}$

Example: Consider $C = NP$, i.e. the oracle is a poly-time NTM (that leaves its result on the oracle tape)

P^{NP} includes both NP and coNP

To solve a problem in NP (resp. coNP) a single call to the oracle is sufficient.

We get



Note: we do not know whether $P^{NP} \neq NP^{NP}$

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Exploiting this idea, we can define a hierarchy of classes of greater and greater apparent difficulty:

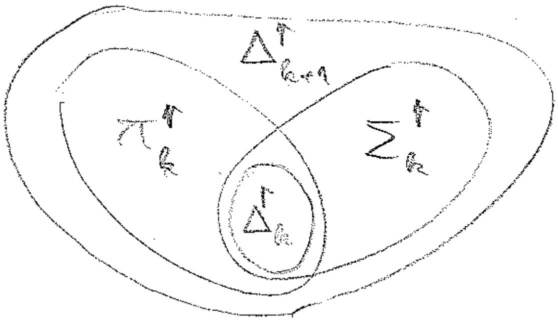
$$\Sigma_0^P = \Pi_0^P = \Delta_0^P = P$$

and for all $k \geq 0$:

$$\Delta_{k+1}^P = P^{\Sigma_k^P}$$

$$\Sigma_{k+1}^P = NP^{\Sigma_k^P}$$

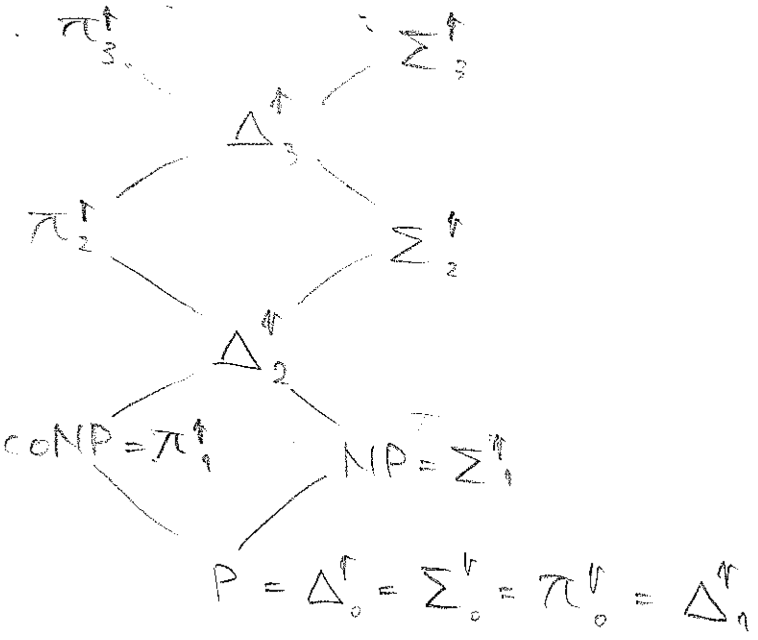
$$\Pi_{k+1}^P = co\text{-}\Sigma_{k+1}^P$$



Note: $\Sigma_1^P = NP^{\Sigma_0^P} = NP^P = NP$

$\Pi_1^P = co\Sigma_1^P = coNP$

Graphically:



We define the polynomial hierarchy $PH = \bigcup_{j=1}^{\infty} \Sigma_j^P$.

It is not known whether the hierarchy is truly infinite, but if it collapses at one level, then it collapses also above.

<u>Theorem</u>: If for some $k \geq 1$, we have $\Sigma_k^P = \Pi_k^P$, then
$\Sigma_j^P = \Pi_j^P = \Sigma_k^P$ for all $j \geq k$

In particular, if $P = NP$, then $NP = \Sigma_1^P = \Pi_1^P$ and so
$\Sigma_j^P = P$ for all $j \geq 0$, i.e. $PH = P$.

We can define completeness for the versions Σ_i^P, Π_i^P, Δ_i^P as we did for NP-completeness.

Are there natural problems that are complete for Σ_i^P, Π_i^P ?

Quantified boolean formulae (QBF)

let X be a set of boolean variables partitioned into

$$X = X_1 \dot\cup \cdots \dot\cup X_i$$

and let F be a propositional formula over X.

Then $\Phi = \exists X_1 \forall X_2 \exists X_3 \cdots \underset{\text{either } \exists \text{ or } \forall}{Q} X_i\ F$ is a quantified boolean formula with i alternations of quantifiers (QBF_i)

Φ is satisfiable if:

- there is an assignment to the variables in X_1 s.t.
  for all — " — X_2
  there is — " — X_3 s.t.
  ⋮
  F is true

$QSAT_i = \{\Phi \mid \Phi$ is a QBF_i and Φ is satisfiable$\}$

Theorem: For all $i \geq 1$ $QSAT_i$ is Σ_i^P-complete.

Note: games where players alternate i moves can be encoded as a formula of QBF_i

Space and time bounded TMs

It turns out that all problems in PH can be solved by a TM that uses at most polynomial space

PSPACE = { L | L = $\mathcal{L}$(M) for some DTM M that uses at most space that is polynomial in its input }

Examples of PSPACE-complete problems

- universality of a regular expression
- emptiness of the intersection of n DFAs (n is part of the input)
- satisfiability of quantified boolean formulas, i.e. QSAT
- board games with a polynomially bounded number of moves (existence of a winning strategy)

We said that $QSAT_i \in \Sigma_i^p$-complete
and $QSAT \in$ PSPACE-complete

We can also define NPSPACE in an analogous way

NPSPACE = { L | L = $\mathcal{L}$(N) for some NTM N that uses at most polynomial space }

Relationship between PSPACE, NPSPACE and P, NP

Easy facts: $P \subseteq PSPACE$
$NP \subseteq NPSPACE$

Follows from the fact that a TM that does at most $P(n)$ steps cannot use more than $P(n)$ tape cells.

Note: in the definition of (N)PSPACE there is nothing that requires the TM to halt, so it could in principle contain non-recursive languages.

Actually this is not the case:

Theorem: If M is a (N)TM with space bound $p(n)$, and $w \in \mathcal{L}(M)$, the w is accepted within $c^{1+p(|w|)}$ steps, for some constant c.

Proof: Idea: M must repeat an ID before making more than $c^{1+p(|w|)}$ moves.

If M repeats an ID and accepts, then there is a shorter sequence of IDs leading to acceptance:

$$\underset{\text{initial ID}}{\alpha} \vdash^* \beta \vdash^* \beta \vdash^* \underset{\text{accepting ID}}{\gamma}$$

$$\Downarrow$$

$$\alpha \vdash^* \beta \vdash^* \gamma$$

Why must M repeat an ID?
- the number of symbols in each position is limited (Γ and Q are finite)
- the number of positions is limited by $p(|w|)$

Let $t = |\Gamma|$, $s = |Q|$, $n = |w|$

with only $p(n)$ tape cells, we have at most

$$K = \underbrace{s}_{\text{states}} \cdot \underbrace{p(n)}_{\text{positions of head}} \cdot \underbrace{t^{p(n)}}_{\text{symbols in the } p(n) \text{ positions}}$$ different IDs

Let $c = t + s$ and consider

$$(t+s)^{1+p(n)} = t^{1+p(n)} + (1+p(n)) \cdot s \cdot t^{p(n)} + \dots \geq K$$

□

This result gives us a way to convert a poly-space TM into one that does at most an exponential number of moves.

__Theorem:__ Let $L \in$ (N)PSPACE.

Then L is accepted by a poly-space bounded (N)TM that makes at most $c^{q(n)}$ steps, for some constant $c > 1$ and polynomial $q(n)$.

__Proof:__ By the previous theorem, we know that $L = \mathcal{L}(M)$ for some (N)TM M that runs in at most $c^{1+p(|w|)}$ steps.

Idea: We construct from M a 2-tape TM M_2 that has a counter in base c on tape 2, and stops at latest when the counter reaches $c^{1+p(|w|)}$

On tape 1, M_2 simulates M.

How much tape uses M_2:

- tape 1: at most $p(|w|)$ cells
- tape 2: at most $1 + p(|w|)$ cells (since the base-c counter counts at most to $c^{1+p(|w|)}$)

We can convert M_2 to a 1-tape TM M_3

- M_3 uses no more than $1+p(n)$ tape cells (for any input of length n)
- M_3 runs in time quadratic in the running time of M_2, i.e. $O(c^{2\cdot p(n)}) < c^{q(n)}$ for $q(n) = 2\cdot p(n) + d$.

□

PSPACE vs. NPSPACE:

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Obviously: PSPACE $\subseteq$ NPSPACE

Surprisingly: PSPACE = NPSPACE !

Idea of proof:

simulation of NTM N with space bound $p(n)$

by DTM D — " — $O(p(n)^2)$

Exploits a deterministic, recursive test for whether a NTM N can have $ID_i \underbrace{\vdash_N^*}_{\leq m \text{ steps}} ID_j$

DTM D tries all intermediate IDs searching for ID_k:

$$ID_i \underbrace{\vdash_N^*}_{\leq \frac{m}{2}} ID_k \underbrace{\vdash_N^*}_{\leq \frac{m}{2}} ID_j$$

We use a recursive boolean function reach(ID_i, ID_j, m):

$$\text{reach}(ID_i, ID_j, 1) = \begin{cases} \text{true, if } ID_i = ID_j \text{ or } ID_i \vdash_N ID_j \\ \text{false, otherwise} \end{cases}$$

$$\text{reach}(ID_i, ID_j, m) = \bigvee_{ID_k} \left(\text{reach}(ID_i, ID_k, \tfrac{m}{2}) \wedge \text{reach}(ID_k, ID_j, \tfrac{m}{2})\right) \quad \text{for } m > 1$$

Note: reach calls itself several times, i.e twice for each possible intermediate ID_k

But - after each ID_k we just need to remember a bit (and the counter used to iterate through ID_k)

- the two calls for the same ID_k can be done in sequence.

What is the depth of the recursive calls?

M does not make more than $c^{p(n)}$ moves

$\Rightarrow$ we can start with $m = c^{p(n)}$

$\Rightarrow$ The recursive call depth is $\leq \log_2 c^{p(n)} = O(p(n))$

Theorem (Savitch's Theorem) PSPACE = NPSPACE

Proof: we only need to show NPSPACE $\subseteq$ PSPACE,
i.e if $L = \mathcal{L}(M)$ for a NTM M with space bound $p(n)$
$L = \mathcal{L}(M)$ for a DTM D — " — $q(n)$
(with $p(n)$, $q(n)$ polynomials)

We can assume that M accepts in $\leq c^{1+p(n)}$ steps (for some constant c)

Given w with $|w| = n$, D repeatedly calls reach(ID_0, ID_j, m),
where - ID_0 is the initial ID of M with input w
- ID_j is an accepting ID using at most $p(n)$ cells
- $m = c^{1+p(n)}$

The depth of the recursive calls of reach is $\leq \log_2(m)$, i.e $O(p(n))$

D can manage its tape as a stack:

for each recursive call, it puts on the stack

ID_0 ... $1+p(n)$ cells

ID_j ... $1+p(n)$ cells

m ... in binary: $\log_2 c^{1+p(n)} = O(p(n))$

⇒ The total length of the tape is

$$O(p(n)\cdot(2+2\,p(n)+O(p(n)))=O(p(n)^2)$$

(Note: D also uses a scratch tape portion to enumerate through the various IDs) □

PSPACE - completeness

We can define PSPACE-hardness (and PSPACE-completeness) as done for NP.

Definition: A language L is PSPACE-hard if for every language $L' \in PSPACE$ we have that $L' <_{poly} L$.

L is PSPACE-complete if:

1) $L \in PSPACE$, and

2) L is PSPACE-hard

Note: the definition uses poly-time and not poly-space reductions, since we want to obtain similar properties as for NP-completeness.

Theorem: Let L be PSPACE-complete

1) If $L \in P$, then $P = PSPACE$
2) If $L \in NP$, then $NP = PSPACE$

Proof: We show (1). The proof for (2) is similar.

Consider a language $L' \in PSPACE$. We show that $L' \in P$.

Since L is PSPACE-complete, we have $L' <_{poly} L$.

Let the poly-time reduction be R and let R take time $q(n)$.

Since $L \in P$, it has a poly-time algorithm.

Let this poly-time algorithm run in time $p(n)$.

Consider a string w for which we want to test whether $w \in L$.

Then $w \in L'$ iff $R(w) \in L$.

Since R takes time $q(|w|)$, $|R(w)| \leq q(|w|)$.

We can test whether $R(w) \in L$ in time $p(|R(w)|) \leq p(q(|w|))$, i.e. polynomial in $|w|$.

Hence, we have a poly-time algorithm for L'.

We get that $PSPACE \subseteq P$. The other direction is obvious. □

We show now a problem that is PSPACE-complete.

Quantified Boolean Formulae (QBF)

We recall the definition of QBF and slightly modify it to make it more suitable for what we want to show here.

A QBF is a boolean expression in which additionally boolean variables may be quantified.

Formally, a QBF is defined inductively as follows:

- 0 (i.e. false) and 1 (i.e. true) are QBFs
- every variable is a QBF
- if E and F are QBFs, then so are

  $\neg E \quad , \quad E \wedge F \quad , \quad E \vee F$
- if F is a QBF that does not include a quantification of variable x, then so are

  $\forall x\,(E) \quad , \quad \exists x\,(E)$

  We say that the scope of x is E.

Note: - We may use parentheses to disambiguate
- For simplicity, we have chosen not to allow multiple quantifications over the same variable. This does not limit expressiveness, but is also not strictly necessary.

Example: $\forall x\,(\exists y\,(x \wedge y) \vee \forall z\,(\neg x \vee z))$

If a variable x is in the scope of $\forall x$ or $\exists x$, then it is said to be <u>bound</u>. Otherwise it is <u>free</u>.

The value of a QBF with no free variables is either 0 or 1.
We can compute the value $v(F)$ of such a QBF F by induction:

- base case: $F = 0$ or $F = 1$

  Then $v(F) = F$

  Note: we cannot have $F = x$, since x would be free
- inductive cases:

  - $F = \neg E$ then E is shorter than F, and we can evaluate it by induction. If $v(E) = 1$, then $v(F) = 0$.
    If $v(E) = 0$ then $v(F) = 1$.

- $F = E \wedge E'$ then both E and E' are shorter than F, and we can evaluate them.

  If $v(E)=1$ and $v(E')=1$, then $v(F)=1$
  otherwise $v(F)=0$.

- $F = E \vee E'$ similar

- $F = \forall x\,(E)$

  - let E_0 be obtained from E by replacing each x by 0
  - let E_1 — " — 1

  Note: E_0 and E_1 - have no free variables
  - are both shorter than F

  Hence we can evaluate E_0 and E_1.

  If $v(E_0)=1$ and $v(E_1)=1$, then $v(F)=1$
  otherwise $v(F)=0$

- $F = \exists x\,(E)$ similar

  ⋮

  If $v(E_0)=0$ and $v(E_1)=0$, then $v(F)=0$
  otherwise $v(F)=1$.

We define the QBF problem:

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$QBF = \{ F \mid F$ is a QBF formula without free variables and $v(F) = 1 \}$

We show now that QBF is PSPACE-complete.

Theorem: QBF $\in$ PSPACE

Proof: We exploit the recursive evaluation procedure, and show that it can be implemented by a TM M that uses only polynomial space.

M keeps on its tape a stack. Each record of the stack contains a formula and an index to the subformula that M is currently working on.

Let F be the formula to evaluate, and let $|F| = n$.

Initially, a record for F is placed on the stack.

If $F = E \wedge E'$, then M proceeds as follows:

1) Place E in a record to the right of the one for F.
2) Recursively evaluate E.
3) If $v(E) = 0$,

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then return $0$ as $v(F)$
else replace the record of $E$ by the one of $E'$
     recursively evaluate $E'$
     return $v(E')$ as $v(F)$
```

If $F = \exists x(E)$, then M proceeds as follows

1) Create E_0 by replacing each occurrence of x in E by 0, and place E_0 in a record to the right of the one for F.
2) Recursively evaluate E_0.
3) If $v(E_0) = 1$

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then return $1$ as $v(F)$
else create $E_1$ by substituting $1$ for $x$ in $E$
     replace the record of $E_0$ by the one of $E_1$
     recursively evaluate $E_1$
     return $v(E_1)$ as $v(F)$
```

The cases for $F = \neg E$, $F = E \vee E'$, $F = \forall x (E)$ are similar [Exercise]

When $F = 0$ or $F = 1$ then F is returned immediately, without creating a further record.

Note:

1) The records to the right of the one for a formula E are for formulas that are shorter than E
2) When two subexpressions have to be evaluated in the cases $E \wedge E'$, $E \vee E'$ and $\exists x (E)$, $\forall x (E)$ the two subexpressions are evaluated in sequence, and the two records for E, E' (resp. E_0, E_1) are never at the same time on the stack.

It follows that for $|F| = n$, there are at most n records on the stack, and each record has length $O(n)$.

Hence, the used tape is at most $O(n^2)$. □

To show PSPACE-hardness of QBF, we encode the computation of a poly-space TM M into a QBF.

Can we directly use variables X_{jtA} to encode that "the symbol in position j of configuration t is A" as done for encoding the computation of a poly-time TM in SAT?

No! Since the number of steps of M is exponential, and we would need exponentially many variables.

Idea: we use quantification, to let a variable represent many different configurations.

<u>Theorem</u>: QBF is PSPACE-hard

Consider a language $L \in$ PSPACE, and let M be a TM s.t. $\mathcal{L}(M) = L$, and let M use at most $p(n)$ space.
Let w be an input string for M with $|w| = n$.

We construct from M and w a QBF E without free variables and size polynomial in n s.t. $v(E) = 1$ iff $w \in \mathcal{L}(M)$.

By a previous theorem, we know that there is a constant c s.t. M accepts an input of length n in $c^{1+p(n)}$ steps.

We encode the computation using variables as follows:

- there is a constant number of configuration symbols:
  $\Rightarrow$ we can encode each explicitly through an index in the variables
- there is a polynomial number of tape cells:
  $\Rightarrow$ we can encode each explicitly through an index in the variables
- there are $c^{1+p(n)}$ IDs
  $\Rightarrow$ we do <u>not</u> encode all of them explicitly, rather we represent an IDs through variables over which we quantify. We call the set of variables y_{iA} representing an ID a "variable ID".

When I is the variable ID represented by $x_1, \dots, x_m$

we use $\exists I$ to denote $\exists x_1 \exists x_2 \cdots \exists x_m$
" " $\forall I$ " " $\forall x_1 \forall x_2 \cdots \forall x_m$

We construct a QBF of the form:

$$\exists I_0 \exists I_f (S \wedge F \wedge N)$$

where

- I_0 is a variable ID representing the initial ID
  I_f — " — accepting ID
- S says "starts right"
  i.e. I_0 is the initial ID of M with input w
- F says "finishes right"
  i.e. I_f is an accepting ID.
- N says "moves right"
  i.e. M moves from I_0 to I_f

Structure of S, F, N:

- starts right: S is the AND of literals using the variables of I_0
  - when the j-th position of the initial ID is A, then y_{jA}
  - — " — is not A, then $\overline{y_{jA}}$

  $\Rightarrow$ $|S|$ is linear in $p(n)$
- finishes right: F is the OR of the variables y_{jA} chosen from those of I_f for which A represents an accepting state, and j is arbitrary.
- moves right: N is based on the recursive splitting of the computation in halves, adding only $O(p(n))$ symbols for each split

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- For variable IDs I with variables y_{jA}

J ... z_{jA}

we use $I=J$ to abbreviate $\bigwedge_{j,A}((y_{jA} \wedge z_{jA}) \vee (\overline{y_{jA}} \wedge \overline{z_{jA}}))$

- We construct $N_i(I,J)$ for $i = 1, 2, 4, 8, 16, \ldots$

to mean $I \vdash_M^* J$ in at most i moves.

The only free variables of $N_i(I,J)$ are those of I, J.

Basis: $N_1(I,J)$ asserts either $I=J$

or $I \vdash_M J$

To encode $I \vdash_M J$, we can proceed as for the proof of Cook's Theorem.

Induction: we construct $N_{2i}(I,J)$ from N_i.

Note: we cannot use

$$N_{2i}(I,J) = \exists K\,(N_i(I,K) \wedge N_i(K,J))$$

since the overall formula would become exponentially long [Verify as an exercise].

Instead, we must use only one copy of N_i to check both $N_i(I,K)$ and $N_i(K,J)$, i.e.

there exists an ID K such that for all variable IDs P, Q:

$(P,Q) = (I,K)$ or $(P,Q) = (K,J)$

implies $N_i(P,Q)$ is true

$$\Rightarrow N_{2i}(I,J) = \exists K\,\forall P\,\forall Q\,(N_i(P,Q) \vee$$
$$(\neg(P=I \wedge Q=K) \wedge \neg(P=K \wedge Q=J))$$

Then $N = N_m(I_0, I_f)$, where m is the smallest power of 2 greater or equal to $c^{1+p(n)}$.

The number of recursive steps to determine N is

$$\log_2(c^{1+p(n)}) = O(p(n))$$

Each recursive step takes time $O(p(n))$.

$\Rightarrow$ N can be constructed in time $O(p(n)^2)$

One can verify that $\exists I_0 \exists I_f (S \wedge F \wedge N)$ has value 1 iff $w \in \mathcal{L}(M)$ $\square$

Further important time and space complexity classes:

EXPTIME = {L | L = $\mathcal{L}$(M) for some exptime DTM M}

EXPSPACE = {L | " – expspace – " }

kEXPTIME = {L | L = $\mathcal{L}$(M) for some DTM M with running time $T(n) = \underbrace{2^{\cdot^{\cdot^{2}}}}_{k \text{ times}}{}^{O(n)}$ }

kEXPSPACE = {L | L = $\mathcal{L}$(M) for some DTM M that, on input of length n, uses space that is at most $\underbrace{2^{\cdot^{\cdot^{2}}}}_{k \text{ times}}{}^{O(n)}$ }

We can define NkEXPTIME
NkEXPSPACE
as for the deterministic classes, but using NTMs instead of DTMs

We have:

kEXPTIME
$\cap$ ← ? open whether inclusion is strict
NkEXPTIME
$\cap$
kEXPSPACE = NkEXPSPACE ← The same proof as for Savitch's Theorem can be used.
$\cap$
(k+1)EXPTIME

Natural problems in these classes are logic related

Note: EXPTIME is the first provable intractable class, i.e. we know:

$\underbrace{P \subseteq NP \subseteq PSPACE \subseteq EXPTIME}_{\subsetneq}$ (we don't know which of these inclusions is strict)