PROPERTIES OF REGULAR LANGUAGES

03/12/2007

E5.1

Exercise 1

Decide which of the following statements is true and which is false. Give a brief explanation of your answer.

a) For all languages L_1 and L_2, it holds that $(L_1^* \cdot L_2^*)^+ = (L_1^+ \cdot L_2^+)^*$

b) If L_1 and L_2 are both non-regular then $L_1 \cup L_2$ could be regular

c) For all languages L_1 and L_2, if $L_1 \subseteq L_2$ then $L_1^* \subseteq L_2^*$.

Solution:

a) False. Consider the languages $L_1 = \{a\}$ and $L_2 = \{b\}$. Then $b \in (L_1^* \cdot L_2^*)^+$ but $b \notin (L_1^+ \cdot L_2^+)^*$.

b) True. Assume that $L_2 = \overline{L_1}$, i.e. $L_1 = \overline{L_2}$. If L_1 is non-regular then so is L_2 because, if L_2 would be regular then, by the closure properties of regular languages, $L_1 = \overline{L_2}$ would be regular too, thus leading to a contradiction. Since $L_1 \cup L_2 = L_1 \cup \overline{L_1} = \overline{L_2} \cup L_2 = \Sigma^*$ we have that the union of two non-regular languages can be regular.

c) True. Given that for all $w \in L_1$ we also have that $w \in L_2$, the argument goes as follows. If $w' \in L_1^*$ then $w' = w_1 w_2 \cdots w_n$ for some $n \in \mathbb{N}$ and $w_i \in L_1$ $(1 \le i \le n)$. But then each w_i is also in L_2 ($w_i \in L_2$ for $1 \le i \le n$) and therefore $w' \in L_2^*$.

Exercise 2

E5.2

Show that the language

$L = \{0^n 1^m 0^{n+m} \mid m, n \geq 0\}$

is not regular.

Solution:

Assume that L is regular.

Then, by the pumping lemma, we would have that:

> there exists n such that for all $w \in L$ such that $|w| \geq n$ there are three strings x, y, z such that $w = xyz$, $|xy| \leq n$, $|y| > 1$, and for all $k > 0$, $xy^k z \in L$.

Now, given some n, let $w = 0^n 1^n 0^{2n}$.

$$w = \underbrace{0\cdots0}_{n}\underbrace{1\cdots1}_{n}\underbrace{0\cdots0}_{2n}$$

Since $|w| = 4n$ we have that $|w| > n$.

In order to apply the pumping lemma we need to find strings x and y such that $|xy| \leq n$. The only possible choices are: $x = 0^a$ and $y = 0^b$ where $b \geq 1$. But then we have that $xz = 0^{n-b} 1^n 0^{2n}$ and thus that $n + n - b \neq 2n$. Therefore, for $k = 0$, $xy^k z \notin L$. Since we assumed that L is regular this is a contradiction. Hence L cannot be regular.

Exercise 3

Show that the language
$L = \{ w \in \{0,1\}^* \mid w \text{ is a palindrome} \}$
is not regular.

[A string w is a palindrome if $w = w^R$ where $(-)^R$ denotes string reversal.]

Solution:

Again, we use the pumping lemma.
Given some n, let $w = 0^n 1 0^n$.
If we consider x, y, z such that

a) $w = xyz$

b) $|xy| \leq n$

c) $|y| \geq 1$

then y can only be a non-empty string of 0's.
Thus, for each $k > 1$, the string xy^kz has more 0's in the left-hand side than in the right-hand side. We conclude that, for $k > 1$, $xy^kz \notin L$.
Therefore we have that L is not regular.

Exercise 4 (4.3.3 from textbook)

Give an algorithm to tell whether a regular language L is universal (i.e. $L = \Sigma^*$?).

Solution:

If L is universal then $\bar{L} = \Sigma^* - L = \emptyset$. Therefore we only need to check whether $\bar{L}$ is empty.

Exercise 5 (4.3.4 from textbook)

Give an algorithm to tell whether two regular languages have at least one string in common.

Solution:

We can check whether the intersection L of the two languages that we denote with L_1 and L_2 is non-empty.

$$L = L_1 \cap L_2 = \overline{\bar{L_1} \cup \bar{L_2}}$$

An automaton accepting L can be easily constructed from automata accepting L_1 and L_2. Note that all automata need to be deterministic, otherwise the complement of a language might not be accepted.