Free University of Bozen-Bolzano – Faculty of Computer Science
Master of Science in Computer Science
Theory of Computing – A.A. 2004/2005

Final exam – 7/6/2005 – Part 1

Solutions

Problem 1.1 [4.5 points] Decide which of the following statements is TRUE and which is FALSE. You must give a brief explanation of your answer to receive full credit.

(a) For all languages L_1 and L_2, it holds that $L_1^* \cap L_2^* = (L_1 \cap L_2)^*$.

(b) If L_1 is regular and L_2 is non-regular, then $L_1 \cup L_2$ must be non-regular.

(c) There exists a language L such that L is not regular but L^* is regular.

Solution:

(a) FALSE. Consider for example $L_1 = \{a\}$ and $L_2 = \{aa\}$. Then $L_1^* \cap L_2^* = (aa)^*$, but $L_1 \cap L_2 = \emptyset$, and hence $(L_1 \cap L_2)^* = \varepsilon$.

(b) FALSE. Consider for example $L_1 = \Sigma^*$ and L_2 some arbitrary language over Σ. Then $L_1 \cup L_2 = \Sigma^*$, which is regular.

(c) TRUE. Consider $L = \{a\} \cup \{a^n \mid n \text{ is prime }\}$. Then L is not regular. On the other hand, $L^* = a^*$, since already $a \in L$ generates in L^* strings of arbitrary length of a's. Hence L^* is regular.

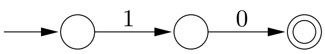
Problem 1.2 [1.5 points] Show that $L^* = L \cdot L^*$ if and only if $\varepsilon \in L$.

Solution: Let $L^+ = L \cdot L^*$. By definition, $L^* = \bigcup_{n\geq 0} L^n$, and $L^+ = L \cdot L^* = \bigcup_{n\geq 1} L^n$. Hence, $L^+ \subseteq L^*$, and L^+ and L^* might differ only due to $L^0 = \varepsilon$. If $\varepsilon \in L$, then also $\varepsilon \in L^+$, and hence $L^+ = L^*$. If $\varepsilon \notin L$, then also $\varepsilon \notin L^+$, while $\varepsilon \in L^*$, and hence $L^+ \neq L^*$.

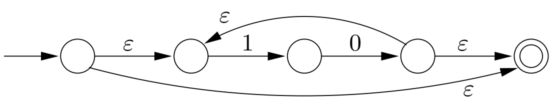
Problem 1.3 [6 points] Consider the regular expression $E = ((1 \cdot 0)^* \cdot 0)^* + (1 \cdot 1)$. Construct an ε-NFA A_ε such that $\mathcal{L}(A_\varepsilon) = \mathcal{L}(E)$. Simplify intermediate results whenever possible. Then, by eliminating ε-transitions from A_ε, construct an NFA A such that $\mathcal{L}(A) = \mathcal{L}(A_\varepsilon)$. Illustrate the steps of the algorithm you have followed to construct A_ε and A.

Solution: Construction of A_ε:

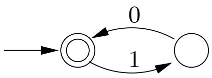
$1 \cdot 0$



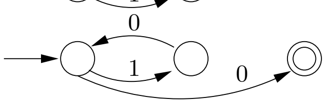
$(1 \cdot 0)^*$



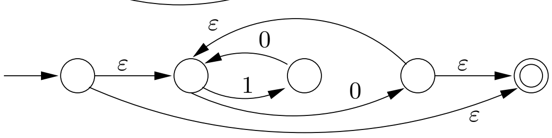
$(1 \cdot 0)^*$ simplified



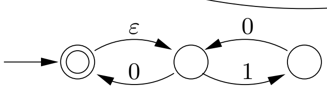
$(1 \cdot 0)^* \cdot 0$



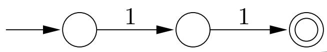
$((1 \cdot 0)^* \cdot 0)^*$



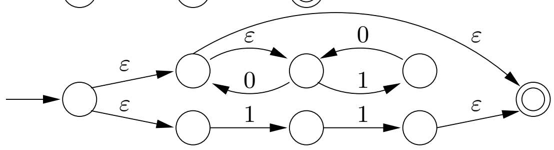
$((1 \cdot 0)^* \cdot 0)^*$ simplified



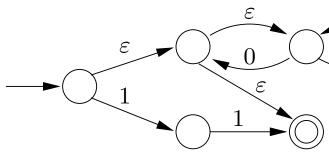
$1 \cdot 1$



$((1 \cdot 0)^* \cdot 0)^* + (1 \cdot 1)$



$((1 \cdot 0)^* \cdot 0)^* + (1 \cdot 1)$ simplified



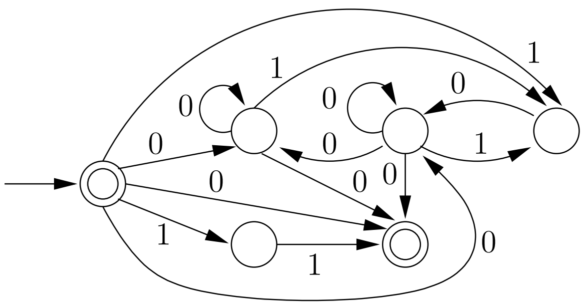
Construction of A: Let $A_\varepsilon = (Q, \Sigma, \delta_\varepsilon, q_0, F_\varepsilon)$. Then $A = (Q, \Sigma, \delta, q_0, F)$, where

$$\delta(q, a) = Eclose\left(\bigcup_{p_i \in Eclose(q)} \delta_\varepsilon(p_i, a) \right), \quad \text{for each } q \in Q \text{ and } a \in \Sigma,$$

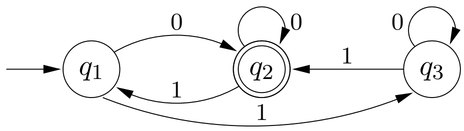
and

$$F = \begin{cases} F_\varepsilon, & \text{if } \varepsilon \notin \mathcal{L}(A_\varepsilon) \\ F_\varepsilon \cup \{q_0\}, & \text{if } \varepsilon \in \mathcal{L}(A_\varepsilon) \end{cases}$$

In our case, we obtain:



Problem 1.4 [6 points] Consider the following DFA A over $\{0, 1\}$:



Construct a regular expression E such that $\mathcal{L}(E) = \mathcal{L}(A)$. Illustrate the steps of the algorithm you have followed to construct E.

Solution: We construct the regular expressions E_{ij}^k inductively. For the inductive case, we use the following rule:

$$E_{ij}^k = E_{ij}^{k-1} + E_{ik}^{k-1} \cdot (E_{kk}^{k-1})^* \cdot E_{kj}^{k-1}$$

The expressions E_{ij}^k are as follows (we have specified only those that are necessary to construct E):

k	E_{11}^k	E_{12}^k	E_{13}^k	E_{21}^k	E_{22}^k	E_{23}^k	E_{31}^k	E_{32}^k	E_{33}^k
0	ε	0	1	1	0	$\emptyset$	$\emptyset$	1	0
1	ε	0	0	1	$0+10$	11	$\emptyset$	1	0
2	–	$0+0(0{+}10)^*(0{+}10)$	$0+0(0{+}10)^*11$	–	–	–	–	$1+1(0{+}10)^*(0{+}10)$	$0+1(0{+}10)^*0$
3	–	$E_{12}^2 + E_{13}^2 \cdot (E_{33}^2)^* \cdot E_{32}^2$	–	–	–	–	–	–	–

Then, $E = E_{12}^3 = (0{+}0(0{+}10)^*(0{+}10)){+}(0{+}0(0{+}10)^*11) \cdot (0{+}1(0{+}10)^*0)^* \cdot (1{+}1(0{+}10)^*(0{+}10))$.

Problem 1.5 [5 points] The *quotient* L_1/L_2 of two languages L_1 and L_2 is defined as

$$L_1/L_2 = \{x \mid \text{ there is } y \in L_2 \text{ such that } xy \in L_1\}$$

For example, if $L_1 = \{w \in \{0,1\}^* \mid w \text{ has an even number of 0's}\}$, $L_2 = \{0\}$, and $L_3 = \{0, 00\}$, then $L_1/L_2 = \{w \in \{0,1\}^* \mid w \text{ has an odd number of 0's}\}$, and $L_1/L_3 = \{0,1\}^*$.

Show that, for an *arbitrary* language L_2, if L_1 is regular, then L_1/L_2 is also regular.

[*Hint*: Start from a DFA A for L_1, and show how to modify the set of final states of A to obtain a DFA for L_1/L_2.]

Solution: Since L_1 is regular, there is a DFA $A = (Q, \Sigma, \delta, q_0, F)$ such that $\mathcal{L}(A) = L_1$. We show that the DFA $A' = (Q, \Sigma, \delta, q_0, F')$, with $F' = \{q \in Q \mid \text{ there is } y \in L_2 \text{ s.t. } \hat{\delta}(q, y) \in F\}$, accepts L_1/L_2. This proves that L_1/L_2 is regular.

Consider a word $x \in L_1/L_2$. By definition of L_1/L_2, there is a word $y \in L_2$ such that $xy \in L_1$. Since $xy \in L_1$, we have that $\hat{\delta}(q_0, xy) \in F$, and moreover there is a state $q \in Q$ such that $\hat{\delta}(q_0, x) = q$. We have that $\hat{\delta}(q, y) = \hat{\delta}(\hat{\delta}(q_0, x), y) = \hat{\delta}(q_0, xy)$. Hence $\hat{\delta}(q, y) \in F$, and since $y \in L_2$, it follows that $q \in F'$, and therefore $x \in \mathcal{L}(A')$.

Conversely, consider a word $x \in \mathcal{L}(A')$. By definition of language accepted by a DFA, we have that $q = \hat{\delta}(q_0, x) \in F'$, and by definition of F', there is a word $y \in L_2$ such that $\hat{\delta}(q, y) \in F$. Hence, $\hat{\delta}(q_0, xy) = \hat{\delta}(\hat{\delta}(q_0, x), y) = \hat{\delta}(q, y) \in F$, and it follows that $xy \in \mathcal{L}(A) = L_1$. By definition of L_1/L_2, we have that $x \in L_1/L_2$.

Note that in the above proof we did not make use of any assumption on L_2, which in fact may be an arbitrary language (possibly non-regular, or even non-recursively enumerable).

Problem 1.6 [3 points] Show that the language $\{uawb \mid u, w \in \{a,b\}^*, \text{ with } |u| = |w|\}$ is context free by exhibiting a context free grammar that generates it.

Solution: A grammar that generates the language $\{uawb \mid u, w \in \{a,b\}^*, \text{ with } |u| = |w|\}$ is $G = (\{S, T, X\}, \{a, b\}, P, S)$, where P consists of the following productions:

$$\begin{array}{rcl} S & \longrightarrow & Tb \\ T & \longrightarrow & a \mid XTX \\ X & \longrightarrow & a \mid b \end{array}$$

Problem 1.7 [4 points] Consider the grammar $G = (\{S, T\}, \{0, 1\}, P, S)$, where P consists of the following productions

$$\begin{array}{rcl} S & \longrightarrow & 0S \mid 1T \mid 0 \\ T & \longrightarrow & 1T \mid 1 \end{array}$$

Show that no string in the language $\mathcal{L}(G)$ contains the substring 10.

Solution: Let α be a sentential form (i.e., a sequence of terminals and non-terminals) derived using G. We show by induction on the length n of the derivation of α that α does not contain any of the substrings 10, $1S$, $S0$, SS, $T0$, or TS. The claim then follows.

Base case: $n = 1$. The possible derivations of length 1 of sentential forms for G are $S \Rightarrow 0S$, $S \Rightarrow 1T$, and $S \Rightarrow 0$. Neither $0S$, nor $1T$, nor 0 contain as substring 10, $1S$, $S0$, SS, $T0$, or TS.

Inductive case: Let $S \stackrel{*}{\Rightarrow} \alpha \Rightarrow \beta$ be a derivation of β of length $n+1$. By inductive hypothesis, α does not contain as substring 10, $1S$, $S0$, SS, $T0$, or TS.

- If the last derivation step uses $S \longrightarrow 0S$, since α does not contain $1S$ or TS, the derivation step cannot introduce 10 or $T0$ in β.

- If the last derivation step uses $S \longrightarrow 1T$, since α does not contain $S0$ or SS, the derivation step cannot introduce $T0$ or TS in β.
- If the last derivation step uses $S \longrightarrow 0$, since α does not contain $1S$, SS, or TS, the derivation step cannot introduce 10, $S0$, or $T0$ in β.
- If the last derivation step uses $T \longrightarrow 1T$, since α does not contain $T0$ or TS, the derivation step cannot introduce $T0$ or TS in β.
- If the last derivation step uses $T \longrightarrow 1$, since α does not contain $T0$ or TS, the derivation step cannot introduce 10 or $1S$ in β.