Modelling Combinatorial Auctions in Linear Logic

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Overview

- Linear logic, some general features, proof-search complexity,
- A model for (multi-unit) combinatorial auctions,
- Modelling agents bids as formulas of linear logic,
- Modelling allocations of goods to bidders as proofs in linear logic,
- Adequacy of the notion of proof to capture allocations,
- Further applications, and some conclusions.

Linear Logic: resource-sensitive account of proofs

(Girard, 1987)
In classical logic sequent calculus, *structural rules* of contraction and weakening define how to deal with hypotheses in a proof:

$$\frac{\Gamma, A, A, \vdash \Delta}{\Gamma, A \vdash \Delta}\ (C) \qquad \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\ (W)$$

W and C determine the behaviour of logical connectives, in particular they make the following two presentations of logical rules are equivalent:

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\ \wedge \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge$$

(*multiplicative and additive presentation*)

- Rejecting structural rules, we are lead to define two conjunctions with different behavior: copying contexts ($\otimes$) or identifying them ($\&$). Linear logic rejects the global validity of structural rules providing a resource sensitive account of proofs.

Example: meaning of linear logic connectives

Meaning of linear logic connectives:

- Price: 27 euros,
- Appetizer: Prosciutto e melone/fichi (depending on season)
- Primo: Spaghetti/Gnocchi,
- Drink: Water (as much as you like)

$$Pz \multimap ((P \otimes M) \oplus (P \otimes F)) \otimes (S \,\&\, G) \otimes !W$$

$A \multimap B$: consuming one A, you get one B;
$A \otimes B$: you have one copy of A and one of B. E.g. $A \otimes B \nvdash A$: in order to sell A and B, we need someone who buys both A and B.
$A \oplus B$: you have one of the two but you cannot chose;
$A \,\&\, B$: you have one of the two and you can chose;
$!A$: use A *ad libitum* (! reintroduce structural rules)

Horn sequents for LL, (Kanovich, 1994)

Proof-search complexity results:

- MLL ($\otimes$, $\parr$): NP-complete (and so is MLL with full weakening (W)) (Lincoln, 1995).
- IMLL, IMLL with (W): intuitionistic versions (single-conclusion sequents): NP-complete
- MALL ($\otimes$, $\parr$, &, $\oplus$) and IMALL are PSPACE-complete (Lincoln et al., 1992).

We will use Horn sequents (HLL) , i.e. sequents must be of the form $X, \Gamma \vdash Y$ where X and Y are tensors of positive atoms, and Γ is one of the following (with X_i, Y_i being tensors of positive atoms):

(*i*) $\otimes$-Horn implications: $(X_1 \multimap Y_1) \otimes \cdots \otimes (X_n \multimap Y_n)$

(*ii*) &-Horn implications: $(X_1 \multimap Y_1) \mathbin{\&} \cdots \mathbin{\&} (X_n \multimap Y_n)$

HLL is NP-complete, and so is HLL + W (Kanovich, 1994)

A model of (multi-unit) combinatorial auctions

- An auctioneer wants to sell elements of a finite multiset of goods $\mathcal{M}$ (with finite multiplicity) to a group of bidders.
- We define *Atoms* $\mathcal{A} = \{p_1, \ldots, p_m\}$ as the elements of $\mathcal{M}$ ignoring their multiplicity. Then, multisets of goods can be defined as tensor formulas. E.g. $p \otimes p \otimes q$.
- Bids: $\langle B_i, w_i \rangle$, with $B_i \subseteq \mathcal{M}$ and a price w_i.
- Bids generate valuations: $v_{\langle B,w \rangle} : \mathcal{P}(\mathcal{M}) \longrightarrow W$, $v_{\langle B,w \rangle}(X) = w$ if $B \subseteq X$, $v_{(B,w)}(X) = 0$ otherwise.
- An allocation $\mathcal{A}$ is a function associating goods to bidders.
- The value of an allocation (here) is given by the sum of the satisfied bids.
- Winner determination problem: finding an allocation that maximizes the revenue.

Bids as LL formulas

We model atomic bids as formulas of the form:

$$B \multimap u^k$$

$$(\textit{if you give me } B, \textit{ I give you } u^k)$$

where B is a tensor product of atoms in $\mathcal{A}$ and u^k is used to model prices symbolically: prices are tensors of a given unit symbol u: $u^k = u \otimes \underbrace{\ldots}_{k-times} \otimes u$

The agreement between the auctioneer and a bidder is interpreted as *modus ponens*:

- Non-free disposal (a bidder is willing to obtain exactly what she demands): (LL)

$$\underbrace{p, q, r}_{goods}, \underbrace{p \otimes q \otimes r \multimap u^k}_{bid} \vdash u^k$$

- Free disposal (a bidder is willing to obtain at least what she demands): (W)

$$\underbrace{p, q, r, s, t}_{goods}, \underbrace{p \otimes q \otimes r \multimap u^k}_{bid} \vdash_W u^k$$

Complex bids: three bidding languages

We can adapt three well known bidding languages for single-unit case to the multi-unit case: XOR, OR (Nisan 2006) and K-additive languages (Chevaleyre et al., 2008). All these languages can be defined using Horn sequents:

- XOR bids: *a bidder would like to get at most one of the bundles she specifies, for the associated price*

$$(B_1 \multimap w_1) \,\&\, \ldots \,\&\, (B_\ell \multimap w_\ell),$$

- OR-bids: *a bidder would pay the sum of the corresponding w_i for each bundle of goods B_i she gets*

$$(B_1 \multimap w_1) \otimes \cdots \otimes (B_\ell \multimap w_\ell),$$

- k-additive bids: *bidders specify weights for the marginal valuations derived from sets of goods*

$$(B_1 \multimap B_1 \otimes w_1) \otimes \cdots \otimes (B_\ell \multimap B_\ell \otimes w_\ell),$$

(*Remark*: a bundle of goods B is available for satisfying different bids)

Valuations defined by formulas

The following definition of valuation induced by bids applies to atomic bids as well as to the more powerful bidding languages:
Every bid formula BID generates a valuation v_{BID} mapping multisets $X \subseteq \mathcal{M}$ to prices:

$$v_{\mathrm{BID}}(X) = \max\{k \mid X, \mathrm{BID} \vdash u^k\}$$

(*the value of is given by the maximal k we can prove using the bid and the multiset of goods X*)

- Simple additive valuation can be expressed in the OR language via:

$$\bigotimes_{i \in \{1,\ldots,m\}} [\underbrace{(p_i \multimap u) \otimes \cdots \otimes (p_i \multimap u)}_{\mathcal{M}(p_i) \text{ times}}]$$

- Simple unit demand valuation, can be expressed in the XOR language via:

$$(p_1 \multimap u) \,\&\, \cdots \,\&\, (p_m \multimap u)$$

Allocation as proof search

There are many logical languages modelling bids, however linear logic can also model procedural aspects: we show how we can model *allocations* as proofs in linear logic:

Firstly, we define bids with indexed resources: e.g. bidder's i bid are expressed as $p^i \otimes q^i \multimap u^k$.
We specify which bidder gets which good by means of the following formula:

$$\mathrm{MAP} := \bigotimes_{p \in \mathcal{A}} [\&_{j \in \mathcal{N}} (p \multimap p^j)]^{\mathcal{M}(p)}$$

(*for each good, the formula choses an agent j who gets it*)

Define an *allocation sequent* for goods $p_1, \ldots, p_m$, bids $\mathrm{BID}_1, \ldots, \mathrm{BID}_n$, and revenue k as the following HLL sequent:

$$\underbrace{p_1, \ldots, p_m}_{goods}, \underbrace{\bigotimes_{p \in \mathcal{A}} [\&_{j \in \mathcal{N}} (p \multimap p^j)]^{\mathcal{M}(p)}}_{who\ gets\ what}, \underbrace{\mathrm{BID}_1, \ldots, \mathrm{BID}_n}_{bids} \vdash \underbrace{u^k}_{revenue}$$

(*Given goods $p_1, \ldots, p_m$ and bids $\mathrm{BID}_1, \ldots, \mathrm{BID}_n$, we can prove the revenue u^k*)

Example

Goods owned by the auctioneer: p, q, r, s.
Bidders: 1,2.
Bids:
$(p^1 \multimap u^3) \otimes (q^1 \multimap u^2)$ (OR bid);
$(r^2 \multimap u^2) \,\&\, (q^2 \multimap u^2)$ (XOR bid)

Question: can we achieve revenue u^7?

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Question: can we achieve revenue u^7?

i.e. can we prove

$$p, q, r, s, \mathrm{MAP}, (p^1 \multimap u^3) \otimes (q^1 \multimap u^2), (r^2 \multimap u^2) \,\&\, (q \multimap u^2) \vdash u^7$$

?

$$p, q, r, s, \mathrm{MAP}, (p^1 \multimap u^3) \otimes (q^1 \multimap u^2), (r^2 \multimap u^2) \,\&\, (q \multimap u^2) \vdash u^7$$

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$$\dfrac{\dfrac{\dfrac{p \vdash p \qquad p^1 \vdash p^1}{p, p \multimap p^1 \vdash p^1} \qquad u^3 \vdash u^3}{p, p \multimap p^1, p^1 \multimap u^3 \vdash u^3} \qquad \dfrac{\dfrac{q \vdash q \qquad q^1 \vdash q^1}{q, q \multimap q^1 \vdash q^1} \qquad u^2 \vdash u^2}{q, q \multimap q^1, q^1 \multimap u^2 \vdash u^2}}{p, q, p \multimap p^1, q \multimap q^1, \underbrace{(p^1 \multimap u^3) \otimes (q^1 \multimap u^2)}_{OR\ bid} \vdash u^3 \otimes u^2} \otimes$$

$$p, q, r, s, \mathrm{MAP}, (p^1 \multimap u^3) \otimes (q^1 \multimap u^2), (r^2 \multimap u^2) \,\&\, (q \multimap u^2) \vdash u^7$$

$$\frac{\dfrac{\dfrac{\dfrac{p \vdash p \qquad p^1 \vdash p^1}{p, p \multimap p^1 \vdash p^1} \qquad u^3 \vdash u^3}{p, p \multimap p^1, p^1 \multimap u^3 \vdash u^3} \qquad \dfrac{\dfrac{q \vdash q \qquad q^1 \vdash q^1}{q, q \multimap q^1 \vdash q^1} \qquad u^2 \vdash u^2}{q, q \multimap q^1, q^1 \multimap u^2 \vdash u^2}}{p, q, p \multimap p^1, q \multimap q^1, \underbrace{(p^1 \multimap u^3) \otimes (q^1 \multimap u^2)}_{OR\ bid} \vdash u^3 \otimes u^2}\otimes}{}$$

$$\frac{\dfrac{\dfrac{r \vdash r \qquad r^2 \vdash r^2}{r, r \multimap r^2 \vdash r^2} \qquad u^2 \vdash u^2}{r, r \multimap r^2, r^2 \multimap u^2 \vdash u^2}}{r, r \multimap r^2, \underbrace{(r^2 \multimap u^2) \,\&\, (q \multimap u^2)}_{XOR\ bid} \vdash u^2}\,\&$$

$$p, q, r, s, \mathrm{MAP}, (p^1 \multimap u^3) \otimes (q^1 \multimap u^2), (r^2 \multimap u^2) \,\&\, (q \multimap u^2) \vdash u^7$$

$$\dfrac{\dfrac{\dfrac{p \vdash p \qquad p^1 \vdash p^1}{p, p \multimap p^1 \vdash p^1} \qquad u^3 \vdash u^3}{p, p \multimap p^1, p^1 \multimap u^3 \vdash u^3} \qquad \dfrac{\dfrac{q \vdash q \qquad q^1 \vdash q^1}{q, q \multimap q^1 \vdash q^1} \qquad u^2 \vdash u^2}{q, q \multimap q^1, q^1 \multimap u^2 \vdash u^2}}{p, q, p \multimap p^1, q \multimap q^1, \underbrace{(p^1 \multimap u^3) \otimes (q^1 \multimap u^2)}_{\textit{OR bid}} \vdash u^3 \otimes u^2}\otimes \qquad \dfrac{\dfrac{\dfrac{r \vdash r \qquad r^2 \vdash r^2}{r, r \multimap r^2 \vdash r^2} \qquad u^2 \vdash u^2}{r, r \multimap r^2, r^2 \multimap u^2 \vdash u^2}}{r, r \multimap r^2, \underbrace{(r^2 \multimap u^2) \,\&\, (q \multimap u^2)}_{\textit{XOR bid}} \vdash u^2}\&$$

$$\dfrac{\cdots}{\underbrace{p, q, r}_{\textit{Goods actually used}}, \underbrace{p \multimap p^1, q \multimap q^1, r \multimap r^2}_{\textit{who gets what}}, \underbrace{(p^1 \multimap u^3) \otimes (q^1 \multimap u^2), (r^2 \multimap u^2) \,\&\, (q \multimap u^2)}_{\textit{bids}} \vdash \underbrace{u^3 \otimes u^2 \otimes u^2}_{\textit{revenue}}}$$

$$p, q, r, s, \textsc{map}, (p^1 \multimap u^3) \otimes (q^1 \multimap u^2), (r^2 \multimap u^2) \,\&\, (q \multimap u^2) \vdash u^7$$

$$\dfrac{\dfrac{\dfrac{p \vdash p \qquad p^1 \vdash p^1}{p, p \multimap p^1 \vdash p^1} \qquad u^3 \vdash u^3}{p, p \multimap p^1, p^1 \multimap u^3 \vdash u^3} \qquad \dfrac{\dfrac{q \vdash q \qquad q^1 \vdash q^1}{q, q \multimap q^1 \vdash q^1} \qquad u^2 \vdash u^2}{q, q \multimap q^1, q^1 \multimap u^2 \vdash u^2}}{p, q, p \multimap p^1, q \multimap q^1, \underbrace{(p^1 \multimap u^3) \otimes (q^1 \multimap u^2)}_{\textit{OR bid}} \vdash u^3 \otimes u^2} \otimes \qquad \dfrac{\dfrac{\dfrac{r \vdash r \qquad r^2 \vdash r^2}{r, r \multimap r^2 \vdash r^2} \qquad u^2 \vdash u^2}{r, r \multimap r^2, r^2 \multimap u^2 \vdash u^2}}{r, r \multimap r^2, \underbrace{(r^2 \multimap u^2) \,\&\, (q \multimap u^2)}_{\textit{XOR bid}} \vdash u^2} \,\&$$

$$\dfrac{\cdots}{\underbrace{p, q, r}_{\textit{Goods actually used}}, \underbrace{p \multimap p^1, q \multimap q^1, r \multimap r^2}_{\textit{who gets what}}, \underbrace{(p^1 \multimap u^3) \otimes (q^1 \multimap u^2), (r^2 \multimap u^2) \,\&\, (q \multimap u^2)}_{\textit{bids}} \vdash \underbrace{u^3 \otimes u^2 \otimes u^2}_{\textit{revenue}}}$$

Then, we build up the $\textsc{map}$ formula:

$$\dfrac{}{\underbrace{p, q, r}_{\textit{Goods actually used}}, \underbrace{\textsc{map}}_{\textit{who gets what}}, \underbrace{(p^1 \multimap u^3) \otimes (q^1 \multimap u^2), (r^2 \multimap u^2) \,\&\, (q \multimap u^2)}_{\textit{bids}} \vdash \underbrace{u^3 \otimes u^2 \otimes u^2}_{\textit{revenue}}} \,\&\, L$$

$$p, q, r, s, \mathrm{MAP}, (p^1 \multimap u^3) \otimes (q^1 \multimap u^2), (r^2 \multimap u^2) \mathbin{\&} (q \multimap u^2) \vdash u^7$$

$$\dfrac{\dfrac{\dfrac{p \vdash p \qquad p^1 \vdash p^1}{p, p \multimap p^1 \vdash p^1} \qquad u^3 \vdash u^3}{p, p \multimap p^1, p^1 \multimap u^3 \vdash u^3} \qquad \dfrac{\dfrac{q \vdash q \qquad q^1 \vdash q^1}{q, q \multimap q^1 \vdash q^1} \qquad u^2 \vdash u^2}{q, q \multimap q^1, q^1 \multimap u^2 \vdash u^2}}{p, q, p \multimap p^1, q \multimap q^1, \underbrace{(p^1 \multimap u^3) \otimes (q^1 \multimap u^2)}_{OR\ bid} \vdash u^3 \otimes u^2} \otimes \qquad \dfrac{\dfrac{\dfrac{r \vdash r \qquad r^2 \vdash r^2}{r, r \multimap r^2 \vdash r^2} \qquad u^2 \vdash u^2}{r, r \multimap r^2, r^2 \multimap u^2 \vdash u^2}}{r, r \multimap r^2, \underbrace{(r^2 \multimap u^2) \mathbin{\&} (q \multimap u^2)}_{XOR\ bid} \vdash u^2} \mathbin{\&}$$

$$\dfrac{\ldots}{\underbrace{p, q, r}_{Goods\ actually\ used}, \underbrace{p \multimap p^1, q \multimap q^1, r \multimap r^2}_{who\ gets\ what}, \underbrace{(p^1 \multimap u^3) \otimes (q^1 \multimap u^2), (r^2 \multimap u^2) \mathbin{\&} (q \multimap u^2)}_{bids} \vdash \underbrace{u^3 \otimes u^2 \otimes u^2}_{revenue}}$$

Then, we build up the MAP formula:

$$\dfrac{}{\underbrace{p, q, r}_{Goods\ actually\ used}, \underbrace{\mathrm{MAP}}_{who\ gets\ what}, \underbrace{(p^1 \multimap u^3) \otimes (q^1 \multimap u^2), (r^2 \multimap u^2) \mathbin{\&} (q \multimap u^2)}_{bids} \vdash \underbrace{u^3 \otimes u^2 \otimes u^2}_{revenue}} \mathbin{\&} \mathrm{L}$$

Unallocated goods are obtained by weakening:

$$\dfrac{}{\underbrace{p, q, r}_{Goods\ actually\ used}, s, \underbrace{\mathrm{MAP}}_{who\ gets\ what}, \underbrace{(p^1 \multimap u^3) \otimes (q^1 \multimap u^2), (r^2 \multimap u^2) \mathbin{\&} (q \multimap u^2)}_{bids} \vdash \underbrace{u^3 \otimes u^2 \otimes u^2}_{revenue}} \text{Weakening}$$

Allocations as proofs:

We can prove the following result:

Proposition

Given n bids in any of the bidding languages introduced (XOR, OR, k-add.), every allocation α with revenue k provides a proof π of an allocation sequent for k, and vice versa, every proof π of an allocation sequent for k provides an allocation α with revenue k.

Remark on Complexity:

- For the three languages presented (OR, XOR, K-add.), checking whether revenue k is attainable is in NP (proof-search for HLL).
- So our form of modelling the problem does not increase complexity with respect to the standard approach (Cramton, Shoham, Steinberg, *Combinatorial Auctions*).
- Of course, the Proposition only provides a method for solving the *decision variant* of the WDP.

Some conclusions

- We saw how fragments of linear logic can be used to model in a principled general framework several bidding languages for multi-unit combinatorial auctions.
  Moreover, the allocation procedure can be modelled within our framework as a proof of a particular sequent, so our logical model takes into account procedural aspects.
- Further applications of our model:
  - Mixed auctions : agents trade transformations of goods (or capabilities), negotiation.
  - Language expressivity : we can specify different kinds of goods: sharable vs. non-sharable (between agents), reusable vs. non-reusable (by a single agents. E.g. menu example $!W$).
  - Formula auctions : agents trade general formulas (generalization of bids).
- Further works: succintness of the languages. We can study different aggregators in the general framework of resource allocation, e.g. egalitarian aggregation and fair division.
  Properties of allocations as properties of proofs (qualitative analysis of allocations, importing techniques from proof theory of LL).