1. Basics of First-Order Logic

Exercise 1.1 In the following first-order sentences, $Bird(x)$ means "x is a bird", $Flies(x)$ means"x flies", $Person(x)$ means "x is a person", and $Mother(x, y)$ means "x is the mother of y". Translate the sentences into English:

1. $\forall x\ (Bird(x) \rightarrow Flies(x))$

   Solution: All birds fly.

2. $\forall x\ \exists y\ (Person(x) \rightarrow Mother(y, x))$

   Solution: All persons have a mother.

3. $\exists x\ \forall y\ (Person(x) \land Mother(x, y))$

   Solution: There exists a person who is the mother of everything.

Exercise 1.2 Convert the following English sentences into sentences of first-order logic. Use meaningful predicate names or state the abbreviation scheme that you are using.

1. All cats are mammals.

   Solution: $\forall x\ (Cat(x) \rightarrow Mammal(x))$

2. No cat is a reptile.

   Solution: $\forall x\ (Cat(x) \rightarrow \neg Reptile(x))$

3. All computer scientists like some operating system.

   Solution: $\forall x\ (CS(x) \rightarrow \exists y\ (Likes(x, y) \land OS(y)))$

4. The only good extraterrestrial is a drunk extraterrestrial.

   Solution: $\forall x\ (ET(x) \land Good(x) \rightarrow Drunk(x))$

5. The Barber of Seville shaves all men who do not shave themselves.

   Solution: $\forall x\ ((Man(x) \land \neg Shaves(x, x)) \rightarrow Shaves(\mathsf{bs}, x))$

6. There are at least two mountains in England.

   Solution: $\exists x, y\ (Mountain(x) \land in(x, \mathsf{e}) \land Mountain(y) \land in(y, \mathsf{e}) \land x \neq y)$

7. No mountain is higher than itself.

   Solution: $\neg \exists x\ (Mount(x) \land Higher(x, x))$

8. There is exactly one coin in the box.

   Solution: $\exists x\ (Coin(x) \land InBox(x) \land \forall y\ (Coin(y) \land InBox(y) \rightarrow y = x))$

9. There are exactly two coins in the box.

   Solution: $\exists x, y\ (Coin(x) \land InBox(x) \land Coin(y) \land InBox(y) \land x \neq y \land$
   $\forall z\ (Coin(z) \land InBox(z) \rightarrow (z = x \lor z = y)))$

10. The largest coin in the box is a quarter.

    Solution: $\forall x\ (Coin(x) \land InBox(x) \land \forall y\ (Coin(y) \land InBox(y) \rightarrow Larger_{=}(x, y)) \rightarrow Quarter(x))$

11. All students get good grades if they study.

    Solution: $\forall x\,(\mathit{Student}(x) \land \mathit{Studies}(x) \rightarrow \mathit{GetsGoodGrade}(x))$

Exercise 1.3 Assume N is intended to mean "is a number"; I is intended to mean "is interesting"; $<$ is intended to mean "is less than"; and 0 is a constant symbol intended to denote zero. Translate into first-order logic sentences the English sentences listed below. If the English sentence is ambiguous, you will need more than one translation.

1. Zero is less than any number.

   Solution: $\forall n\,(N(n) \rightarrow (0 < n))$

2. If any number is interesting, then zero is interesting.

   Solution: There are two possible interpretations of this sentence:

   - $\exists n\,(N(n) \land I(n)) \rightarrow I(0)$ (here "any" is interpreted as "some")
   - $\forall n\,(N(n) \rightarrow I(n)) \rightarrow I(0)$ (here "any" is interpreted as "all")

3. No number is less than zero.

   Solution: $\neg\exists n\,(N(n) \land (n < 0))$ or, equivalently,
   $\forall n\,\neg(N(n) \land (n < 0))$, or, equivalently
   $\forall n\,(N(n) \rightarrow \neg(n < 0))$

4. Any uninteresting number with the property that all smaller numbers are interesting certainly is interesting.

   Solution: $\forall n\,\big(N(n) \land \neg I(n) \land \forall m\,((N(m) \land (m < n)) \rightarrow I(m))\big) \rightarrow I(n)$
   (which is a contradiction!)

5. There is no number such that all numbers are less than it.

   Solution: $\neg\exists n\,\big(N(n) \land \forall m\,(N(m) \rightarrow (m < n))\big)$

6. There is no number such that no number is less than it.

   Solution: $\neg\exists n\,\big(N(n) \land \neg\exists m\,(N(m) \land (m < n))\big)$

Exercise 1.4 For each of the following English sentences, write a corresponding sentence in FOL.

1. P is a person; T is a time, $F(x,y)$ means that you can fool x at time y.

   (a) You can fool some of the people all of the time.
   Solution: Two interpretations: $\exists p\,\forall t\,(P(p) \land (T(t) \rightarrow F(p,t)))$ or
   $\forall t\,(T(t) \rightarrow \exists p\,(P(p) \land F(p,t)))$

   (b) You can fool all of the people some of the time.
   Solution: Two interpretations: $\exists t\,\forall p\,(T(t) \land (P(p) \rightarrow F(p,t)))$ or
   $\forall p\,(P(p) \rightarrow \exists t\,(T(t) \land F(p,t)))$

   (c) You can't fool all of the people all of the time.
   Solution: $\neg\big(\forall p, t\,(P(p) \land T(t) \rightarrow F(p,t))\big)$

2. J is a job; a designates Adam; $D(x,y)$ means that x can do y right.

(a) Adam can't do every job right.
Solution: $\neg\forall j\ (J(j) \rightarrow D(\mathsf{a}, j))$

(b) Adam can't do any job right.
Solution: $\forall j\ (J(j) \rightarrow \neg D(\mathsf{a}, j))$

3. Nobody likes everybody. ($L(x, y)$ means x likes y.)
Solution: $\neg(\exists x\ \forall y\ L(x, y))$

Exercise 1.5 Consider the following English sentences. Can you formalize them in first-order logic? If yes, how?

1. "There are three critics who admire only one another."
Solution: $\exists x_1, x_2, x_3((\bigwedge_{1\leq i<j\leq 3}(x_i \neq x_j)) \wedge (\bigwedge_{1\leq i\leq 3} \mathit{Critics}(x_i)) \wedge (\bigwedge_{1\leq i,j\leq 3} \mathit{Admires}(x_i, x_j)) \wedge \forall y((\bigvee_{1\leq i\leq 3} \mathit{Admires}(x_i, y)) \rightarrow (\bigvee_{1\leq i\leq 3} y = x_i)))$

2. "There are some critics who admire only one another."
Solution: We can interpret this statement as defining a clique on a graph, where there is an edge between two critics iff they admire one another. Since FOL is not powerful enough to express cliques, this sentence cannot be expressed either.

3. "It is not the case that there are some natural numbers smaller than 5 among which none is least."
Solution: We define the following auxiliary n-ary formulas:

$$\textsc{Distinct}_n(x_1, \ldots, x_n) = \bigwedge_{1\leq i<j\leq n} x_i \neq x_j$$

$$\textsc{Smaller5}_n(x_1, \ldots, x_n) = \bigwedge_{i=1}^{n} N(x_i) \wedge 0 \leq x_i \wedge x_i < 5$$

$$\textsc{NoLeast}_n(x_1, \ldots, x_n) = \bigwedge_{i=1}^{n} \neg\Big(\bigwedge_{j=1}^{n} x_i \leq x_j\Big)$$

Then our sentence can be encoded as follows:

$$\neg\Big(\big(\exists x_1\ \textsc{Smaller5}_1(x_1) \wedge \textsc{NoLeast}_1(x_1)\big) \bigvee$$
$$\big(\exists x_1, x_2\ \textsc{Smaller5}_2(x_1, x_2) \wedge \textsc{Distinct}_2(x_1, x_2) \wedge \textsc{NoLeast}_2(x_1, x_2)\big) \bigvee$$
$$\big(\exists x_1, x_2, x_3\ \textsc{Smaller5}_3(x_1, x_2, x_3) \wedge \textsc{Distinct}_3(x_1, x_2, x_3) \wedge \textsc{NoLeast}_3(x_1, x_2, x_3)\big) \bigvee$$
$$\ldots$$
$$\big(\exists x_1, \ldots, x_5\ \textsc{Smaller5}_5(x_1, \ldots, x_5) \wedge \textsc{Distinct}_5(x_1, \ldots, x_5) \wedge \textsc{NoLeast}_5(x_1, \ldots, x_5)\big)\Big)$$

Or equivalently,

$$\neg\Big(\bigvee_{k=1}^{5} \big(\exists x_1, \ldots, x_k\ \textsc{Smaller5}_k(x_1, \ldots, x_k) \wedge \textsc{Distinct}_k(x_1, \ldots, x_k) \wedge \textsc{NoLeast}_k(x_1, \ldots, x_k)\big)\Big)$$

4. "It is not the case that there are some numbers among which none is least."
Solution: This sentence is not expressible in FOL as one needs to be able to quantify over sets of numbers, but this is not allowed in FOL.

Exercise 1.6 For each group of sentences, either write an interpretation under which the last sentence is false and all the rest are true or explain why such an interpretation does not exist.

1. $\forall x\ (P(x) \rightarrow Q(x))$
   $\forall x\ (R(x) \rightarrow Q(x))$
   $\exists x\ (R(x) \land P(x))$

   Solution: $\mathcal{I} = \langle \Delta^\mathcal{I}, \cdot^\mathcal{I} \rangle$
   $\Delta^\mathcal{I} = \{a\}$
   $P^\mathcal{I} = \{\},\ R^\mathcal{I} = \{\},\ Q^\mathcal{I} = \{\}$

2. $\forall x\ \exists y\ P(x, y)$
   $\exists y\ \forall x\ P(x, y)$

   Solution: $\mathcal{I} = \langle \Delta^\mathcal{I}, \cdot^\mathcal{I} \rangle$
   $\Delta^\mathcal{I} = \{a, b\}$
   $P^\mathcal{I} = \{(a, b), (b, a)\}$

3. $\forall x\ (P(x) \rightarrow Q(\mathsf{a}))$
   $(\forall x\ P(x)) \rightarrow Q(\mathsf{a})$

   Solution: Such an interpretation does not exist as the first sentence implies the second

Exercise 1.7 For each group of sentences, write an interpretation in which all sentences are true.

1. $\forall x\ (P(x) \lor Q(x)) \rightarrow \exists x\ R(x)$
   $\forall x\ (R(x) \rightarrow Q(x))$
   $\exists x\ (P(x) \land \neg Q(x))$

   Solution: $\mathcal{I} = \langle \Delta^\mathcal{I}, \cdot^\mathcal{I} \rangle$
   $\Delta^\mathcal{I} = \{a, b\}$
   $P^\mathcal{I} = \{a\},\ R^\mathcal{I} = \{b\},\ Q^\mathcal{I} = \{b\}$

   or $\mathcal{I} = \langle \Delta^\mathcal{I}, \cdot^\mathcal{I} \rangle$
   $\Delta^\mathcal{I} = \{a, b\}$
   $P^\mathcal{I} = \{a\},\ R^\mathcal{I} = \{\},\ Q^\mathcal{I} = \{\}$

2. $\forall x\ \neg P(x, x)$
   $\forall x, y, z\ (P(x, y) \land P(y, z) \rightarrow P(x, z))$
   $\forall x\ \exists y\ P(x, y)$

   Solution: $\mathcal{I} = \langle \mathbb{N}, \cdot^\mathcal{I} \rangle$
   $P^\mathcal{I} = \{(x, y) \mid x, y \in \mathbb{N},\ x < y\}$

3. $\forall x\ \exists y\ P(x, y)$
   $\forall x\ (Q(x) \rightarrow \exists y\ P(y, x))$
   $\exists x\ Q(x)$
   $\forall x\ \neg P(x, x)$

   Solution: $\mathcal{I} = \langle \Delta^\mathcal{I}, \cdot^\mathcal{I} \rangle$
   $\Delta^\mathcal{I} = \{a, b\}$
   $Q^\mathcal{I} = \{a\},\ P^\mathcal{I} = \{(a, b), (b, a)\}$