E 10.1

Exercise :

Show that CSAT remains NP-complete even if it is restricted to instances in which each variable appears at most three times. Let's call this variant 3VARCSAT

Solution: We show how to reduce CSAT to 3VARCSAT.

Given a formula F in CNF, we construct a formula E in CNF where each variable appears at most three times and such that E is satisfiable iff F is satisfiable.

Let x be a variable appearing in F k times, with $k > 3$. Then we construct from F a formula F_x as follows:

1) We replace the i-th occurrence of x in F with x_i, for $i \in \{1, \dots, k\}$, where each x_i is a fresh variable.
2) We add the following clauses, ensuring that all variables $x_1, \dots, x_k$ are assigned the same truth value

$$(\overline{x}_1 \vee x_2) \wedge (\overline{x}_2 \vee x_3) \wedge \dots \wedge (\overline{x}_{k-1} \vee x_k) \wedge (\overline{x}_k \vee x_1)$$

We have that F_x can be constructed from F in polynomial time, and F_x is satisfiable iff F is satisfiable.

Then, the formula E is obtained from F by repeating the above transformation for each variable x occurring in F more than three times.

We have that:

1) Each variable appears in E at most three times
2) E can be constructed from F in polynomial time
3) E is satisfiable iff F is satisfiable □

Exercise:

Consider 2VARCSAT, i.e. the variant of CSAT in which each variable appears at most two times.

What is the complexity of 2VARCSAT?

Solution: 2VARCSAT is in P.

Let F be an instance of 2VARCSAT.

Notice that for each variable x, we have one of 3 cases:

1) x appears only positively in F (one or two times)
2) x — " — negatively in F (— " —)
3) x appears one time positively and one time negatively in F

From this, we obtain the following algorithm to decide the satisfiability of F:

Input: set F of clauses over variables $x_1, \ldots, x_n$,
    with each x_i appearing in at most two clauses
Output: YES, if F is satisfiable, NO otherwise
For each variable $x \in \{x_1, \ldots, x_n\}$
  if x appears only positively in F (case 1), or
     x appears only negatively in F (case 2), or
     x appears only in one clause of F
  then remove from F the clause(s) in which x_i appears
  else let $C = x \vee C_{rest}$ and $C' = \bar{x} \vee C'_{rest}$
(case 3)   be the two clauses of F in which x appears
      if C_{rest} and C'_{rest} are both empty
        then answer NO
        else remove from F both C and C', and
              replace them with the clause $C_{rest} \vee C'_{rest}$
Answer YES

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The algorithm runs in polynomial time,
since the for-loop is executed n times, and each iteration is at most linear.

Note that a variable x might be removed from F before it is considered in the i-th iteration of the for loop. In this case, F is not changed in that iteration.

Moreover:

- For cases (1) and (2) the clauses containing x can be trivially satisfied by making x true/false.
- For case (3), the algorithm applies a resolution step to x, and replaces the clauses C and C' with their resolvent.
- By applying a resolution step to a variable x_i, for another variable x_j that has not yet been considered (i.e. $j > i$) and that occurred positively and negatively in two different clauses, the two occurrences of x_j might be merged into a single clause $C_{rest} \vee C'_{rest}$.
  This clause can be removed, since it is trivially satisfied by every truth assignment.

Exercise: 2SAT is in P

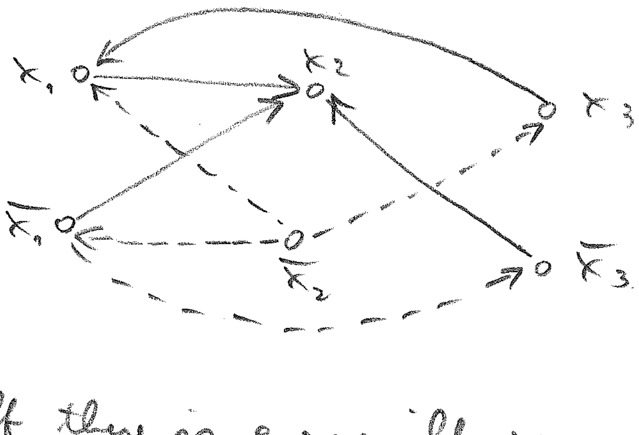
Idea: we show that 2SAT can be encoded as a graph reachability problem, and then use an algorithm for graph reachability

1) Encoding of 2SAT as a directed graph reachability problem

Let ϕ be an instance of 2SAT. We define a graph $G(\phi)$ as follows:
- one node for each variable and one node for each negated variable
- for each clause $\alpha \vee \beta$ two edges $\bar{\alpha} \to \beta$, $\bar{\beta} \to \alpha$

(note: $\alpha \vee \beta \equiv \bar{\alpha} \to \beta \equiv \bar{\beta} \to \alpha$)

Example: $(x_1 + x_2) \cdot$
$(x_1 + \bar{x}_3) \cdot$
$(\bar{x}_1 + x_2) \cdot$
$(x_2 + x_3)$



Then ϕ is unsatisfiable iff there is a variable x such that $G(\phi)$ contains two paths $x \to \dots \to \bar{x}$, $\bar{x} \to \dots \to x$

"$\Leftarrow$" Suppose that ϕ has a satisfying truth assignment T. Assume that $T(x) = $ true (a similar argument holds for $T(x) = $ false)

Since $T(x) = $ true and $T(\bar{x}) = $ false, and there is a path $x \to \dots \to \bar{x}$, there must be an edge $\alpha \to \beta$ along this path with $T(\alpha) = $ true and $T(\beta) = $ false.

However, since $\alpha \rightarrow \beta$ is an edge of $G(\Phi)$, it follows that $\bar{\alpha} + \beta$ is a clause of Φ. This clause is not satisfied by T_1 which is a contradiction.

"$\Leftarrow$" Let $G(\Phi)$ be a graph that does not contain any node x with

$$\begin{tikzcd} x \arrow[r] & \cdots \arrow[r] & \bar{x} \arrow[ll, bend left, dashed] \end{tikzcd}$$

We construct from such a graph a satisfying truth assignment T.

Repeat the following step as often as possible:

Choose a node α such that

- $T(\alpha)$ is not yet defined, and
- there is no path $\alpha \xrightarrow{*} \bar{\alpha}$

For every node β that is reachable from α (including α itself)

1) set $T(\beta) = $ true
2) set $T(\bar{\beta}) = $ false

(Note: 2) means to assign false to all predecessors of $\bar{\alpha}$)

$$\alpha \cdots\xrightarrow{*}\cdots \beta \qquad \bar{\alpha} \cdots\xleftarrow{*}\cdots \bar{\beta}$$

Observe: 1) the truth assignment T is well defined, i.e., we never have both $T(\beta) = $ true and $T(\bar{\beta}) = $ true or $T(\beta) = $ false and $T(\bar{\beta}) = $ false

T would not be well defined, if we had both

$$\begin{tikzcd} \alpha \arrow[r, "*"] \arrow[dr, "*"'] & \beta \\ & \bar{\beta} \end{tikzcd}$$

(for some β)

But this cannot happen, since we would have

$$\begin{tikzcd} \alpha \arrow[r, "*"] \arrow[dr] & \beta \arrow[dl, dashed, "*"] \\ \bar{\alpha} & \bar{\beta} \arrow[l, dashed] \end{tikzcd}$$

Hence, we would have $\alpha \xrightarrow{*} \bar{\alpha}$

2) We assign to all nodes a truth value, since there is no path

$$\begin{tikzcd} \alpha \arrow[r, "*"] & \bar{\alpha} \arrow[l, bend left, "*"] \end{tikzcd}$$

3) The truth assignment satisfies all clauses of F, since each clause corresponds to an implication, and there is no $\alpha \overset{*}{\rightleftarrows} \bar{\alpha}$.