Introduction to Computability

OPTIONAL PART ↓

Question: Using a counting argument, we have seen that there are functions that cannot be computed (or, in other words, problems that cannot be solved by any algorithm).
How can we exhibit a specific problem of this form?

Solution: we need a formal definition of algorithm

Let us start with something we know: Java
Can we show that there is no Java program that solves a specific problem?

Hello-World problem:

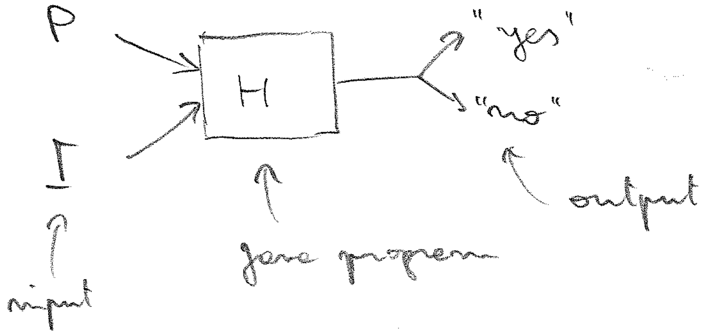
Your first Java program HW:

```
public class HW {
    public static void main (String[] args) {
        System.out.println("Hello, world");
    }
}
```

The first 12 characters output by HW are "Hello, world".

Hello-world problem (HWP): given an arbitrary Java program P and an input I for P, does P(I) print "Hello, world" as its first 12 characters?

Consider a solution to HWP:



Does such a program H exist?

- we could scan P for println statements
- but, how do we know whether they are executed?

To give you an idea how difficult this can become, consider <u>Fermat's last theorem</u>:

The equation $x^n + y^n = z^n$ has no integer solution for $n \geq 3$.

For $n = 2$: a solution is $x = 3$, $y = 4$, $z = 5$

For $n \geq 3$: mathematicians have believed that the theorem is true, but no proof was found until recently (proof given by Wiles is very complex, and still under verification)

Consider a simple java program P_1 that:

1) reads input n
2) for all possible x, y, z do
   if $(x^n + y^n = z^n)$
      println ("Hello, world");

Consider input $n = 3$: P_1 prints "Hello, world" only if F.L.T. is false, otherwise P_1 loops forever.

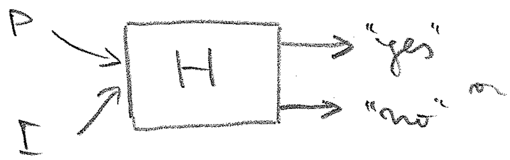
$\Rightarrow$ If we could solve HWP, we would also have proved or disproved F.L.T.

This would be too nice !! Where is the problem ?

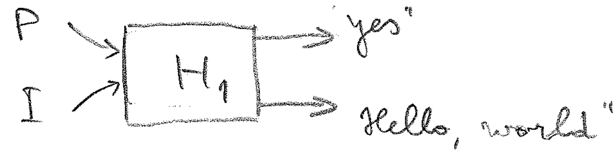
<u>Theorem</u>: There is no Java program H that decides HWP.

<u>Proof</u>: assume H exists and derive a contradiction.

Consider H :

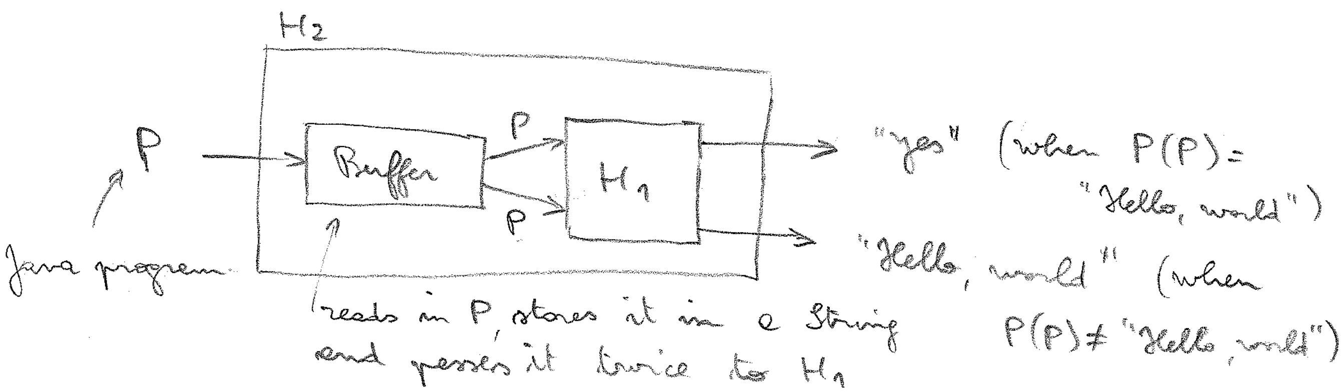


We modify H to H_1 s.t. H_1 prints "Hello, world" instead of "no"



(Note: we have to modify the println statements in H)

We modify H_1 to H_2, which takes only input P and feeds it to H_1 as both P and I:



Let us consider $H_2(P)$ when $P = H_2$.

suppose $H_2(H_2)$ = "yes" $\Rightarrow$ P(P) = "Hello, world"
suppose $H_2(H_2)$ = "Hello, world" $\Rightarrow$ P(P) $\neq$ "Hello, world"

But $P = H_2$ $\Rightarrow$ Contradiction $\Rightarrow$ H, H_1, H_2 cannot exist !

Q.e.d.

We have shown HWP to be <u>undecidable</u>, i.e., there cannot be an algorithm (or a program) that solves it.

We can show that other problems are undecidable by "reducing" HWP to them

<u>Reductions</u>

<u>Foo-problem</u>: given a program R and its input z, does R ever call a function named foo while executing on input z.

Idea: we <u>reduce</u> the HWP to the foo-problem, i.e. we show that if it's possible to solve the foo-problem on (R, z), then we can solve HWP on (Q, y), for any program Q with input y.
Since HWP is undecidable, so is the foo-problem.

Suppose there is a program F that takes as input (R, z) and decides the foo-problem for (R, z).
We show how F can be used to construct H that decides HWP on input (Q, y)

Idea: apply modifications to Q

1) rename function foo in Q (if present) to pippo.
   $\Rightarrow Q_1$
2) add a dummy function foo to Q_1 $\Rightarrow Q_2$
3) modify Q_2 to store all its output in some array A
   $\Rightarrow Q_3$
4) modify Q_3 so that after every println statement it checks array A to see if "Hello, world" has been printed. If yes, then call function foo $\Rightarrow Q_4$

Note: We can write a Java program that takes as input a Java sourcefile and modifies it as specified above.

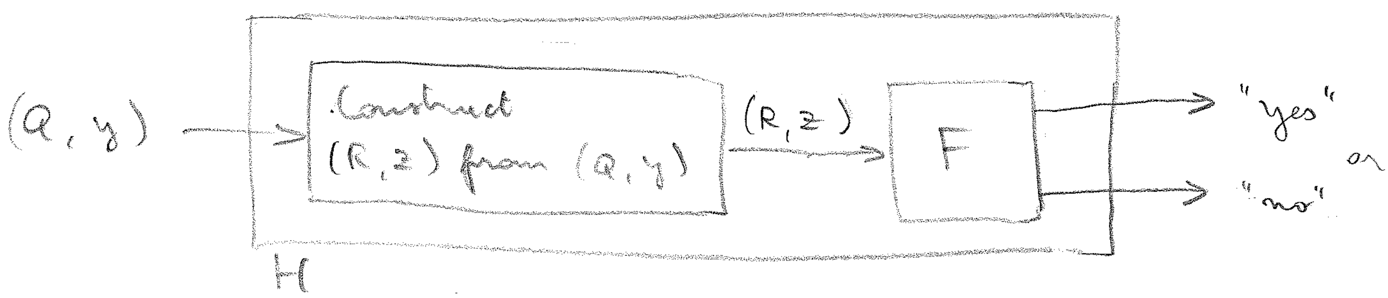
Let $R = Q_4$ and $z = y$

We have by construction:

$Q(y)$ prints "Hello, world" $\Longleftrightarrow$
$R(z)$ calls function foo.

Hence, we can use F that solves foo-problem on $R(z)$ to construct H that solves HWP on $Q(y)$.

Schematically:



But: since H does not exist, also F cannot exist.

Q.e.d

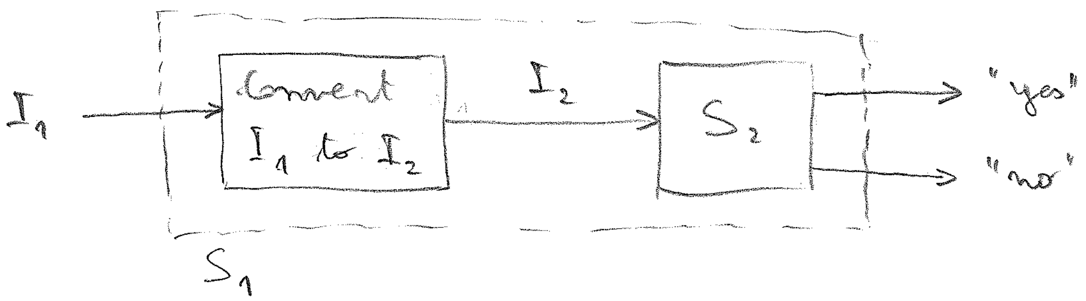
Showing undecidability by reduction from undecidable problem

Problem P_1 taking input I_1 known to be undecidable
— " — P_2 — " — I_2 to show undecidable.

Reduction: convert I_1 to I_2 such that

$$P_1(I_1) = \text{"yes"} \quad \text{iff} \quad P_2(I_2) = \text{"yes"}$$

given solution program S_2 for P_2, we could obtain
— " — S_1 for P_1



Since S_1 does exist, we obtain that S_2 cannot exist
$\Rightarrow$ P_2 is undecidable.

END OF OPTIONAL PART

Existence of undecidable problems:

While it was tricky to show that a specific problem is undecidable, it is rather easy to show that there are infinitely many undecidable problems.

We use a counting argument:

- a problem P is a language over Σ (for some finite Σ)
  (the strings in the language represent those instances of P for which the answer is "yes")
  $\Rightarrow$ there are uncountably many problems
- an algorithm is a string over Σ' (for some finite Σ')
  $\Rightarrow$ there are countably many algorithms

$\Rightarrow$ There must be (uncountably many) problems for which there is no algorithm.

Turing Machines

Java (or C, Pascal, ...) programs are not well-suited to develop a theory of computation:

- run-time environment and run-time errors
- complex language constructs
- finite memory
- "state" of the computation is complicated to represent
- would need to show that the results for a specific programming language are in fact general

⇒ We resort to an abstract computing device, the Turing Machine (TM)

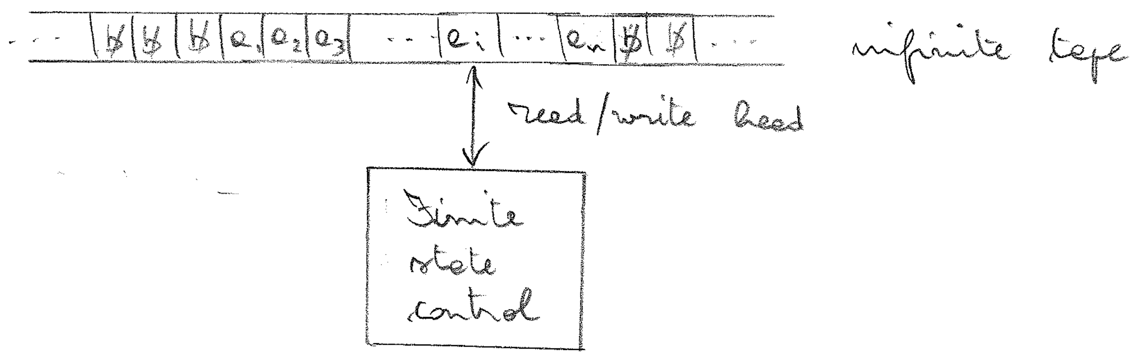
- simple and universal programming language
- state of computation is easy to describe
- unbounded memory
- can simulate any known computing device

Church-Turing hypothesis:

All reasonably powerful computation models are equivalent to TMs (but not more powerful).

⇒ TMs model anything we can compute.

The TM:



Programmed by specifying transitions

move depends on – current state (finitely many)
– symbol under the tape head

effects of a move: – new state
– write new symbol on tape cell under the head
– move head left/right/stay

Observations:

relationship to real computers: CPU ⟷ finite state control
memory ⟷ tape

"differences" (features lost in the abstraction)

– no random access memory
– limited instruction set

However: a TM can simulate a computer (with a cubic increase in running time – see book 8.6)

<u>Definition</u> a TM $M = (Q, \Sigma, \Gamma, \delta, q_0, \not{b}, F)$

Q ... set of states (finite)

Σ ... input alphabet (finite)

$F \subseteq Q$ ... final states

$q_0 \in Q$ ... initial state

Γ ... tape alphabet (finite)

$\not{b} \in \Gamma$ ... blank symbol

Conditions: $\Sigma \subseteq \Gamma$, since input is written initially on tape

$\not{b} \in \Gamma - \Sigma$, since the rest of the tape is blank

Initially: - state q_0
- tape contains w surrounded by $\not{b}$
- tape head is at the leftmost cell of the input

Transitions: $\delta : Q \times \Gamma \longrightarrow Q \times \Gamma \times \{L, R, S\}$

$\delta(q, x) = (p, y, d)$ means that

if M is in state q and tape head is over symbol x,
then M - changes state to p
- replaces x by y on the tape
- moves tape head by one cell in direction d
(left for L, right for R, S for stay in place)

The TM is deterministic:

for each $\delta(q, x)$ we have at most one move
($\delta(q, x)$ could also be undefined)

<u>Acceptance</u>: w is accepted by TM M if M, when started with w on the tape, eventually enters a final state

We can assume that all final states are halting, i.e. no transition is defined for them

<u>Rejection</u>: - halts in non final state (i.e., no transition defined)
- never halts (infinite loop)

Difference between FA / PDA and TM:

FA / PDA scans over w and accepts/rejects when it has reached its end

TM can move back and forth over w and accepts/rejects when it halts or rejects if it loops forever

13/10/2010

Example: $L = \{ w \#^* w t \mid w \in \{0,1\}^+, \ t \in \{0,1,\#\}^* \}$

initially

$\not b$	$\not b$	$\not b$	$\cdots w \cdots$	$\#$	$\cdots$	$\#$	$\cdots w \cdots$	$\cdots t \cdots$	$\not b$	$\not b$	$\cdots$

TM idea: - remember (in the state) leftmost symbol, and erase it
- move to leftmost symbol after #'s
- if the two don't match, then reject
- otherwise replace the symbol by #, move left and start again

$M = (Q, \Sigma, \Gamma, \delta, q_0, \not b, F)$ 14/10/2014

$Q = \{q_0, q_1, \ldots, q_7\}$ $\quad F = \{q_7\}$

$\Sigma = \{0, 1, \#\}$ $\quad \Gamma = \{0, 1, \#, \not b\}$

$\delta(q_0, 0) = (q_1, \not b, R)$ } Erase 0 and look for matching 0

$\delta(q_0, 1) = (q_2, \not b, R)$ } — " — 1 — " — 1

$\delta(q_1, 0) = (q_1, 0, R)$
$\delta(q_1, 1) = (q_1, 1, R)$
$\delta(q_1, \#) = (q_3, \#, R)$
} Skip over 0's and 1's, till # is found (remembering 0)

$\delta(q_2, 0) = (q_2, 0, R)$
$\delta(q_2, 1) = (q_2, 1, R)$
$\delta(q_2, \#) = (q_4, \#, R)$
} — " — (remembering 1)

2.11

$\delta(q_3, \#) = (q_3, \#, R)$
$\delta(q_3, 0) = (q_5, \#, L)$

Skip over #'s, look for 0, and replace it by #.
Note: if after #'s a 1 or a $\not b$ is found, M halts and rejects

$\delta(q_4, \#) = (q_4, \#, R)$
$\delta(q_4, 1) = (q_5, \#, L)$

as previous ones, replacing 0/1 with 1/0.

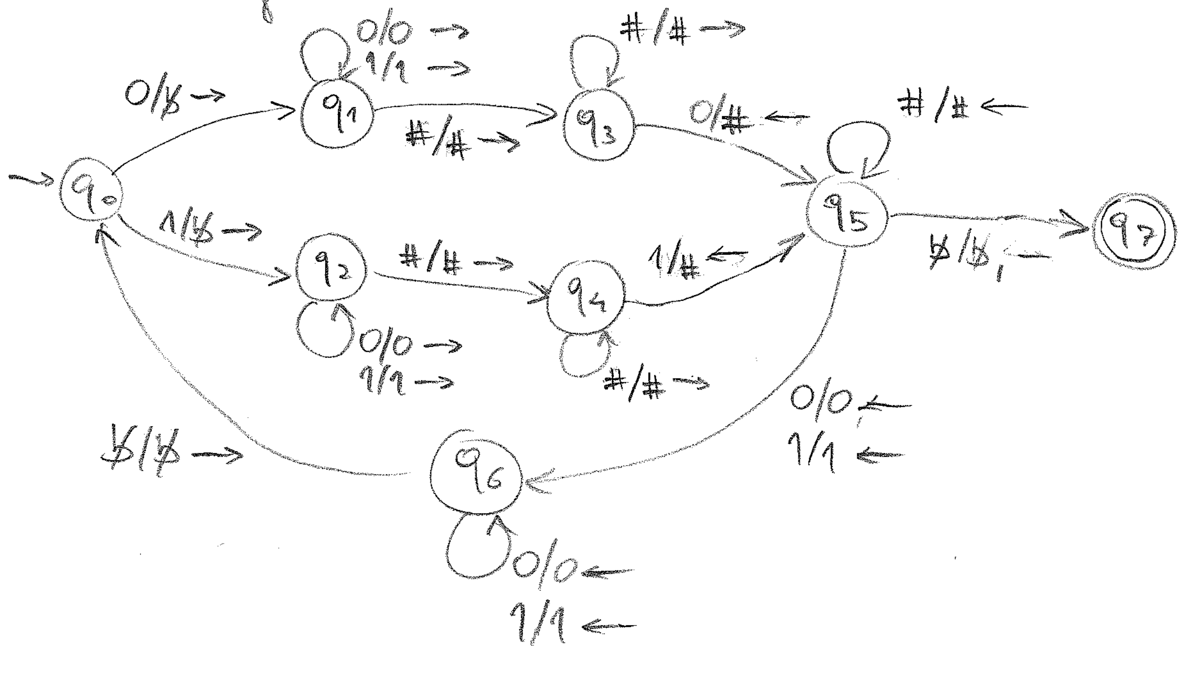
$\delta(q_5, \#) = (q_5, \#, L)$
$\delta(q_5, 0) = (q_6, 0, L)$
$\delta(q_5, 1) = (q_6, 1, L)$
$\delta(q_5, \not b) = (q_7, \not b, S)$

Move left skipping #'s.
If to the left of the #'s a 0 or 1 is found, move to q_6 to skip them also. If $\not b$ is found, accept

$\delta(q_6, 0) = (q_6, 0, L)$
$\delta(q_6, 1) = (q_6, 1, L)$
$\delta(q_6, \not b) = (q_0, \not b, R)$

Move left, skipping 0's and 1's, and restart again.

Transition diagram



represents $\delta(q, x) = (r, y, d)$

Instantaneous description (I.D.) or configuration of a TM

describes the current situation of TM and tape.

$I.D. = \alpha_1 q \alpha_2$ with $q \in Q$, $\alpha_1, \alpha_2 \in \Gamma^*$

means:
- non-blank portion of tape contains $\alpha_1 \alpha_2$
- head is on leftmost symbol of α_2
- machine is in state q

Corresponds to

state q

BLANKS	α_1	α_2	BLANKS

Let $ID = \Gamma^* \times Q \times \Gamma^*$ be the set of instantaneous descriptions. We use a relation $\vdash_M \subseteq ID \times ID$ to describe the transitions of a TM M. (when M is clear from the context, we abbreviate $\vdash_M$ with $\vdash$)

Example:
$$q_0 01\#01 \vdash q_1 1\#01 \vdash 1 q_1 \#01 \vdash$$
$$\vdash 1\# q_3 01 \vdash 1 q_5 \#\#1 \vdash$$
$$\vdash q_5 1\#\#1 \vdash q_6 \not{b} 1\#\#1 \vdash$$
$$\vdash q_0 1\#\#1 \vdash \dots \vdash$$
$$\vdash q_5 \not{b} \#\#\# \vdash q_7 \#\#\# \quad \leftarrow \text{accepts}$$

<u>Note</u>: we can define $\vdash_M$ formally, making use of δ. [See next page]

Making use of the closure $\vdash^*$ of $\vdash$ we can define the language accepted by a TM

<u>Definition</u>: Let $M = (Q, \Sigma, \Gamma, \delta, q_0, \not{b}, F)$ be a TM. Then the language $\mathcal{L}(M)$ accepted by M is
$$\mathcal{L}(M) = \{ w \in \Sigma^* \mid q_0 w \vdash^* \alpha_1 q \alpha_2 \text{ with } q \in F \text{ and } \alpha_1, \alpha_2 \in \Gamma^* \}$$

2.12 b)

Relation $\vdash_M \subseteq ID \times ID$ describes the moves of a TM

$M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$

where $ID = \Gamma^* \times Q \times \Gamma^*$

- Let $\delta(q, X) = (p, Y, L)$ be a leftward move of M

  Then $X_1 \cdots X_{i-1}\, q\, X\, X_{i+1} \cdots X_n \vdash_M X_1 \cdots X_{i-2}\, p\, X_{i-1}\, Y\, X_{i+1} \cdots X_n$

  note: the head is now at cell $i-1$

  There are two exceptions to this general case

  1) if $i = 1$, then M moves to the blank to the left of X

     $q X X_2 \cdots X_n \vdash_M p B Y X_2 \cdots X_n$

  2) if $i = n$ and $Y = B$, then the symbol B written over X is not represented in the resulting ID

     $X_1 \cdots X_{n-1}\, q\, X \vdash_M X_1 \cdots X_{n-2}\, p\, X_{n-1}$

- Similarly, we can define when $ID_1 \vdash_M ID_2$ for a rightward move $\delta(q, X) = (p, Y, R)$ of M

  [Exercise]

Notes:

1) We have used TMs for language recognition, which in turn corresponds to solving decision problems

- We can, however, consider also TMs as computing functions:
  - the output (result of the function) is left on the tape

2) The class of languages accepted by TMs are called recursively enumerable
   - for a string w in the language
     - the TM halts on input w in a final state
   - for a string w not in the language
     - the TM may halt in a non-final state, or
     - it may loop forever

Those languages for which the TM always halts (regardless of whether it accepts or not) are called recursive:
- these languages correspond to recursive functions
- TMs that always halt are a good model of algorithms and they correspond to decidable problems

Programming techniques for TMs

We present some notational conveniences that make it easier to write TM programs

Idea: use structured states and tape symbols

1) Storage in the state: ("CPU register")

Idea: state names are a tuple of the form

$[q, D_1, \dots, D_k]$

D_i ... acts as stored symbol

q ... control portion of the state

Example: TM $M = (Q, \Sigma, \Gamma, \delta, q_s, B\!\!\!/, F)$ for $L = 01^* + 10^*$

Idea: M remembers the first symbol and checks that it does not reappear

$Q = \{ [q_i, e] \mid i \in \{0,1\},\ e \in \{0, 1, -\} \} =$

$\{ [q_0, -], [q_0, 0], [q_0, 1], [q_1, -], [q_1, 0], [q_1, 1] \}$

$\Sigma = \{0, 1\}$ $\qquad \Gamma = \{0, 1, B\!\!\!/\}$

$q_s = [q_0, B\!\!\!/]$ $\qquad F = \{[q_1, -]\}$

Meaning of $[q_i, e]$

- control portion q_i:

  q_0 ... M has not yet read its first symbol

  q_1 ... M has read its first symbol

- data portion e: e is the first symbol read transitions:

$$\delta([q_0, -], e) = ([q_1, e], e, R) \quad , \text{ for } e \in \{0, 1\}$$

... M remembers in $[q_1, e]$ that it has read e

$$\left.\begin{array}{l} \delta([q_1, 0], 1) = ([q_1, 0], 1, R) \\ \delta([q_1, 1], 0) = ([q_1, 1], 0, R) \end{array}\right\} \text{M moves right as long as it does not see the first symbol}$$

$$\delta([q_1, e], \not b) = ([q_1, -], \not b, R), \text{ for } e \in \{0, 1\}$$

... M accepts when it reaches the first $\not b$

2) <u>Multiple tracks</u>:

Idea: view tape as having multiple tracks, i.e. each symbol in Γ has multiple components

	0	*	$\not b$	
...	1	0	0	...
	a	a	c	

the symbols on the tape are $\begin{bmatrix}0\\1\\a\end{bmatrix}, \begin{bmatrix}*\\0\\a\end{bmatrix}, \begin{bmatrix}\not b\\0\\c\end{bmatrix}$

<u>Example</u>: $L = \{ww \mid w \in \{0,1\}^+\}$

We first need to find midpoint, and then we can match corresponding symbols.

To find midpoint: we view tape as 2 tracks

			*		
0	1	1	0	1	1

← used to put markers on symbols

Hence: $\Gamma = \left\{ \begin{bmatrix}\not b\\\not b\end{bmatrix}, \begin{bmatrix}\not b\\0\end{bmatrix}, \begin{bmatrix}\not b\\1\end{bmatrix}, \begin{bmatrix}*\\0\end{bmatrix}, \begin{bmatrix}*\\1\end{bmatrix} \right\}$

(note: we need no * over $\not b$)

We put markers on two outermost symbols and move them inwards:

$$\delta(q_0, [{\not b \atop i}]) = (q_1, [{* \atop i}], R)$$
$$\delta(q_1, [{\not b \atop i}]) = (q_1, [{\not b \atop i}], R)$$

move right till end or first marked symbol

$$\delta(q_1, [{\not b \atop \not b}]) = (q_2, [{\not b \atop \not b}], L)$$
$$\delta(q_1, [{* \atop i}]) = (q_2, [{\not b \atop i}], L)$$
$$\delta(q_2, [{\not b \atop i}]) = (q_3, [{* \atop i}], L)$$

move rightmost mark one symbol to the left

$$\delta(q_3, [{\not b \atop i}]) = (q_3, [{\not b \atop i}], L)$$
$$\delta(q_3, [{* \atop i}]) = (q_0, [{\not b \atop i}], R)$$

move left till end or first marked symbol

Note: we have each of the above for $i \in \{0, 1\}$

At the end: head is over first symbol of second w, with a $*$ above it, in state q_0.

3) <u>Subroutines / procedure calls</u> 15/10/2014

<u>Example</u>: shifting over

given: $ID_1 = \alpha\, q_i\, x\, \beta$

want: $ID_2 = \alpha \,\Box\, q_j\, x\, \beta$

for $x \in \Gamma$, $\alpha, \beta \in \Gamma^*$, $\Box \in \Gamma$

Subroutine for shifting over can be used repeatedly to create space in the middle of the tape

E.g. to implement a counter

$\$0\$ \to \$1\$ \to \$\Box 1\$ \to \$01\$ \to \$10\$ \to$
$\to \$11\$ \to \$\Box 11\$ \to \$011\$ \to \dots$

Procedure call: $\delta(q_i, x) = ([p, x], [\overset{j}{\square}], R)$, $\forall x \in \Gamma$ (2.17) a

- remember return state q_j, and erased symbol x
- state p calls procedure

Procedure p for shifting

1) shift 1 cell to the right

$$\delta([p, x], y) = ([p, y], x, R) \qquad \forall x, y \in \Gamma \text{ with } y \neq \not{b}$$

2) till we have reached end of β

$$\delta([p, y], \not{b}) = (r, y, L) \qquad \forall y \in \Gamma$$

3) return to calling point by moving left

$$\delta(r, y) = (r, y, L) \qquad \forall y \neq [\overset{j}{\square}]$$

4) exit and return to state q_j

$$\delta(r, [\overset{j}{\square}]) = (q_j, \square, R)$$

In fact, we can implement arbitrary complex procedures, with any kind of parameter passing

$\boxed{\text{Exercise:}}$ redesign the TMs you have seen so far to take advantage of storage in the state, multiple tracks, and subroutines

$\boxed{\text{Exercise}}$ Implement a procedure call to copy a string to the end of the input, i.e. given $ID_1 = \alpha \not{b}\, q_i \beta \not{b} \gamma$

we want $ID_2 = \alpha \not{b}\, q_j \beta \not{b} \gamma \not{b} \beta$

Example of computation for shifting over

State

q_i

$\not b$	0	0	0	a	b	c	$\not b$	...
				↑				

we want to place □ after the 0's

$$\delta(q_i, a) = \left([q, a], \begin{bmatrix} \dot{\jmath} \\ \square \end{bmatrix}, R\right)$$

$[q, a]$

...	$\not b$	0	0	0	$\begin{smallmatrix} \dot{\jmath} \\ \square \end{smallmatrix}$	b	c	$\not b$	...
						↑			

$$\delta([q, a], b) = ([q, b], a, R)$$

$[q, b]$

...	$\not b$	0	0	0	$\begin{smallmatrix} \dot{\jmath} \\ \square \end{smallmatrix}$	a	c	$\not b$	...
							↑		

$$\delta([q, b], c) = ([q, c], b, R)$$

$[q, c]$

...	$\not b$	0	0	0	$\begin{smallmatrix} \dot{\jmath} \\ \square \end{smallmatrix}$	a	b	$\not b$	...
								↑	

$$\delta([q, c], \not b) = (r, c, L)$$

r

...	$\not b$	0	0	0	$\begin{smallmatrix} \dot{\jmath} \\ \square \end{smallmatrix}$	a	b	c	$\not b$	...
							↑			

$$\delta(r, x) = (r, x, L) \qquad x \neq \begin{bmatrix} \dot{\jmath} \\ \square \end{bmatrix}$$

r

...	$\not b$	0	0	0	$\begin{smallmatrix} \dot{\jmath} \\ \square \end{smallmatrix}$	a	b	c	$\not b$	...
					↑					

$$\delta\left(r, \begin{bmatrix} \dot{\jmath} \\ \square \end{bmatrix}\right) = (q_j, \square, R)$$

q_j

...	$\not b$	0	0	0	□	a	b	c	$\not b$	...
						↑				

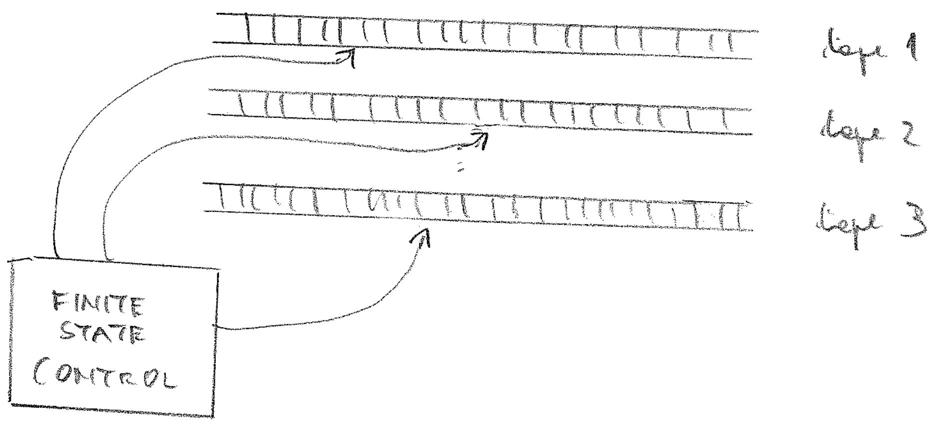
Extensions to the basic TM

Note: if the TM seen so far can compute all that can be computed, then it should not become more expressive by extending it

We consider two extensions: - multiple tapes
- non-determinism

and show that both can be captured by the basic T.M.

1) Multi-tape T.M.



Initially: input w is on tape 1 with tape-head on the leftmost symbol. Other tapes are all blank.

Transitions: specify behaviour of each head independently

$$\delta(q, x_1, \dots, x_k) = (p, (y_1, d_1), \dots, (y_k, d_k))$$

x_i ... symbol under head i

y_i ... new symbol written to head i

d_i ... direction in which head i moves

To simulate k-tape TM M_k with a 1-tape TM M_1, we use $2k$ tracks in M_1: for each tape of M_k

- one track of M_1 to store tape content
- one track of M_1 to mark head position with $*$

A	B	A	C	B	A		tape 1
			$*$				head 1
0	0	1	1	1	0		tape 2
	$*$						head 2
b	b	a	b	a	b		tape 3
					$*$		head 3

Each transition of M_k is simulated by a series of transitions of M_1: $\delta(q, x_1, \ldots, x_k) = (p, (y_1, d_1), \ldots, (y_k, d_k))$

- start at leftmost head position marker
- sweep right and remember in appropriate "CPU registers" the symbols x_i under each head (note: these are exactly k, and hence finitely many)
- knowing all x_i's, sweep left, change each x_i to y_i, and move the marker for tape i according to d_i

Note: M_1 needs to remember always how many of the k heads are to its left (uses an additional CPU-register)

The final states of M_1 are those that have in the state-component a final state of M_k.

We can verify that we can construct M_1 so that
$\mathcal{L}(M_1) = \mathcal{L}(M_k)$

(details are straightforward, but cumbersome)

$\boxed{\text{Exercise}}$ Provide the details of the construction to convert a 2-tape TM to a 1-tape TM

Simulation speed:

Note: - enhancements do not affect the expressive power of a TM
- they do affect its efficiency

Definition: a TM is said to have running time $T(n)$ if it halts within $T(n)$ steps on all inputs of length n.

Note: $T(n)$ could be infinite

Theorem: If M_k has running time $T(n)$, then M_1 will simulate it with running time $O(T(n)^2)$.

Proof: Consider input w of length n.
- M_k runs at most $T(n)$ time on it.
- At each step, leftmost and rightmost heads can drift apart by at most 2 additional cells.
- It follows that after $T(n)$ steps the k heads cannot be more than $2 \cdot T(n)$ apart, and M_k uses $\leq 2 \cdot T(n)$ tape cells

Consider M_1:
- makes two sweeps for each transition of M_k
- each sweep takes at most $O(T(n))$
- number of transitions of M_k is $\leq T(n)$

It follows that the total running time is $O(T(n)^2)$.

2) Nondeterministic TMs (NTM)

In a (deterministic) TM, $\delta(q, x)$ is unique or undefined

In a NTM, $\delta(q, x)$ is a finite set of triples

$$\delta(q, x) = \{(p_1, y_1, d_1), \ldots, (p_k, y_k, d_k)\}$$

At each step, the NTM can non-deterministically choose which transition to make.

As for other ND devices: a string w is accepted if the NTM has at least one execution leading to a final state.

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Example: $\Sigma = \{0, 1, \ldots, 9\}$

$L = \{ w \in \Sigma^* \mid$ a 0 appears i positions to the left of some i in w, with $0 < i \leq 9\} =$

$= \{ w \in \Sigma^* \mid \exists j > 0 \text{ s.t. } w_{j-w_j} = 0 \}$

(w_i indicates the i-th character of w)

Ex. $02146 \notin L$

$5810855442 1 \ldots$

$0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10 \ldots$

$w_7 = 4$

$w_3 = w_{7-4} = 0$

NTM N s.t. $L(M) = L$

$Q = \{q_0, f, [p, 0], [p, 1], \ldots, [p, 9]\}$

$F = \{f\}$

$\Gamma = \{0, 1, \ldots, 9, b\!\!\!/\}$

Idea for N: - scan w from left to right,
- guess at some $w_j = i$,
- store i in CPU register, and
- move i steps left to find 0

Transitions:

- $\delta(q_0, 0) = \{(q_0, 0, R)\}$ (since $w_j \geq 1$)
- $\forall i > 0 : \delta(q_0, i) = \{(q_0, i, R), \underbrace{([\uparrow, i]}_{\text{guess}}, i, L)\}$
- $\forall i \geq 2, \forall x \in \Gamma : \delta([\uparrow, i], x) = \{[\uparrow, i-1], x, L)$
- accepting: $\delta([\uparrow, 1], 0\} = \{(f, 0, R\})$

Execution traces on input $w = 103332$

$q_0 103332 \vdash 1 q_0 03332 \vdash 10 q_0 3332 \vdash 103 q_0 332 \vdash$
$\vdash 10 [\uparrow, 3] 3332 \vdash 1 [\uparrow, 2] 03332 \vdash [\uparrow, 1] 103332$

$\Rightarrow$ reject

$q_0 103332 \vdash^* 1033 q_0 32 \vdash 103 [\uparrow, 3] 332 \vdash$
$\vdash 10 [\uparrow, 2] 3332 \vdash 1 [\uparrow, 1] 03332 \vdash 10 f 3332$

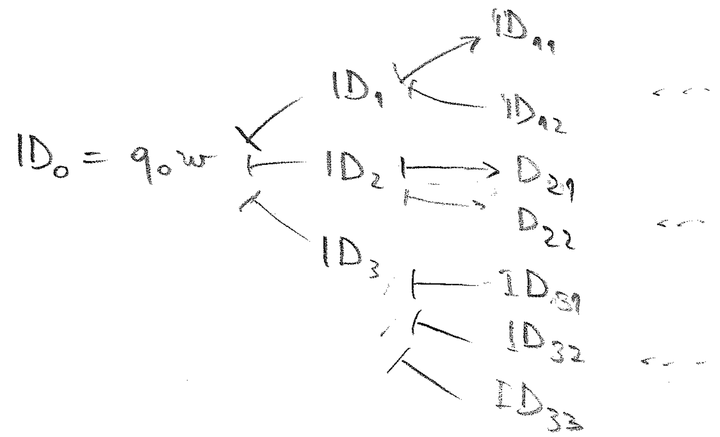
$\Rightarrow$ accept

<u>Theorem</u>: Let N be a NTM. Then there exists a DTM D s.t. $\mathcal{L}(D) = \mathcal{L}(N)$

<u>Proof</u>: Given N, we show how to construct a multi-tape DTM that can simulate the execution of N on some input w. We can then convert the multi-tape DTM to a single-tape DTM

Idea for the simulation:

Consider the execution tree of N on some input w



DTM D will perform a breadth-first search of the execution tree, systematically enumerating the IDs, until it finds an accepting one.

We use two tapes:

tape 2: is for working

tape 1: contains a sequence of ID's of N in BFS order

- $*$ used to separate two ID's
- $\hat{*}$ marks next ID to be explored
    - ID's to the left of $\hat{*}$ have been explored
    - ID's to the right of $\hat{*}$ are still to be explored
- initially, only $ID_0 = q_0 w$ is on the tape
- we can use multiple tracks for convenience

"BFS order" means that the sequence of IDs is managed as a queue:

- $\hat{*}$ determines the beginning of the queue
- the end of tape 1 represents the end of the queue

Algorithm: repeat the following steps

step 0: examine current ID_c (the one after $\hat{*}$) and read q, a from it

if $q \in F$, then accept and halt

step 1: let $\delta(q, a)$ have k possible transitions

- copy ID_c onto tape 2
- make k new copies of ID_c and place them at the end of tape 1

step 2: modify the k copies of ID_c on tape 1 to become the k possible outcomes of $\delta(q, a)$ on ID_c

step 3: move $\hat{*}$ right past ID_c.
clean up tape 2
return to step 0

It is possible to verify:
- the above steps can all be implemented in a DTM
- the construction is correct, i.e. $w \in \mathcal{L}(D)$ iff $w \in \mathcal{L}(N)$

Evolution of tape 1:

1) $\hat{*}\ ID_0\ *$
2) $\hat{*}\ ID_0 * ID_0 * ID_0 * ID_0\ *$
3) $\hat{*}\ ID_0 * ID_1 * ID_2 * ID_3\ *$
4) $*\ ID_0\ \hat{*}\ ID_1 * ID_2 * ID_3\ *$
5) $*\ ID_0\ \hat{*}\ ID_1 * ID_2 * ID_3 * ID_1 * ID_1\ *$
6) $*$ — " — $* ID_{11} * ID_{12}\ *$
7) $* ID_0 * ID_1\ \hat{*}\ ID_2 * ID_3 * ID_{11} * ID_{12}\ *$

⋮

Simulation time:

Let NTM N have running time $T(n)$.

What is the running time of D?

Let m be the maximum number of non-det. choices for each transition (i.e., the maximum size of $\delta(q, x)$)

Consider execution tree of N on w.

let $t = T(|w|)$ $\Rightarrow$ exec. tree has at most t levels

size of the tree is $\leq 1 + m + m^2 + \cdots + m^t =$

$$= \frac{m^{t+1} - 1}{m - 1} = O(m^t)$$

Thus D has at most $O(m^t)$ iterations of steps 0-3.

Each iteration requires at most $O(m^t)$ steps

$\Rightarrow$ Total running time is $m^{O(t)}$, i.e. exponential