Exercise: Reduction from CSAT to 3SAT

E8.1

(see textbook 10.3.4)

Given a CNF formula $E = C_1 \cdot C_2 \cdots C_k$

with each $C_i = \sum_{j=1}^{k_i} l_{ij}$,

we construct a 3-CNF formula F as follows.

For each clause C_i of E

1) if $C_i = (l)$ (i.e., a single literal)

introduce two new variables u_i, v_i, and replace C_i by 4 clauses

$(l + u_i + v_i) \cdot$
$(l + u_i + \overline{v_i}) \cdot$
$(l + \overline{u_i} + v_i) \cdot$
$(l + \overline{u_i} + \overline{v_i})$

Since u_i, v_i appear in all 4 combinations, the 4 clauses can be satisfied only if l is true

2) if $C_i = (l_1 + l_2)$

introduce a new variable z_i, and replace C_i by 2 clause

$(l_1 + l_2 + z_i) \cdot$
$(l_1 + l_2 + \overline{z_i})$

as in 1

3) if $C_i = (l_1 + l_2 + l_3)$, just leave it

4) if $C_i = (l_1 + l_2 + \cdots + l_m)$ with $m \geq 4$

introduce $y_1, y_2, \ldots, y_{m-3}$ and replace C_i by

$(l_1 + l_2 + y_1) \cdot (l_3 + \overline{y_1} + y_2) \cdot (l_4 + \overline{y_2} + y_3) \cdots$
$+ (l_{m-2} + \overline{y_{m-4}} + y_{m-3}) \cdot (l_{m-1} + l_m + \overline{y_{m-3}})$

- An assignment T satisfying E makes at least one literal of C_i true. Let it be l_j.
  Then, by making $y_1, \dots, y_{j-2}$ true and $y_{j-1}, \dots, y_{m-3}$ false,
  we satisfy all clauses replacing C_i.
  Thus we can extend T to satisfy F.
- Conversely, if T makes all l_j of C_i false, then not all new clauses can be satisfied.
  Why? each y_j can make at most 1 clause true, but there are $m-2$ clauses and $m-3$ y_j's.

The 3-CNF formula F is linear in E and can be constructed in linear time

We get: CSAT $<_{poly}$ 3-SAT

$\Rightarrow$ from CSAT NP-hard, we get 3-SAT NP-hard

We also know that 3SAT $\in$ NP, since SAT $\in$ NP.

$\Rightarrow$ 3-SAT is NP-complete

E8.3

Exercise (10.3.2): The problem 4TA - SAT is defined as follows:

Give a prop. formula E, does E have at least 4 satisfying truth assignments?

Show that 4TA - SAT is NP-complete.

Proof:

1) 4 TA - SAT is in NP

We devise a non-deterministic poly-time algorithm.

1) Guess 4 truth-assignments T_1, T_2, T_3, T_4

2) Check that T_1, T_2, T_3, T_4 all satisfy E.

Note that both steps require time polynomial in the size of E

2) 4TA - SAT is NP-hard

We show this by reducing SAT to 4TA - SAT.

Let E be a prop. formula, and let $x_1, \dots, x_n$ be all variables in E.

We construct a new formula E' s.t.

$$E \in SAT \Leftrightarrow E' \in 4TA\text{-}SAT$$

Let y_1, y_2 be two new variables. Then

$$E' = E \vee ((x_1 \wedge x_2 \wedge \dots \wedge x_n) \wedge ((y_1 \wedge y_2) \vee (y_1 \wedge \overline{y_2}) \vee (\overline{y_1} \wedge y_2)))$$

E8.4

Consider the truth assignments for $x_1, \dots, x_n, y_1, y_2$

						Case 1 $E \notin SAT$		Case 2 $E \in SAT$	
	x_1	x_j	x_n	y_1	y_2	E	E'	E	E'
1)	T	...	T	T	T	F	T	?	T
2)	T	...	T	T	F	F	T	?	T
3)	T	...	T	F	T	F	T	?	T
4)	T	...	T	F	F	F	F	⋮	⋮
	...	F	...	...		F	F	E	T
2^{n+2}		⋮				F	F	⋮	⋮
							$= 3$		≥ 4

Alternative solution:

$$E' = E \wedge (y_1 \vee y_2 \vee y_3)$$

- If E is unsatisfiable, then E' is unsatisfiable, and hence $E' \notin 4TA\text{-}SAT$

- If E is satisfiable, then E' has at least 7 satisfying truth assignments: these are obtained by combining
  - a TA for $x_1, \dots, x_n$ satisfying E with
  - the 7 TAs for y_1, y_2, y_3 satisfying $y_1 \vee y_2 \vee y_3$