PRIMITIVE RECURSIVE FUNCTIONS

E6.1

Exercise 1

Show that multiplication is a primitive recursive function.

Solution:

$$\begin{cases} mult(x,0) = g(x) = 0 \\ mult(x, y+1) = h(x, y, mult(x,y)) = mult(x,y) + x \end{cases}$$

where $g = z$ and $h = add \circ (p_3^{(3)}, p_1^{(3)})$

Exercise 2

Let $g(x,y)$ be a primitive recursive function. Then the following functions obtained from g are also PR.

a) $f(x, y, z_1, \dots, z_n) = g(x,y)$

b) $f(x,y) = g(y,x)$

c) $f(x) = g(x,x)$

Solution:

a) $f = g \circ (p_1^{(n+2)}, p_2^{(n+2)})$

b) $f = g \circ (p_2^{(2)}, p_1^{(2)})$

c) $f = g \circ (p_1^{(1)}, p_1^{(1)})$

E6.2

Exercise 3

Let $p(x,z)$ be a primitive recursive predicate. Show that the following functions are primitive recursive.

a) $f_1(x, y_0, y)$ = the first value z in $[y_0, y]$ for which $p(x,z)$ is true

b) $f_2(x,y)$ = the second value z in $[0,y]$ for which $p(x,z)$ is true

c) $f_3(x,y)$ = the largest value z in $[0,y]$ for which $p(x,z)$ is true

If there is no value z in the range such that $p(x,z)$ is true, then f_i is $y+1$.

Solution:

a) $f_1(x,y_0,y) = \mu z \leq y\, [p(x,z) \cdot ge(z, y_0)]$
The PRF ge ("greater than or equal to") is used to enforce the lower bound; multiplication $\cdot$ works as "boolean and".

b) $f_2(x,y) = \mu z \leq y\, [p(x,z) \cdot gt(z, \mu z' \leq y\, [p(x,z')])]$
The PRF gt ("greater than") makes sure we skip the first value.

c) Let $f'(x,y) = y \dot{-} \underbrace{\mu z \leq y\, [p(x, y \dot{-} z)]}_{\substack{\text{reverses the order of examination} \\ \text{(i.e. we go from } y \text{ down to } 0)}}$

Then:

$$f_3(x,y) = eq(y+1, \mu z \leq y\, [p(x,z)]) \cdot (y+1) + neq(y+1, \mu z \leq y\, [p(x,z)]) \cdot f'(x,y)$$

It checks whether there is a z such that $z \leq y$ and $p(x,z) = true$, and outputs $f'(x,y)$ if it is the case and $y+1$ otherwise.

Exercise 4

Consider integer division $\mathrm{div}(x,y)$: it's not defined for 0, hence not total and hence not PR. Let

$$\mathrm{quo}(x,y) = \begin{cases} 0 & \text{if } y=0 \\ \mathrm{div}(x,y) & \text{otherwise} \end{cases}$$

a) Define $\mathrm{quo}(x,y)$ using bounded minimization.

b) Show that remainder, divides, number of divisors, and prime are primitive recursive.

Solution:

a) $$\mathrm{quo}(x,y) = \mathrm{sg}(y) \cdot \mu z \leq x \,[\mathrm{gt}((z+1)\cdot y, x)]$$

b) Remainder:

$$\mathrm{rem}(x,y) = x \mathbin{\dot{-}} (y \cdot \mathrm{quo}(x,y))$$

Divides:

$$\mathrm{divides}(x,y) = \begin{cases} 1 & \text{if } x>0, y>0, \text{ and } y \text{ is a divisor of } x \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{divides}(x,y) = \mathrm{eq}(\mathrm{rem}(x,y), 0) \cdot \mathrm{sg}(x)$$

Number of divisors:

$$\mathrm{ndivisors}(x)\cancel{y)} = \sum_{y=0}^{x} \mathrm{divides}(x,y)$$

Prime:

$$\mathrm{prime}(x) = \begin{cases} 1 & \text{if } x \text{ is prime} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{prime}(x) = \mathrm{eq}(\mathrm{ndivisors}(x), 2)$$

Exercise 5

Show that the function $pn(i)$ computing the i-th prime is PR by exploiting the fact that $pn(x+1) \leq pn(x)! + 1$.

Solution:

$$\begin{cases} pn(0) = 2 \\ pn(x+1) = \mu z \leq (pn(x)! + 1)[\, prime(z) \cdot gt(z, pn(x)) \,] \end{cases}$$

Exercise 6

Show that the Ackermann function

$$\begin{cases} A(0,y) = y+1 \\ A(x+1,0) = A(x,1) \\ A(x+1,y+1) = A(x, A(x+1,y)) \end{cases}$$

is defined for every pair $x, y \in \mathbb{N}$.

Solution :

By induction on x (main induction)

Base case : $A(0,y) = y+1$

Inductive step : By induction on y (secondary induction)
$A(x+1,y)$

Base case : $A(x+1,0) = A(x,1)$ and the main induction hypothesis applies
$A(x+1,0)$

Inductive step : By the secondary induction hypothesis $A(x+1,y)$ is defined;
$A(x+1,y+1)$

thus for $A(x+1,y+1) = A(x, A(x+1,y))$ the main induction hypothesis applies

Exercise 7

Define a primitive recursive function $f: \mathbb{N} \to \mathbb{N}$ that counts the number of occurrences of the digit 5 in a natural number.

Solution:

We need some auxiliary primitive recursive functions

- exponential $m^n : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

$$exp(m,n) = \begin{cases} exp(m,0) = 1 \\ exp(m,n+1) = exp(m,n) \cdot m \end{cases}$$

- length (number of digits) $: \mathbb{N} \to \mathbb{N}$

$length(n) = (\mu z \leq n [gt(10^{z+1}, n)]) + 1$

examples: $length(0) = length(1) = \ldots = length(9) = 1$, $length(10) = 2, \ldots$

$f: \mathbb{N} \to \mathbb{N}$ is then defined as follows

$$f(n) = \Sigma_{i=1}^{length(n)} \; eq(5, rem(quo(n, 10^{i-1}), 10))$$

Example:
$$\begin{aligned} f(253) &= eq(5, \underbrace{rem(\underbrace{quo(253,1)}_{253}, 10)}_{3}) & 0 \\ & & + \\ &+ eq(5, \underbrace{rem(\underbrace{quo(253,10)}_{25}, 10)}_{5}) & 1 \\ & & + \\ &+ eq(5, \underbrace{rem(\underbrace{quo(253,100)}_{2}, 10)}_{2}) & 0 \\ & & \overline{\;1\;} \\ &= 0 + 1 + 0 = 1 \end{aligned}$$

Exercise 8

Define a primitive recursive function $f : \mathbb{N} \to \mathbb{N}$ that reverses the digits of a natural number, i.e. $f(253) = 352$, $f(5524) = 4255$.

Solution:

$$f(n) = \sum_{i=1}^{length(n)} \left(quo\left(rem\left(n, 10^{length(n)-i+1}\right), 10^{length(n)-i}\right) \cdot 10^{i-1}\right.$$

Example:
$$\begin{aligned} f(5524) &= \underbrace{quo(\underbrace{rem(5524, 10000)}_{5524}, 1000)}_{5} \cdot 1 \\ &+ \underbrace{quo(\underbrace{rem(5524, 1000)}_{524}, 100)}_{5} \cdot 10 \\ &+ \underbrace{quo(\underbrace{rem(5524, 100)}_{24}, 10)}_{2} \cdot 100 \\ &+ \underbrace{quo(\underbrace{rem(5524, 10)}_{4}, 1)}_{4} \cdot 1000 \\ &= 5 + 50 + 200 + 4000 = 4255 \end{aligned}$$

E6.7

Exercise 1: Show that the following functions are PRFs and can be defined through primitive recursion.
Explicitly give the functions g and h.

1) Predecessor $pred(x)$: $\begin{cases} pred(0) = 0 \\ pred(y+1) = y \end{cases}$

$pred(0) = 0 \qquad g() = 0$

$pred(y+1) = h(y, pred(y)) = y \qquad h = \pi_1^{(2)}$

2) Proper subtraction $sub(x,y)$, or $x \dot{-} y$

$\begin{cases} sub(x,0) = x \\ sub(x, y+1) = pred(sub(x,y)) \end{cases}$

$sub(x,0) = g(x) = \pi_1^{(1)}(x) = x$

$sub(x, y+1) = h(x, y, sub(x,y))$

$h = pred \circ \pi_3^{(3)}$

3) Sign $sg(x)$: $\begin{cases} sg(0) = 0 \\ sg(y+1) = 1 \end{cases}$

$sg(0) = g() \qquad g() = 0$

$sg(y+1) = h(y, sg(y)) \qquad h = s \circ z \circ \pi_1^{(2)}$

Some other useful functions:

less than: $lt(x,y) = sg(y \dot{-} x)$

greater than: $gt(x,y) = sg(x \dot{-} y)$