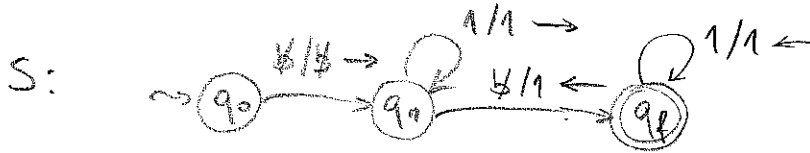
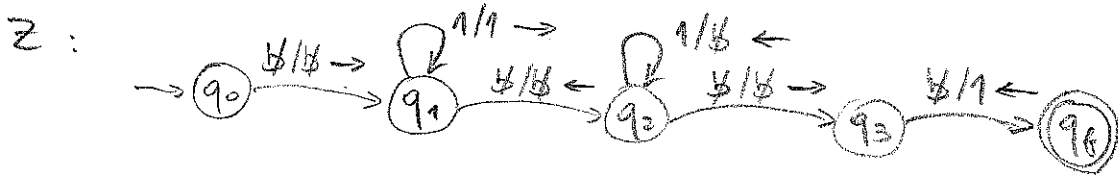


1) Construct a TM computing the successor function

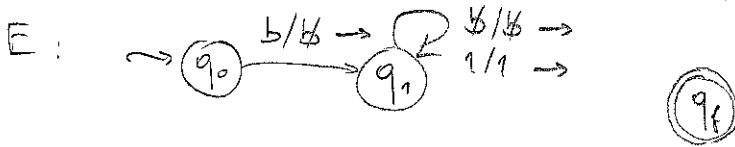
$$s(n) = n + 1$$



2) Construct a TM computing the zero function $z(n) = 0$



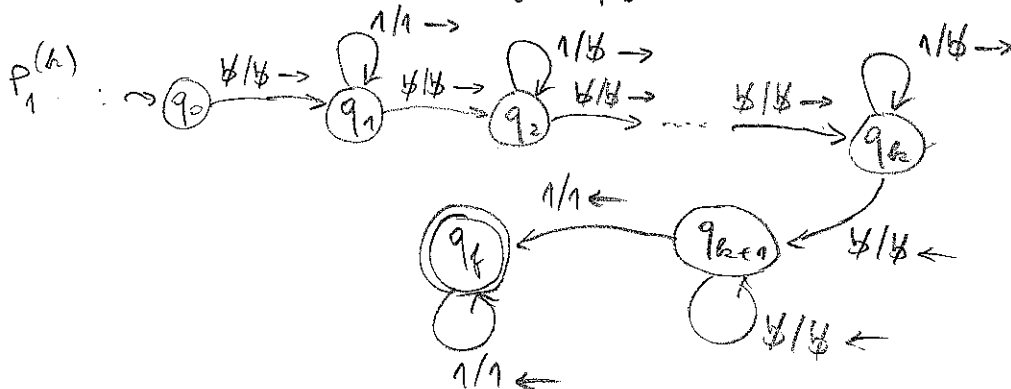
3) Construct a TM computing the empty function $e(n) \uparrow$ i.e., the function that is undefined for every $n \in \mathbb{N}$



The k -variable projection function $\pi_i^{(k)}$ is defined as

$$\pi_i^{(k)}(a_1, \dots, a_k) = a_i \quad (\text{for } 1 \leq i \leq k)$$

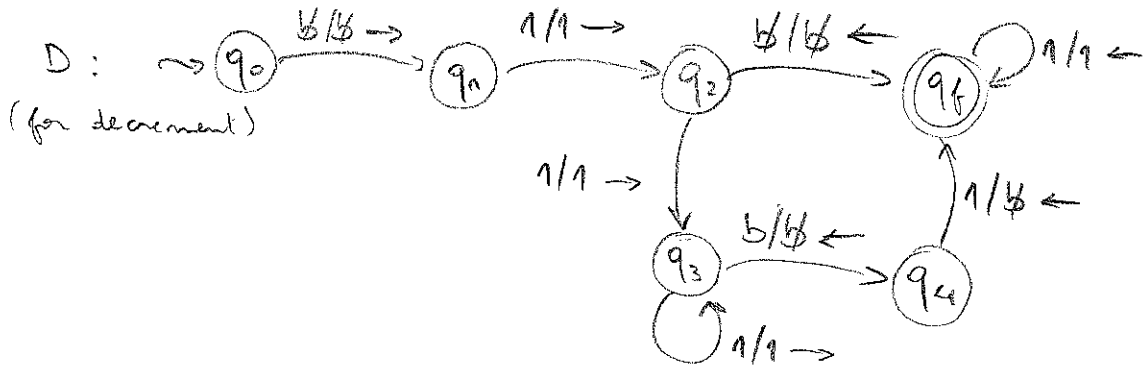
4) Construct a TM computing $\pi_1^{(k)}$



Note $\pi_1^{(1)}$ is also called the identity function $id(n) = n$

5) Construct a TM computing the predecessor function

$$\text{pred}(n) = \begin{cases} 0 & \text{if } n=0 \\ n-1 & \text{if } n>0 \end{cases}$$

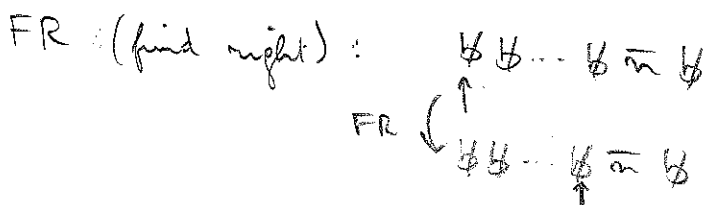
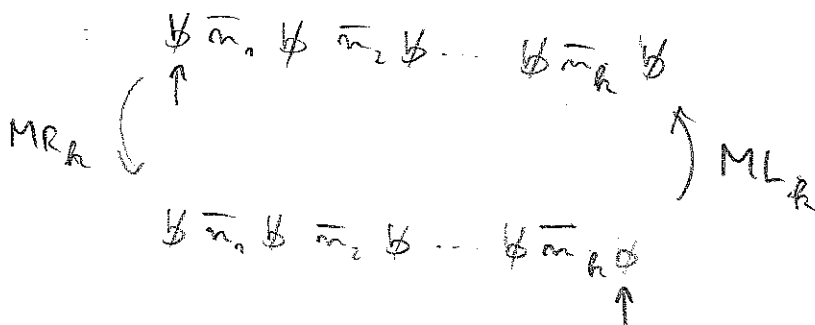
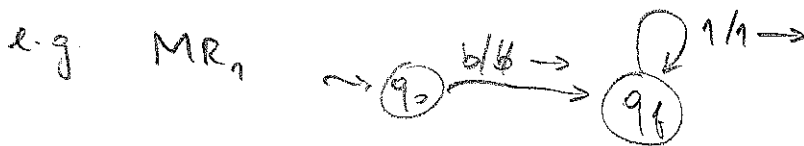


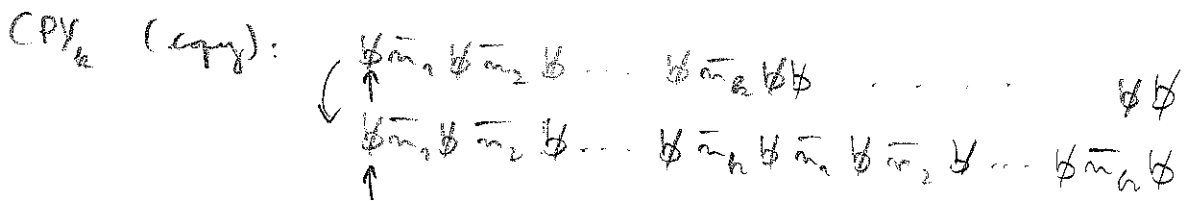
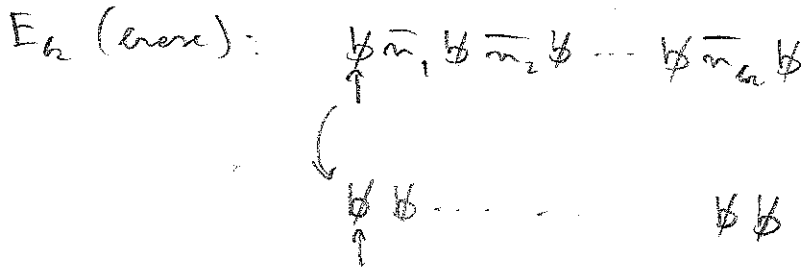
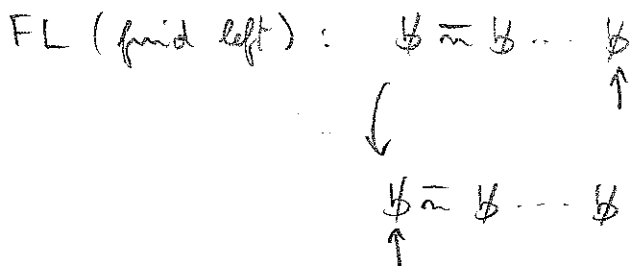
6) Using sequential composition, construct a TM computing the constant function $c(n) = 1$



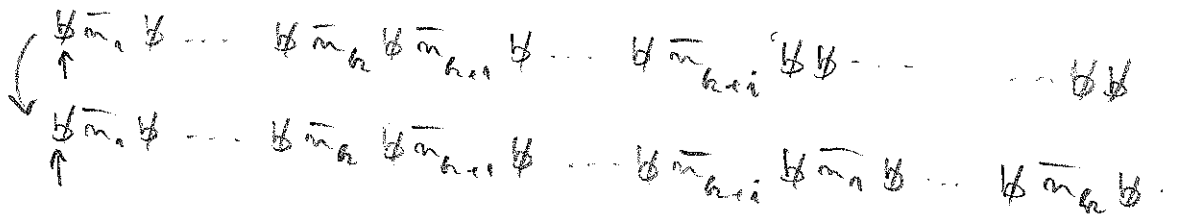
We now define some TMs that can be used as macros:

MR_k : move the tape head to the right through k consecutive natural numbers



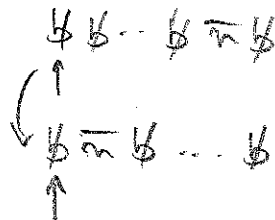


$CPY_{k,i}$ (copy through i numbers)

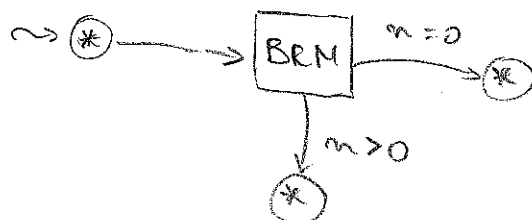


for CPY_k and $CPY_{k,i}$ the blank portion following $\bar{m}_1 \emptyset \dots \emptyset \bar{m}_k$ is assumed to be long enough to contain the copy.

T (truncate)



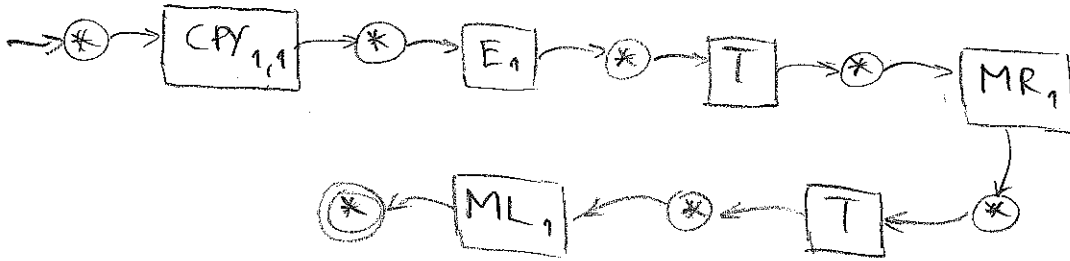
BRN (branch on zero)



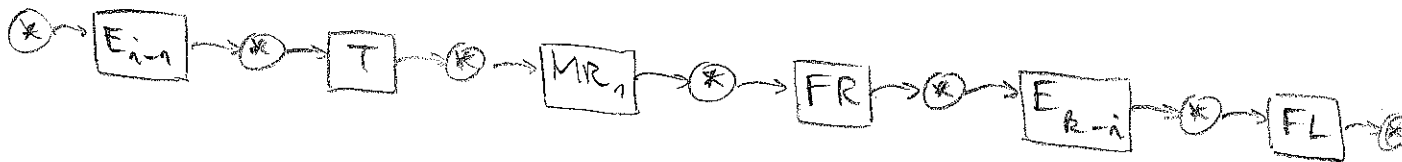
it does not alter the tape or change the bead position

7) Using composition and the defined macros, construct

a TM INT (interchange): $\$ \bar{n} \$ m \$ b^{n+1} \$$
 \uparrow
 $\$ \bar{m} \$ n \$ b^{m+1} \$$



8) Using composition and the defined macros, construct a TM for $p_i(b)$



Function composition:

E 5.6

$$\text{Let } g_i : \mathbb{N}^k \rightarrow \mathbb{N} \text{ for } 1 \leq i \leq m$$

$$h : \mathbb{N}^m \rightarrow \mathbb{N}$$

The composition of h with g_1, \dots, g_m , written

$$f = h \circ (g_1, \dots, g_m)$$

is the function $f : \mathbb{N}^k \rightarrow \mathbb{N}$ defined by

$$f(x_1, \dots, x_k) = h(g_1(x_1, \dots, x_k), \dots, g_m(x_1, \dots, x_k))$$

The function $f(x_1, \dots, x_k)$ is undefined if either

1) $g_i(x_1, \dots, x_k) \uparrow$ for some $i \in \{1, \dots, m\}$

2) $g_i(x_1, \dots, x_k) = y_i$ for $i \in \{1, \dots, m\}$ and $h(y_1, \dots, y_m) \uparrow$

We show that the composition of Turing-computable functions is also Turing-computable.

Exercise: given $g_1 : \mathbb{N}^3 \rightarrow \mathbb{N}$, $g_2 : \mathbb{N}^3 \rightarrow \mathbb{N}$, $h : \mathbb{N}^2 \rightarrow \mathbb{N}$

and TMs G_1, G_2, H computing respectively g_1, g_2, h ,

construct a TM computing $h \circ (g_1, g_2)$.

Use macros and TM composition, and construct the configurations obtained after every macro application.

macro configuration

$$\underline{\$} \bar{m}_1 \$ \bar{m}_2 \$ \bar{m}_3 \$$$

$$CPY_3 \quad \underline{\$} \bar{m}_1 \$ \bar{m}_2 \$ \bar{m}_3 \$ \bar{m}_1 \$ \bar{m}_2 \$ \bar{m}_3 \$$$

$$MR_3 \quad \$ - \dots - \underline{\$} - \dots - \$$$

$$G_1 \quad \$ - \dots - \underline{\$} \bar{y}_1 \$ \quad (\text{where } y_1 = g_1(m_1, m_2, m_3))$$

$$ML_3 \quad \underline{\$} \bar{m}_1 \$ \bar{m}_2 \$ \bar{m}_3 \$ \bar{y}_1 \$$$

$$CPY_{3,1} \quad \underline{\$} - \dots - \underline{\$} \bar{m}_1 \$ \bar{m}_2 \$ \bar{m}_3 \$$$

$$MR_4 \quad \$ - \dots - \underline{\$} - \dots - \$$$

$$G_2 \quad \$ - \dots - \underline{\$} \bar{y}_1 \$ \bar{y}_2 \$ \quad (\text{where } y_2 = g_2(m_1, m_2, m_3))$$

$$ML_1 \quad \$ - \dots - \underline{\$} \bar{y}_1 \$ \bar{y}_2 \$$$

$$H \quad \$ - \dots - \underline{\$} \bar{z} \$ \quad (\text{where } z = h(y_1, y_2))$$

$$ML_3 \quad \underline{\$} - \dots - \underline{\$} \bar{z} \$$$

$$E_3 \quad \underline{\$} \$ \dots \underline{\$} \bar{z} \$$$

$$T \quad \$ \bar{z} \$$$