

PRIMITIVE RECURSIVE FUNCTIONS

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E 7.1

Exercise 1

Show that multiplication is a primitive recursive function.

Solution:

$$\begin{cases} \text{mult}(x, 0) = g(x) = 0 \end{cases}$$

$$\begin{cases} \text{mult}(x, y+1) = h(x, y, \text{mult}(x, y)) = \text{mult}(x, y) + x \end{cases}$$

where $g = z$ and $h = \text{add} \circ (P_3^{(3)}, P_1^{(3)})$

Exercise 2

Let $g(x, y)$ be a primitive recursive function. Then the following functions \square obtained from g are also PR.

a) $f(x, y, z_1, \dots, z_n) = g(x, y)$

b) $f(x, y) = g(y, x)$

c) $f(x) = g(x, x)$

Solution:

a) $f = g \circ (P_1^{(n+2)}, P_2^{(n+2)})$

b) $f = g \circ (P_2^{(2)}, P_1^{(2)})$

c) $f = g \circ (P_1^{(1)}, P_1^{(1)})$

Exercise 3

E7.2

Let $p(x, z)$ be a primitive recursive predicate. Show that the following functions are primitive recursive.

a) $f_1(x, y_0, y) =$ the first value z in $[y_0, y]$ for which $p(x, z)$ is true

b) $f_2(x, y) =$ the second value z in $[0, y]$ for which $p(x, z)$ is true

c) $f_3(x, y) =$ the largest value z in $[0, y]$ for which $p(x, z)$ is true

If there is no value z in the range such that $p(x, z)$ is true, then f_i is $y + 1$.

Solution:

$$a) f_1(x, y_0, y) = \mu z \leq y [p(x, z) \cdot \text{ge}(z, y_0)]$$

The PRF ge ("greater^{than} or equal to") is used to enforce the lower bound; multiplication works as "boolean and".

$$b) f_2(x, y) = \mu z \leq y [p(x, z) \cdot \text{gt}(z, \mu z' \leq y [p(x, z')])]$$

The PRF gt ("greater than") makes sure we skip the first value.

$$c) \text{ Let } f'(x, y) = \mu z \leq y [p(x, y - z)]$$

reverses the order of examination
(i.e. we go from y down to 0)

Then:

$$f_3(x, y) = \text{eq}(y+1, \mu z \leq y [p(x, z)]) \cdot (y+1) \\ + \text{neq}(y+1, \mu z \leq y [p(x, z)]) \cdot f'(x, y)$$

It checks whether there is a z such that $z \leq y$ and $p(x, z) = \text{true}$, and outputs $f'(x, y)$ if it is the case and $y+1$ otherwise.

Exercise 4

Consider integer division $\text{div}(x, y)$: it's not defined for 0, hence not total and hence not PR. Let

$$\text{quo}(x, y) = \begin{cases} 0 & \text{if } y=0 \\ \text{div}(x, y) & \text{otherwise} \end{cases}$$

- a) Define $\text{quo}(x, y)$ using bounded minimization.
 b) Show that remainder, divides, number of divisors, and prime are primitive recursive.

Solution:

a) $\text{quo}(x, y) = \text{sg}(y) \cdot \mu z \leq x [\text{gt}((z+1) \cdot y, x)]$

b) Remainder:

$$\text{rem}(x, y) = x - (y \cdot \text{quo}(x, y))$$

Divides:

$$\text{divides}(x, y) = \begin{cases} 1 & \text{if } x > 0, y > 0, \text{ and } y \text{ is a divisor of } x \\ 0 & \text{otherwise} \end{cases}$$

$$\text{divides}(x, y) = \text{eq}(\text{rem}(x, y), 0) \cdot \text{sg}(x)$$

Number of divisors:

$$\text{ndivisors}(x, y) = \sum_{i=0}^x \text{divides}(x, y)$$

Prime:

$$\text{prime}(x) = \begin{cases} 1 & \text{if } x \text{ is prime} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{prime}(x) = \text{eq}(\text{ndivisors}(x), 2)$$

Exercise 5

Show that the function $pn(i)$ computing the i -th prime is PR by exploiting the fact that $pn(x+1) \leq pn(x)! + 1$.

Solution:

$$\begin{cases} pn(0) = 2 \\ pn(x+1) = \mu z \leq (pn(x)! + 1) [\text{prime}(z) \cdot gt(z, pn(x))] \end{cases}$$

Exercise 6

Show that the Ackermann function

$$\begin{cases} A(0, y) = y + 1 \\ A(x+1, 0) = A(x, 1) \\ A(x+1, y+1) = A(x, A(x+1, y)) \end{cases}$$

is defined for every pair $x, y \in \mathbb{N}$.

Solution:

By induction on x (main induction).

Base case: $A(0, y) = y + 1$

Inductive step: By induction on y (secondary induction)

$A(x+1, y)$

Base case: $A(x+1, 0) = A(x, 1)$ and the

$A(x+1, 0)$

main induction hypothesis applies

Inductive step: By the secondary induction

$A(x+1, y+1)$

hypothesis $A(x+1, y)$ is defined;

thus for $A(x+1, y+1) = A(x, A(x+1, y))$ the main induction hypothesis applies

Exercise 7

Define a primitive recursive function $f: \mathbb{N} \rightarrow \mathbb{N}$ that counts the number of occurrences of the digit 5 in a natural number.

Solution:

We need some auxiliary primitive recursive functions

- exponential $m^n: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$

$$\text{exp}(m, n) = \begin{cases} \text{exp}(m, 0) = 1 \\ \text{exp}(m, n+1) = \text{exp}(m, n) \cdot m \end{cases}$$

- length (number of digits) : $\mathbb{N} \rightarrow \mathbb{N}$

$$\text{length}(n) = (\mu z \leq n [\text{gt}(10^{z+1}, n)]) + 1$$

examples : $\text{length}(0) = \text{length}(1) = \dots = \text{length}(9) = 1$, $\text{length}(10) = 2, \dots$

$f: \mathbb{N} \rightarrow \mathbb{N}$ is then defined as follows

$$f(n) = \sum_{i=1}^{\text{length}(n)} \text{eq}(5, \text{rem}(\text{quo}(n, 10^{i-1}), 10))$$

Example : $f(253) =$

$\text{eq}(5, \text{rem}(\text{quo}(253, 1), 10))$	$\underbrace{253}_{3}$	0
	+	
$+ \text{eq}(5, \text{rem}(\text{quo}(253, 10), 10))$	$\underbrace{25}_{5}$	1
	+	
$+ \text{eq}(5, \text{rem}(\text{quo}(253, 100), 10))$	$\underbrace{2}_{2}$	0
		1

$= 0 + 1 + 0 = 1$

Exercise 8

Define a primitive recursive function $f: \mathbb{N} \rightarrow \mathbb{N}$ that reverses the digits of a natural number, i.e. $f(253) = 352$, $f(5524) = 4255$.

Solution:

$$f(n) = \sum_{i=1}^{\text{length}(n)} (\text{quo}(\text{rem}(n, 10^{\text{length}(n)-i+1}), 10^{\text{length}(n)-i}) \cdot 10^{i-1})$$

Example: $f(5524) =$

$$\begin{aligned} & \underbrace{\text{quo}(\text{rem}(5524, 10000), 1000)}_5 \cdot 1 \\ & + \underbrace{\text{quo}(\text{rem}(5524, 1000), 100)}_5 \cdot 10 \\ & + \underbrace{\text{quo}(\text{rem}(5524, 100), 10)}_2 \cdot 100 \\ & + \underbrace{\text{quo}(\text{rem}(5524, 10), 1)}_4 \cdot 1000 \\ & = 5 + 50 + 200 + 4000 = 4255 \end{aligned}$$