5. Reasoning in Description Logics

Exercise 5.1 Let $\mathcal{T}$ be a TBox consisting of concept inclusions of the form $A_1 \sqsubseteq A_2$ and concept disjointness assertion of the form $A_1 \sqsubseteq \neg A_2$, for atomic concepts A_1 and A_2.

Describe an algorithm for checking concept satisfiability with respect to $\mathcal{T}$, i.e., whether for some concept A it holds that A is satisfiable with respect to $\mathcal{T}$.

Exercise 5.2 Consider TBoxes $\mathcal{T}$ consisting of axioms of the form $B_1 \sqsubseteq B_2$, where

$$B_1, B_2 \ ::= \ A \ \mid \ \exists R \ \mid \ \exists R^-,$$

A denotes an atomic concept, and R an atomic role.

1. Describe an algorithm for checking subsumption with respect to a given $\mathcal{T}$, i.e., whether for two concepts B_1 and B_2 it holds that $\mathcal{T} \models B_1 \sqsubseteq B_2$.
2. Let $\mathcal{A} = \{A_0(a)\}$ and $\mathcal{T}$ a(n arbitrary) TBox of the above form. Can we determine whether $\langle \mathcal{T}, \mathcal{A} \rangle$ is satisfiable?

Exercise 5.3 Show that concept satisfiability in $\mathcal{ALC}$ is NP-hard.

Hint: show the claim by reduction from SAT.

Exercise 5.4 Let q_n, for $n \geq 2$, be a Boolean conjunctive query with n existential variables of the form $\exists x_1, \ldots, x_n.\ P(x_1, x_2) \wedge \cdots \wedge P(x_{n-1}, x_n)$. Given $n \geq 2$:

1. construct an $\mathcal{ALC}$ KB $\mathcal{K}_n$ such that $\mathcal{K}_n \models q_n$.
2. construct an $\mathcal{ALC}$ KB $\mathcal{K}'_{2^n}$ of size polynomial in n such that $\mathcal{K}'_{2^n} \models q_{2^n}$ and $\mathcal{K}'_{2^n} \not\models q_{2^n+1}$.

   Hint: $\mathcal{K}'_{2^n}$ "implements" a binary counter by means of n atomic concepts representing the bits of the counter, and such that the models of $\mathcal{K}'_{2^n}$ contain a P-chain of objects of length 2^n.