6/12/2011
E 8.1

Exercise: Reduction from 3-SAT to CSAT

(see textbook 10.3.4)

Given a CNF formula $E = C_1 \cdot C_2 \cdots C_k$

with each $C_i = \sum_{j=1}^{k_i} l_{ij}$,

we construct a 3-CNF formula F as follows.

For each clause C_i of E

1) if $C_i = (l)$ (i.e., a single literal)
introduce two new variables u, v, and replace C_i by 4 clauses

$(l + u + v) \cdot$
$(l + u + \bar{v}) \cdot$
$(l + \bar{u} + v) \cdot$
$(l + \bar{u} + \bar{v})$

Since u, v appear in all 4 combinations, the 4 clauses can be satisfied only if l is true

2) if $C_i = (l_1 + l_2)$
introduce a new variable z, and replace C_i by 2 clause

$(l_1 + l_2 + z) \cdot$
$(l_1 + l_2 + \bar{z})$

as in 1

3) if $C_i = (l_1 + l_2 + l_3)$, just leave it

4) if $C_i = (l_1 + l_2 + \cdots + l_m)$ with $m \geq 4$
introduce $y_1, y_2, \ldots, y_{m-3}$ and replace C_i by

$(l_1 + l_2 + y_1) \cdot (l_3 + \bar{y}_1 + y_2) \cdot (l_4 + \bar{y}_2 + y_3) \cdots$
$+ (l_{m-2} + \bar{y}_{m-4} + y_{m-3}) \cdot (l_{m-1} + l_m + \bar{y}_{m-3})$

- An assignment T satisfying E makes at least one literal of C_i true. Let it be 'l_j'.
  Then, by making $y_1, \dots, y_{j-2}$ true and
  $y_{j-1}, \dots, y_{m-3}$ false,
  we satisfy all clauses replacing C_i.
  Thus we can extend T to satisfy F.
- Conversely, if T makes all l_j of C_i false, then not all new clauses can be satisfied.
  Why? each y_j can make at most 1 clause true, but there are $m-2$ clauses and
  $m-3$ y_j's.

The 3-CNF formula F is linear in E and can be constructed in linear time

We get: CSAT $<_{poly}$ 3-SAT

$\Rightarrow$ from CSAT NP-hard, we get 3-SAT NP-hard

We also know 3-SAT $\in$ P (since SAT $\in$ P)

$\Rightarrow$ 3-SAT is NP-complete

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Exercise: 2SAT is in P

Idea: we show that 2SAT can be encoded as a graph reachability problem, and then use an algorithm for graph reachability

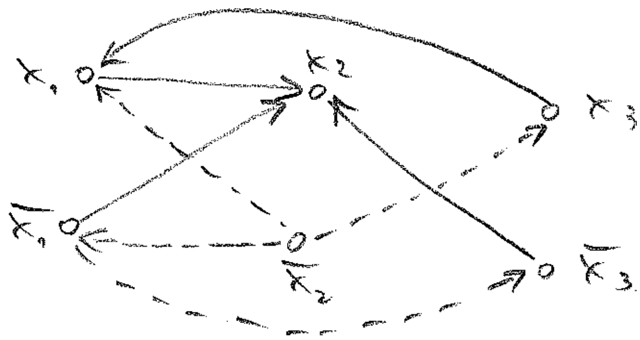
1) Encoding of 2SAT as a directed graph reachability problem

Let ϕ be an instance of 2SAT. We define a graph $G(\phi)$ as follows: – one node for each variable and for each negated variable

– for each clause $\alpha \vee \beta$ two edges $\bar{\alpha} \to \beta$
$\bar{\beta} \to \alpha$

(note: $\alpha \vee \beta \equiv \bar{\alpha} \to \beta \equiv \bar{\beta} \vee \alpha$)

Example: $(x_1 + x_2) \cdot$
$(x_1 + \bar{x}_3) \cdot$
$(\bar{x}_1 + x_2) \cdot$
$(x_2 + x_3)$



Then ϕ is unsatisfiable iff there is a variable x such that $G(\phi)$ contains two paths $x \to \cdots \to \bar{x}$, $\bar{x} \to \cdots \to x$

'$\Leftarrow$' Suppose that ϕ has a satisfying truth assignment T. Assume that $T(x) = \text{true}$ (a similar argument holds for $T(x) = \text{false}$)

Since $T(x) = \text{true}$ and $T(\bar{x}) = \text{false}$, and there is a path $x \to \cdots \to \bar{x}$, there must be an edge $\alpha \to \beta$ along this path with $T(\alpha) = \text{true}$ and $T(\beta) = \text{false}$.

However, since $\alpha \to \beta$ is an edge of $G(\Phi)$, it follows that $\bar{\alpha} + \beta$ is a clause of Φ. This clause is not satisfied by T_1 which is a contradiction.

"$\Rightarrow$" Let $G(\Phi)$ be a graph that does not contain any node x with $x \to \dots \to \bar{x} \to \dots \to x$

We construct from such a graph a satisfying truth assignment T

Repeat the following step as often as possible:

Choose a node α such that

- $T(\alpha)$ is not yet defined, and
- there is no path $\alpha \xrightarrow{*} \bar{\alpha}$

For every node β that is reachable from α

1) set $T(\beta)$ = true
2) set $T(\bar{\beta})$ = false

(Note: 2) means to assign false to all predecessors of $\bar{\alpha}$) $\alpha \xrightarrow{*} \beta$, $\bar{\alpha} \xleftarrow{*} \bar{\beta}$

Observe: 1) the truth assignment T is well defined, i.e., we never have both $T(\beta)$ = true and $T(\bar{\beta})$ = true or $T(\beta)$ = false and $T(\bar{\beta})$ = false

T would not be well defined, if we had both $\alpha \xrightarrow{*} \beta$ and $\alpha \xrightarrow{*} \bar{\beta}$ (for some β)

But this cannot happen, since we would have

$$\begin{array}{ccc} \alpha & \xrightarrow{*} & \beta \\ & \searrow^{*} \swarrow & \\ \bar{\alpha} & \longleftarrow & \bar{\beta} \end{array}$$

Hence, we would have $\alpha \xrightarrow{*} \bar{\alpha}$

2) We assign to <u>all</u> nodes a truth value, since there is no path $\alpha \xrightarrow{*} \bar{\alpha} \xrightarrow{*} \alpha$

3) The truth assignment satisfies all clauses of F, since each clause corresponds to an implication, and there is no $\alpha \leftrightarrows \bar{\alpha}$.

<u>Exercise</u>: Let $G = (V, E)$ be an undirected graph E 8.5

A vertex cover C of G is a subset of the vertices V s.t. every edge of G touches at least one of the nodes of C.

The vertex cover problem:

input: - graph $G = (V, E)$

- integer k

output: yes iff G has a vertex cover of size $\leq k$

Vertex-cover is NP-complete:

Proof:

in NP: easy

- guess a subset C of V of size $\leq k$
- check in poly-time that it is a vertex-cover

NP-hard by reduction from 3-SAT

We define a poly-time reduction R that:

- takes as input a 3-CNF formula F
- constructs a graph $G = (V, E)$ and an integer k

such that:

F is satisfiable $\Leftrightarrow$ G admits a vertex cover with k nodes

Let $F = C_1 \cdot \dots \cdot C_m$ be a 3-CNF formula over variables $\{x_1, \dots, x_n\}$

We construct $G = (V, E)$ as constituted by various components

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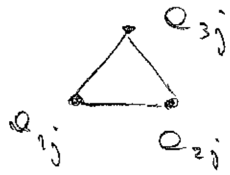
- For each variable x_i, we have a truth-setting component $T_i = (V_i, E_i)$ with $V_i = \{x_i, \bar{x}_i\}$

  $E_i = \{\{x_i, \bar{x}_i\}\}$

  undirected edge

  note: at least one of x_i, $\bar{x}_i$ will be in every vertex cover to cover $\{x_i, \bar{x}_i\}$

- For each clause C_j in F we have a satisfaction testing component $S_j = (V'_j, E'_j)$



  note: at least two of V'_j will be in every vertex cover to cover E'_j

- We have a communication component, which is the only part that depends on which literals are in which clauses

  Let $C_j = l_{1j} + l_{2j} + l_{3j}$

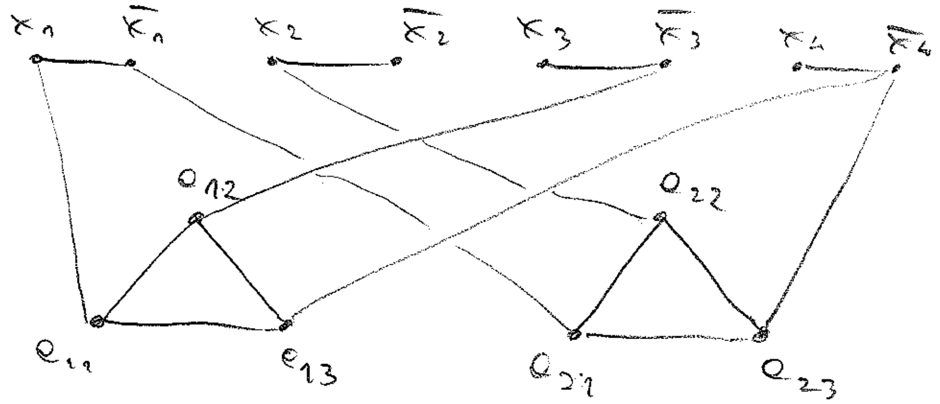
  Then we have $E''_j = \{\{a_{1j}, l_{1j}\}, \{a_{2j}, l_{2j}\}, \{a_{3j}, l_{3j}\}\}$

We then set $K = n + 2m$

\# clauses

\# variables

Example: $F = (x_1 + \overline{x}_3 + \overline{x}_4) \cdot (\overline{x}_1 + x_2 + \overline{x}_4)$



$k = n + 2m = 4 + 2 \cdot 2 = 8$

We show that F is satisfiable $\Leftrightarrow$ G has a vertex cover of size $\leq k$

"$\Leftarrow$" Let $V' \subseteq V$ be a vertex cover for G with $|V'| \leq k$.

We said that V contains at least
- one vertex for each variable
- 2 vertices for each clause

This is already $k = n + 2m$

$\Rightarrow$ at least is actually exactly

We use V' to obtain the truth assignment γ

we set x_i = true if $x_i \in V'$

x_i = false if $x_i \notin V'$

To show that γ is a truth assignment that satisfies F, we exploit that all arcs of the communication components are covered by V'.

Consider a clause $C_j = l_{1j} + l_{2j} + l_{3j}$

- two of the arcs in E''_j are covered by the choice of two among a_{1j}, a_{2j}, a_{3j} in V'.

W.l.o.g., let these be a_{1j}, a_{2j}

- the third arc is then covered by the literal l_{3j} (connected to e_{3j}) which has to be in V'

Since, by definition of γ, l_{3j} = true, C_j is satisfied

"$\Rightarrow$" Let γ be a truth assignment that satisfies F.

We define a subset $V' \subseteq V$ as follows

- $x_i \in V'$ iff $\gamma(x_i)$ = true
  $\overline{x}_i \in V'$ iff $\gamma(x_i)$ = false

Since γ satisfies F, for each communication component $E''_j = \{\{e_{1j}, l_{1j}\}, \{e_{2j}, l_{2j}\}, \{e_{3j}, l_{3j}\}\}$, one of the three edges $\{e_{ij}, l_{ij}\}$ is covered in V' by l_{ij}.
W.l.o.g., let $i = 1$. Then $\{e_{2j}, l_{2j}\}$, $\{e_{3j}, l_{3j}\}$ can be covered by having $e_{2j} \in V'$ and $e_{3j} \in V'$.

We get that V' contains $n + 2m$ vertices.