

Turing Machines computing functions:

We consider number-theoretic functions $f: \mathbb{N}^k \rightarrow \mathbb{N}$

- natural numbers are represented in unary

n is represented by $\bar{n} = 1^{n+1}$

e.g.

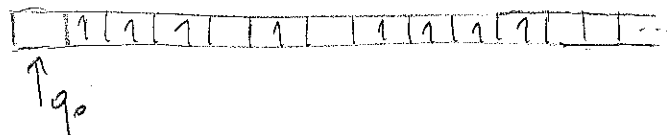
n	\bar{n}
0	1
1	11
2	111
\vdots	\vdots

Hence, the input alphabet of a TM is $\Sigma = \{1\}$

- consecutive numbers on the input are separated by a $\$$

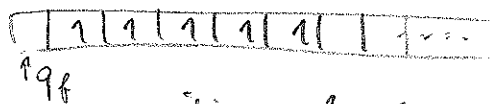
- the TM starts its computation with the head placed on a $\$$ preceding the first number

e.g. for $f(2, 0, 3)$



- when the computation terminates, the input has been replaced on the tape with the result of the function

e.g. if $f(2, 0, 3) = 4$

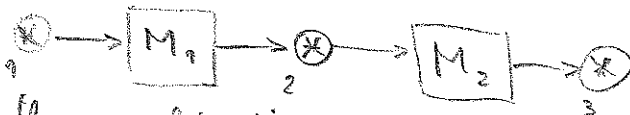


We make some assumptions to ease composition of TMs:

- 1) there is a single final state q_f
- 2) the only transition from q_0 is $\delta(q_0, \$) = (q_1, \$, R)$
- 3) there are no transitions entering q_0 or of the form $\delta(q_f, \$)$
- 4) the computation loops whenever $f(n) \uparrow$, i.e. $f(n)$ is undefined

This allows us to sequentially compose TMs:

represented by the diagram



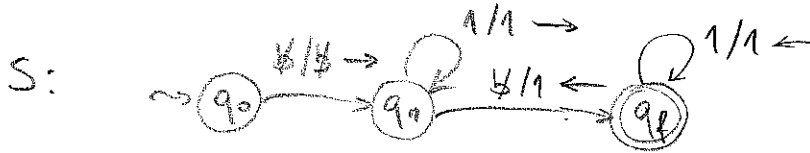
1... initial state of M_1 end of the combination

3... final state of M_2

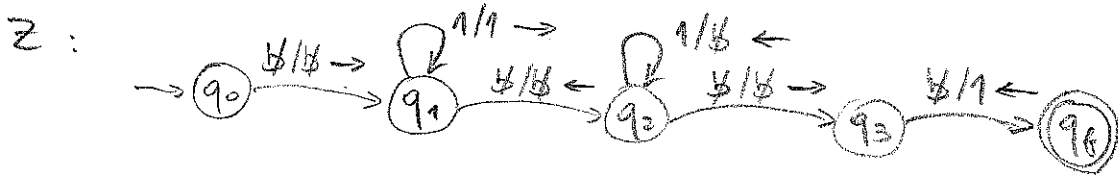
2... final state of M_1 is also the initial state of M_2

1) Construct a TM computing the successor function

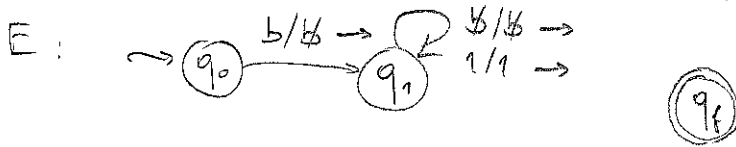
$$s(n) = n + 1$$



2) Construct a TM computing the zero function $z(n) = 0$



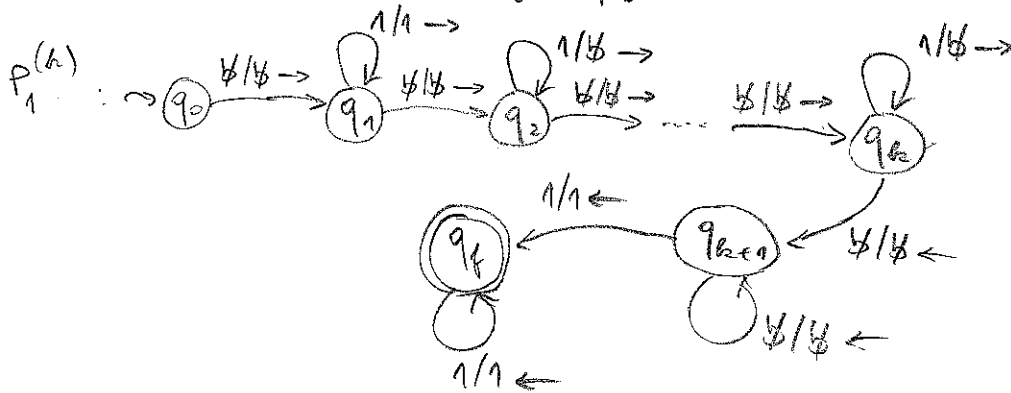
3) Construct a TM computing the empty function $e(n) \uparrow$ i.e., the function that is undefined for every $n \in \mathbb{N}$



The k -variable projection function $\pi_i^{(k)}$ is defined as

$$\pi_i^{(k)}(a_1, \dots, a_k) = a_i \quad (\text{for } 1 \leq i \leq k)$$

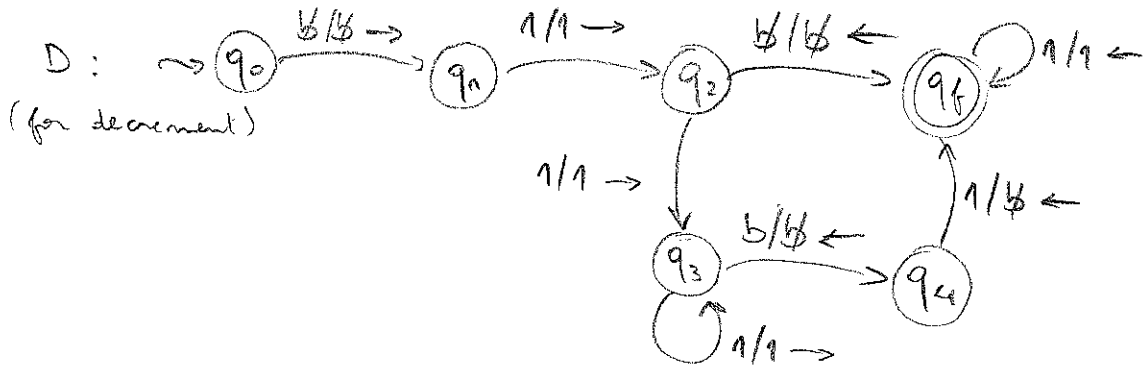
4) Construct a TM computing $\pi_1^{(k)}$



Note $\pi_1^{(1)}$ is also called the identity function $id(n) = n$

5) Construct a TM computing the predecessor function

$$\text{pred}(n) = \begin{cases} 0 & \text{if } n=0 \\ n-1 & \text{if } n>0 \end{cases}$$

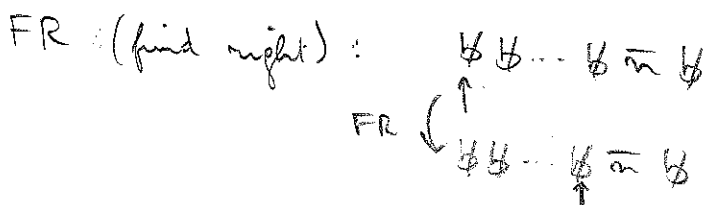
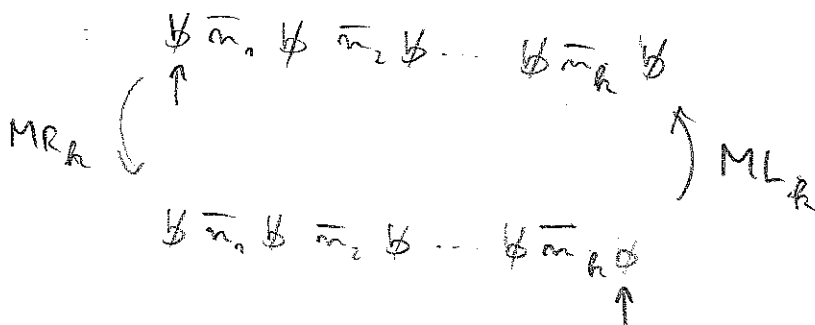
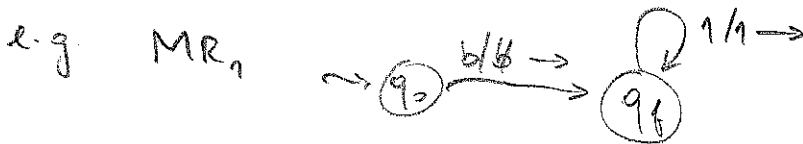


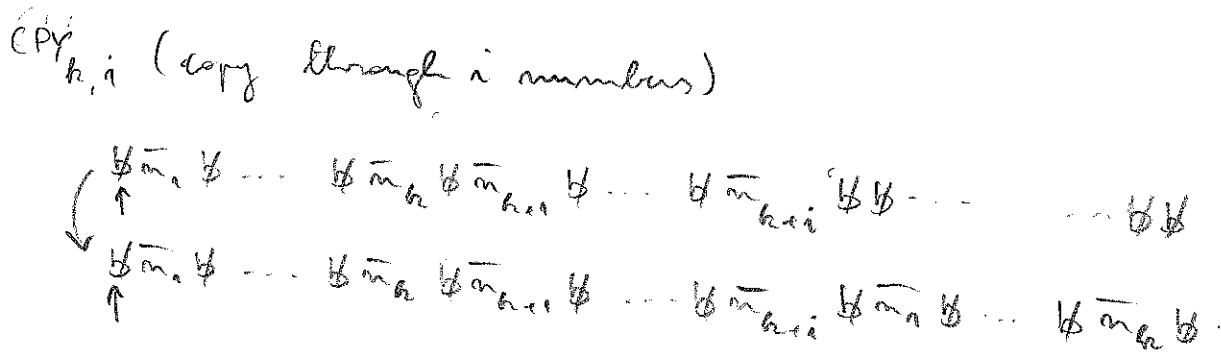
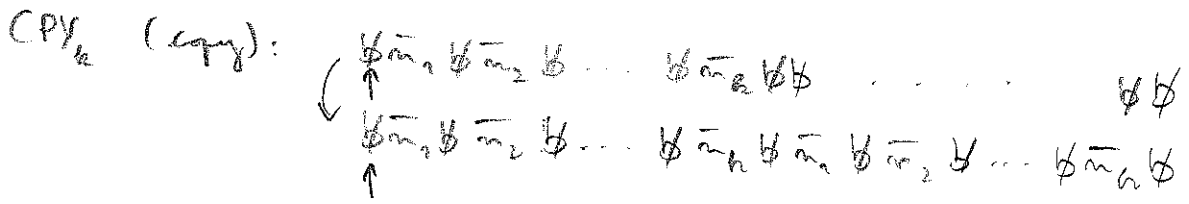
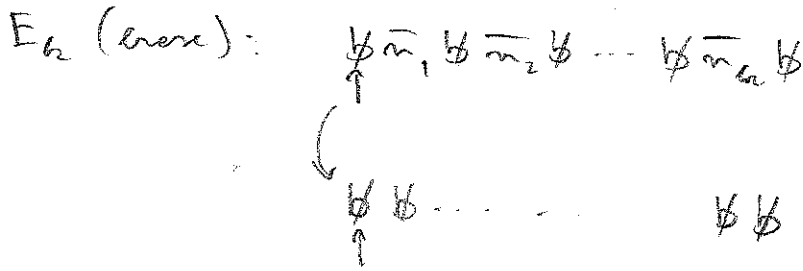
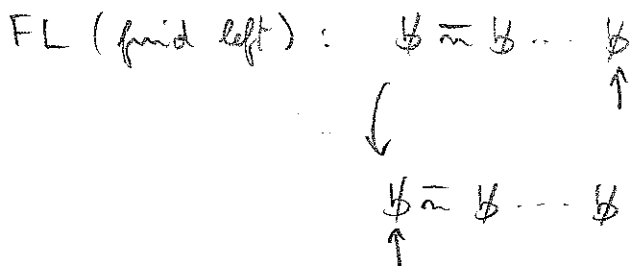
6) Using sequential composition, construct a TM computing the constant function $c(n) = 1$



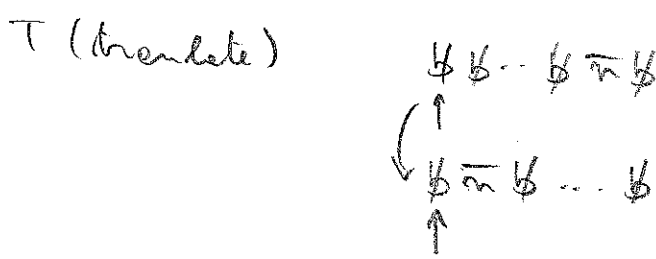
We now define some TMs that can be used as macros:

MR_k : move the tape head to the right through k consecutive natural numbers

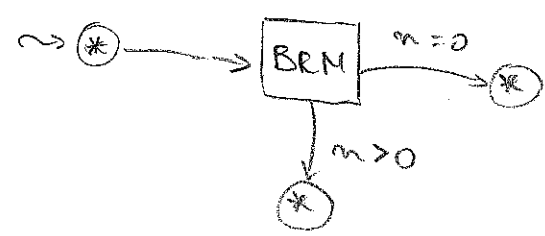




for CPY_k and $CPY_{k,i}$ the blank portion following $\bar{m}_1 \emptyset \dots \emptyset \bar{m}_k$ is assumed to be long enough to contain the copy.



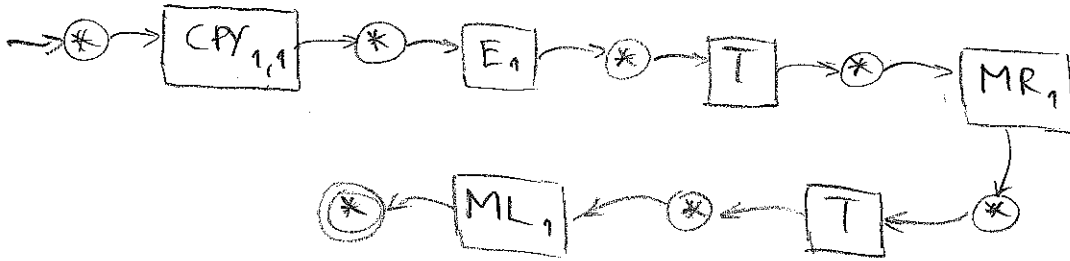
BRN (branch on zero)



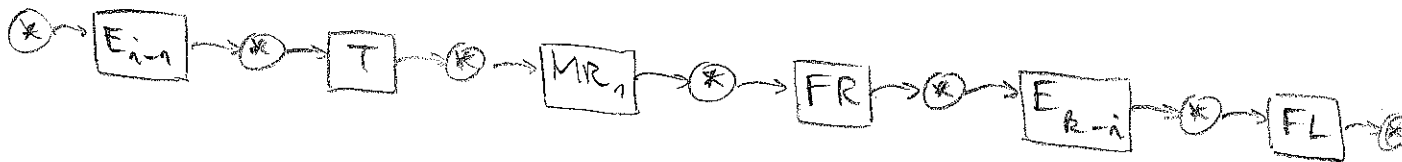
it does not alter the tape or change the head position

7) Using composition and the defined macros, construct a TM INT (interchange):

$$\begin{array}{c} \$ \bar{n} \$ m \$ b^{n+1} \$ \\ \uparrow \\ \$ \bar{m} \$ n \$ b^{m+1} \$ \end{array}$$



8) Using composition and the defined macros, construct a TM for $p_i(b)$



Function composition:

E 5.6

$$\text{Let } g_i : \mathbb{N}^k \rightarrow \mathbb{N} \text{ for } 1 \leq i \leq m$$

$$h : \mathbb{N}^m \rightarrow \mathbb{N}$$

The composition of h with g_1, \dots, g_m , written

$$f = h \circ (g_1, \dots, g_m)$$

is the function $f : \mathbb{N}^k \rightarrow \mathbb{N}$ defined by

$$f(x_1, \dots, x_k) = h(g_1(x_1, \dots, x_k), \dots, g_m(x_1, \dots, x_k))$$

The function $f(x_1, \dots, x_k)$ is undefined if either

1) $g_i(x_1, \dots, x_k) \uparrow$ for some $i \in \{1, \dots, m\}$

2) $g_i(x_1, \dots, x_k) = y_i$ for $i \in \{1, \dots, m\}$ and $h(y_1, \dots, y_m) \uparrow$

We show that the composition of Turing-computable functions is also Turing-computable.

Exercise: given $g_1 : \mathbb{N}^3 \rightarrow \mathbb{N}$, $g_2 : \mathbb{N}^3 \rightarrow \mathbb{N}$, $h : \mathbb{N}^2 \rightarrow \mathbb{N}$

and TMs G_1, G_2, H computing respectively g_1, g_2, h ,

construct a TM computing $h \circ (g_1, g_2)$.

Use macros and TM composition, and construct the configurations obtained after every macro application.

macro configuration

$$\underline{\$} \bar{m}_1 \$ \bar{m}_2 \$ \bar{m}_3 \$$$

$$CPY_3 \quad \underline{\$} \bar{m}_1 \$ \bar{m}_2 \$ \bar{m}_3 \$ \bar{m}_1 \$ \bar{m}_2 \$ \bar{m}_3 \$$$

$$MR_3 \quad \$ - \dots - \underline{\$} - \dots - \$$$

$$G_1 \quad \$ - \dots - \underline{\$} \bar{y}_1 \$ \quad (\text{where } y_1 = g_1(m_1, m_2, m_3))$$

$$ML_3 \quad \underline{\$} \bar{m}_1 \$ \bar{m}_2 \$ \bar{m}_3 \$ \bar{y}_1 \$$$

$$CPY_{3,1} \quad \underline{\$} - \dots - \underline{\$} \bar{m}_1 \$ \bar{m}_2 \$ \bar{m}_3 \$$$

$$MR_4 \quad \$ - \dots - \underline{\$} - \dots - \$$$

$$G_2 \quad \$ - \dots - \underline{\$} \bar{y}_1 \$ \bar{y}_2 \$ \quad (\text{where } y_2 = g_2(m_1, m_2, m_3))$$

$$ML_1 \quad \$ - \dots - \underline{\$} \bar{y}_1 \$ \bar{y}_2 \$$$

$$H \quad \$ - \dots - \underline{\$} \bar{z} \$ \quad (\text{where } z = h(y_1, y_2))$$

$$ML_3 \quad \underline{\$} - \dots - \underline{\$} \bar{z} \$$$

$$E_3 \quad \underline{\$} \$ \dots \underline{\$} \bar{z} \$$$

$$T \quad \$ \bar{z} \$$$