

Exercise (Example 8.2 from textbook)

Construct a Turing Machine accepting the language

$$\{0^n 1^n \mid n \geq 1\}$$

Solution

The idea is that the TM M that we construct reads the leftmost 0, turns it into x , and moves right until it reaches a 1, that is turned into y . Then the head moves left again to the leftmost 0 (on the right to a x), and starts again until all 0's and 1's are turned into x 's and y 's respectively.

If the input is not in 0^*1^* , M will fail to find a move and it won't accept. If M changes the last 0 and the last 1 in the same round, it will go into the final state and accept.

$$Q = \{q_0, q_1, q_2, q_3, q_4\}$$

$$\Sigma = \{0, 1\}$$

$$\Gamma = \{0, 1, x, y, \epsilon\}$$

(ϵ denotes blank symbol)

q_0 : start state

$$F = \{q_4\}$$

In q_0 is the state in which M is when the head precedes the leftmost 0. In state q_1 , M moves right skipping 0's and 1's until it gets to a 1. In state q_2 , M moves left while skipping y 's and 0's again, until it gets to a x and goes again in q_0 .

Starting from q_0 , if a Y is read instead of a 0 ,
 it goes in q_3 and moves right: if a 1 is found,
 then there are more 1 's than 0 's; if a b is read,
 then the initial string is accepted (transition to q_4).

	0	1	X	Y	b
q_0	(q_1, X, R)	—	—	(q_3, Y, R)	—
q_1	$(q_2, 0, R)$	(q_2, Y, L)	—	(q_1, Y, R)	—
q_2	$(q_2, 0, L)$	—	(q_0, X, R)	(q_2, Y, L)	—
q_3	—	—	—	(q_3, Y, R)	(q_4, b, R)
q_4	—	—	—	—	—

Exercise

Show the computation of the TM above when the
 input string is:

- (a) 00
- (b) 000111

Solution

(a) $q_0 00 \vdash X q_1 0 \vdash X 0 q_1$
 and the TM halts

(b) $q_0 000111 \vdash X q_1 00111 \vdash X 0 q_1 0111 \vdash$
 $X 0 0 q_1 111 \vdash X 0 q_2 0 Y 11 \vdash X q_2 0 0 Y 11 \vdash q_2 X 0 0 Y 11 \vdash$
 $X q_0 0 0 Y 11 \vdash X X q_1 0 Y 11 \vdash X X 0 q_2 Y 11 \vdash X X 0 Y q_2 11 \vdash$
 $X X 0 q_2 Y Y 1 \vdash X X q_2 0 Y Y 1 \vdash X q_2 X 0 Y Y 1 \vdash X X q_0 0 Y Y 1 \vdash$
 $X X X q_1 Y Y 1 \vdash X X X Y q_2 Y 1 \vdash X X X Y Y q_2 1 \vdash X X X Y q_2 Y Y \vdash$
 $X X X q_2 Y Y Y \vdash X X q_2 X Y Y Y \vdash X X X q_0 Y Y Y \vdash X X X Y q_3 Y Y \vdash$
 $X X X Y Y q_3 Y \vdash X X X Y Y Y q_3 b \vdash X X X Y Y Y b q_4 b$

Exercise (8.2.3 from textbook) :

Design a Turing Machine that takes as input a number N in binary and turns it into $N+1$ (in binary); the number N is preceded by the symbol $\$$, which may be destroyed during the computation. For example, $\$111$ is turned into 1000 ; $\$1001$ is turned into $\$1010$.

solution

The idea is to toggle the rightmost digit, and, from right to left, all consecutive 1's until we get to the first 0 (which is also toggled). If there is no 0 to be toggled, a 1 is added on the left of the first digit (i.e., in place of the $\$$).

We need three states, where only q_2 is the final state; we briefly describe what the TM does in the different states.

q_0 : the TM goes right until it reaches $\$$, after the rightmost digit. When $\$$ is reached, the TM goes into q_1 .

q_1 : goes left toggling all 1's and the first 0 (from right); when 0 or $\$$ is reached, the symbol is turned into 1.

q_2 : final state; the TM does nothing.

	$\$$	0	1	\bar{b}
q_0	$(q_0, \$, R)$	$(q_0, 0, R)$	$(q_0, 1, R)$	(q_1, \bar{b}, L)
q_1	$(q_2, 1, L)$	$(q_2, 1, L)$	$(q_1, 0, L)$	—
q_2	—	—	—	—

Exercise (8.22 from textbook)

Design Turing machines accepting the following languages:

$$\{w \in \{0,1\}^* \mid w \text{ has an equal number of 0's and 1's}\}$$

Solution

The idea is that the head of our TM M moves back and forth on the tape, "deleting" one 0 for each 1; if there are no 0's and 1's in the end, the string is accepted.

When in state q_1 , M has found a 1 and looks for a 0; in state q_2 it's the other way around.

Note that the head never moves left of any x , so that there are never unmatched 0's and 1's on the left of an x .

From initial state q_0 , M picks up a 0 or a 1 and turns it into x . The only final state is q_4 . In state q_3 , M moves head left looking for the rightmost x .

	0	1	$\bar{0}$	x	$\bar{1}$
q_0	(q_2, x, R)	(q_1, x, R)	$(q_4, \bar{0}, R)$	—	$(q_0, \bar{1}, R)$
q_1	$(q_3, \bar{1}, L)$	$(q_1, 1, R)$	—	—	$(q_2, \bar{1}, R)$
q_2	$(q_2, 0, R)$	$(q_3, \bar{1}, L)$	—	—	$(q_2, \bar{1}, R)$
q_3	$(q_3, 0, L)$	$(q_3, 1, L)$	—	(q_0, x, R)	$(q_3, \bar{1}, L)$
q_4	—	—	—	—	—

Exercise (8.4.5 from textbook)

Suppose you have a tape with all b 's except a single $\$$, with the head in some (unknown) position. (a) Write a NTM able to enter into a final state (starting from initial state) by scanning $\$$.

(b) Then, write a deterministic TM doing the same job.

Solution

(a) The TM just needs to guess whether $\$$ is on the left or on the right. We call q, q_f the two states (q_f is final).

$$\delta(q, b) = \{(q, b, L), (q, b, R)\}$$

$$\delta(q, \$) = \{(q_f, \$, R)\}$$

(b) The deterministic TM goes back and forth examining one new position on the tape on the left, and then one on the right; marked symbols are turned from b to $\#$.

	b	$\#$	$\$$
q_0	$(q_1, \#, L)$	$(q_0, \#, R)$	$(q_2, \$, R)$
q_1	$(q_0, \#, R)$	$(q_1, \#, L)$	$(q_2, \$, R)$
q_2	—	—	—

In q_0 , the TM looks for the next b on the right, while in q_1 it looks for the next one on the left. When a b is reached, it is turned into $\#$ and the search starts over in the opposite direction.