15/12/2010
E 9.1

Exercise (10.3.2): The problem 4TA-SAT is defined as follows:

Given a prop. formula E, does E have at least 4 satisfying truth assignments?

Show that 4TA-SAT is NP-complete.

Proof:

1) 4TA-SAT is in NP

We devise a non-deterministic poly-time algorithm.

1) Guess 4 truth-assignments T_1, T_2, T_3, T_4

2) Check that T_1, T_2, T_3, T_4 all satisfy E.

Note that both steps require time polynomial in the size of E

2) 4TA-SAT is NP-hard

We show this by reducing SAT to 4TA-SAT.

Let E be a prop. formula, and let $x_1, \dots, x_n$ be all variables in E.

We construct a new formula E' s.t.

$$E \in SAT \Leftrightarrow E' \in 4TA\text{-}SAT$$

Let y_1, y_2 be two new variables. Then

$$E' = E \vee ((x_1 \wedge x_2 \wedge \dots \wedge x_n) \wedge ((y_1 \wedge y_2) \vee (y_1 \wedge \overline{y_2}) \vee (\overline{y_1} \wedge y_2)))$$

E 9.2

Consider the truth assignments for $x_1, \dots, x_n, y_1, y_2$

	x_1	x_j	x_n	y_1	y_2	Case 1 $E \notin SAT$: E	Case 1 $E \notin SAT$: E'	Case 2 $E \in SAT$: E	Case 2 $E \in SAT$: E'
1)	T	...	T	T	T	F	T	?	T
2)	T	...	T	T	F	F	T	?	T
3)	T	...	T	F	T	F	T	?	T
4)	T	...	T	F	F	F	F	⋮	⋮
⋮	...	F	...	...		F	F	E	T
		⋮				⋮	⋮	⋮	⋮
2^{n+2}						F	F		
							$=3$		≥ 4

Alternative solution:

$$E' = E \wedge (y_1 \vee y_2 \vee y_3)$$

- If E is unsatisfiable, then E' is unsatisfiable, and hence $E' \notin$ 47A-SAT
- If E is satisfiable, then E' has at least 7 satisfying truth assignments: these are obtained by combining
  - a TA for $x_1, \dots, x_n$ satisfying E with
  - the 7 TAs for y_1, y_2, y_3 satisfying $y_1 \vee y_2 \vee y_3$

Exercise 11.1.1 b)

Consider the problem FALSE-SAT:

Given a boolean expression E that is false when all its variables are made false, is there some other truth assignment that makes E false, besides all-false?

Decide whether the problem is in NP or coNP.

Describe its complement.

If the problem or its complement is NP-complete, prove it.

Proof:

The problem is NP-complete.

- In NP: given a boolean expression E, we need to check:
  1) that E is false when all variables are assigned false
  2) that there is some other truth-assignment making E false

  (1) can be done in poly-time by a DTM
  (2) can be done in poly-time by a NTM
  guess a truth-assignment T different from all false, and answer yes if under T, E evaluates to false

- NP-hard: by a reduction from SAT
  Let E be a boolean expression with variables $x_1, \dots, x_n$.
  We construct an expression E' s.t. E ∈ SAT iff E' ∈ FALSE-SAT
  1) Test if E is true when all variables are false (polynomial)
  If so, E ∈ SAT, and we convert it to a fixed expression that is in FALSE-SAT, e.g. $x \wedge y$.

2) Otherwise, let E' be $\neg E \wedge (x_1 \vee x_2 \vee \cdots \vee x_n)$. E 9.4

Clearly, the reduction is poly-time.

We have that E' is false when all of $x_1, \ldots, x_n$ are false.

Notice that in case (2), E is false when all variables are false.

Hence, if $E \in SAT$, then it is satisfied by a truth assignment T different from all-false.

Thus, $\neg E$ is made false by T, and $E' \in$ FALSE-SAT.

Conversely, if $E' \in$ FALSE-SAT, then since $x_1 \vee \cdots \vee x_n$ is false only for the all-false truth assignment, there must be some other truth-assignment T that makes $\neg E$ false. Then T makes E true, and $E \in SAT$.