Basic notations and definitions

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Sets

Set:

- explicit notation e.g. $V = \{a, e, i, o, u\}$

  informally, we also use ... e.g. $\mathbb{N} = \{0, 1, 2, \ldots\}$

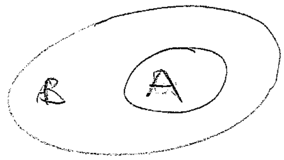
- using a set former, i.e. $\{x \mid E(x)\}$

  where $E(x)$ is a boolean expression depending on x

  e.g. $\{x \mid x \in \mathbb{N} \land x \geq 10 \land x \leq 50\}$

Subset: $A \subseteq B$ denotes that A is a subset of B (or A is contained in B)

i.e. $\forall x$: if $x \in A$ then $x \in B$



$A \subset B$ means $A \subseteq B$ and $A \neq B$

N.B. We may have sets whose elements are themselves sets

e.g. $A = \{\{0,1\}, \{0,2\}\}$

$B = \{\{0,1\}, \{0,2\}, \{1,2,3\}\}$

If $A \subseteq B$, this does not imply anything wrt the containment between $X \in A$ and $Y \in B$, e.g. $X \subseteq Y$

Powerset: of a set A: denoted 2^A

$$2^A = \{X \mid X \subseteq A\}$$

N.B. $X \in 2^A \iff X \subseteq A$

Set operations:

- intersection : $A \cap B = \{x \mid x \in A \wedge x \in B\}$
- union : $A \cup B = \{x \mid x \in A \vee x \in B\}$
- difference $A \setminus B = \{x \mid x \in A \wedge x \notin B\}$

When we refer to an implicit universe U, we may denote with $\bar{A}$ the complement of A (wrt U)

i.e. $\bar{A} = U \setminus A$ (e.g. $U = \mathbb{N}$ or $U = \Sigma^*$)

Cartesian product of sets $A_1, A_2, \dots, A_n$

$$A_1 \times A_2 \times \dots \times A_n = \{(x_1, \dots, x_n) \mid x_1 \in A_1 \wedge \dots \wedge x_n \in A_n\}$$

... set of n-tuples of elements respectively of $A_1, \dots, A_n$

Relations

- binary relation between two sets A and B

  $R \subseteq A \times B$

  e.g. $\leq \subseteq \mathbb{N} \times \mathbb{N}$ is defined as

  $$\leq = \{(x, y) \mid x \in \mathbb{N}, y \in \mathbb{N}, \exists k \in \mathbb{N} \text{ s.t. } x + k = y\}$$

- we may use infix notation: $(x, y) \in R \Leftrightarrow x R y$
  $(x, y) \in \leq \Leftrightarrow x \leq y$

- a relation $R \subseteq S \times S$, for some set S, is called a precedence relation
  - reflexive: $\forall a \in S: a R a$
  - symmetric: $\forall a, b \in S$: if aRb then bRa
  - transitive: $\forall a, b, c \in S$: if aRb and bRc then aRc
  - antisymmetric: $\forall a, b$: if aRb and bRa then $a = b$

- Types of precedence relations:
  - equivalence: reflexive, symmetric, and transitive
  - preorder: reflexive and transitive
  - partial order: antisymmetric preorder
  - total order on S: for all $x, y \in S$ either $x R y$ or $y R x$

When $\prec \in S \times S$ is a partial order (on S), we say also that $(S, \prec)$ is a *partially ordered* set.

- minimal element $x \in S$: $\forall y \in S : y \nprec x$
- maximal — " — — " — $x \nprec y$

- Transitive closure of $R \subseteq S \times S$, denoted R^+

$R^+ = \bigcup_{i \in \mathbb{N}, i \geq 1} R^i$, with

$$\begin{cases} R^1 = R \\ R^{i+1} = \{(a, c) \mid \exists b . (a, b) \in R^i \wedge (b, c) \in R\} \end{cases}$$

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Functions:

Consider an n-ary relation $R \subseteq A_1 \times \cdots \times A_n$ and $k < n$.
Then R is a k-argument *function* if
for each k-tuple $(x_1, \ldots, x_k) \in A_1 \times \cdots \times A_k$
there is a unique $(n-k)$-tuple $(x_{k+1}, \ldots, x_n) \in A_{k+1} \times \cdots \times A_n$
such that $(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) \in R$

We denote this as $R : A_1 \times \cdots \times A_k \to A_{k+1} \times \cdots \times A_n$

$A_1 \times \cdots \times A_k$ ... domain of R

$A_{k+1} \times \cdots \times A_n$ ... co-domain of R

We may use $\vec{x}$ to denote an n-tuple of elements, i.e.

$\vec{x} = (x_1, \dots, x_n)$ (where n depends on the context)

For simplicity we consider now just functions $f: A \to B$ (but the same holds for $f: A_1 \times \cdots \times A_k \to A_{k+1} \times \cdots \times A_n$)

Each $f: A \to B$ is also a relation $f \subseteq A \times B$.

The converse does in general not hold.

But we can associate to each $R \subseteq A \times B$ a function

$$f_R : A \to 2^B \quad \text{with } f_R(x) = \{ y \mid x R y \}$$

$f: A \to B$ is
- injective if $f(x_1) = f(x_2)$ implies $x_1 = x_2$
- surjective if $\forall y \in B, \exists x \in A : f(x) = y$
- bijective if it both injective and surjective

For $D \subseteq A$, $f(D)$ denotes the image of D via f, i.e.

$$f(D) = \{ y \mid \exists x \in D . f(x) = y \}$$

f^{-1} denotes the <u>inverse</u> of f.

f^{-1} may not be a function.

But we can always define for $D \subseteq B$ the <u>inverse image</u> of D

$$f^{-1}(D) = \{ x \mid x \in A \land f(x) \in D \}$$

Partial functions:

$f : A \to B$ is total if it is defined for every $x \in A$,

i.e. if $\forall x \in A : \exists y \in B : f(x) = y$ (i.e., $x f y$)

If f is not defined for some $x \in A$ it is called partial
(we denote partial functions with greek letters)

We use $A \to B$ to denote the set of total functions from A to B.

We use $\varphi(x)\downarrow$ when φ is defined on x
— " — $\varphi(x)\uparrow$ — " — is not defined — " —

Domain of φ : $\mathrm{dom}(\varphi) = \{x \mid \varphi(x)\downarrow\}$

Range of φ : $\mathrm{range}(\varphi) = \{x \mid \exists y.\ \varphi(y) = x \neq \uparrow\}$

(where $\uparrow$ denotes the undefined value)

Cardinality of sets:

$|S|$ denotes the cardinality of a set S

- when S is finite, then $|S|$ is the number of its elements
- when S is infinite, defining $|S|$ is more complicated

Definitions:

- A and B are equinumerous if there is a bijection $f : A \to B$, written $A \approx B$.
- Then $|S|$ denotes the collection of sets Y such that $Y \approx S$.
- $|A| \leq |B|$ if there is an injection $f : A \to B$

Easy: if $A \subseteq B$ then $|A| \leq |B|$ ($A < B$ if $A \leq B$ but $A \not\approx B$

Basic definitions about languages:

- **Alphabet**: finite, nonempty set of symbols: Σ

  e.g. $\Sigma = \{0, 1\}$

  $\Sigma = \{a, b, \dots, z\}$

  $\Sigma =$ set of Unicode characters

- **String**: finite sequence of symbols from Σ

  $w = a_1 a_2 \dots a_n$, with $a_i \in \Sigma$ for $i \in \{1, \dots, n\}$

  e.g. · 01101

  · ciaociao

- empty string: denoted ε: string with no symbols

- length of a string = number of (positions for) symbols in the string

  denoted $|w|$ If $w = a_1 \cdots a_n$, then $|w| = n$

  e.g. $|\varepsilon| = 0$ ε is the only string of length 0

  $|b| = 1$

  $|ciaociao| = 8$

  Notice: strictly speaking, the number of symbols in ciaociao is 4

- **Powers of an alphabet**:

  $\Sigma^k = \underbrace{\Sigma \times \Sigma \times \cdots \times \Sigma}_{k \text{ times}}$ ... set of all strings over Σ of length k

  e.g. $\Sigma^0 = \{\varepsilon\}$

  $\{0,1\}^1 = \{0,1\}$ ← what is the difference between lhs and rhs?

  $\{0,1\}^2 = \{00, 01, 10, 11\}$

<u>Closure</u> of an alphabet Σ : Σ^* is the set of all finite strings over Σ

i.e. $\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \ldots$

also $\Sigma^+ = \Sigma^1 \cup \Sigma^2 \ldots$ hence $\Sigma^* = \Sigma^0 \cup \Sigma^+$

Note: all strings in Σ^* are finite
Σ^* is an infinite set

e.g. $\Sigma = \{0, 1\}$
$\Sigma^* = \{0,1\}^* = \{\varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, \ldots\}$

<u>Concatenation</u> of two strings:

$x = a_1 a_2 \cdots a_m \in \Sigma^*$
$y = b_1 b_2 \cdots b_n \in \Sigma^*$

$\Rightarrow$ $x \cdot y = a_1 \cdots a_m b_1 \cdots b_n$ (we may omit the $\cdot$)

Note: $\varepsilon \cdot x = x \cdot \varepsilon = x$, i.e. ε is the identity for conc.
$|x \cdot y| = |x| + |y|$

<u>Language</u> L over Σ : is any subset of Σ^* (i.e. $L \subseteq \Sigma^*$)

Note: L contains only finite strings, but it may be infinite

<u>Examples</u>:

$\begin{cases} \Sigma = \{a, b, \ldots, z\} \\ L = \text{set of all English words} \end{cases}$

$\begin{cases} \Sigma = \text{Unicode characters} \\ L = \text{compilable Java programs} \end{cases}$

$\begin{cases} \Sigma = \{0, 1\} \\ L = \{\varepsilon, 01, 0011, 000111, \ldots\} \end{cases}$ all string with equal # of 0 and 1, with all 0's preceding the 1's

$\emptyset$ the empty language ($\neq \{\varepsilon\}$)