PROPERTIES OF REGULAR LANGUAGES

3/12/2010
E7.1

EXERCISE 1

Decide which of the following statements is true and which is false. Give a brief explanation of your answer.

a) For all languages L_1 and L_2, it holds that $(L_1^* \cdot L_2^*)^+ = (L_1^+ \cdot L_2^+)^*$.

b) If L_1 and L_2 are both not regular then $L_1 \cup L_2$ could be regular.

c) For all languages L_1 and L_2, if $L_1 \subseteq L_2$ then $L_1^* \subseteq L_2^*$.

EXERCISE 2

Show that the following languages are not regular.

a) $\{0^n 1^m 0^{n+m} \mid m, n \geq 0\}$ b) $\{w \in \{0,1\}^* \mid w \text{ is a palindrome}\}$

SOLUTIONS (12/12/2008)

1) a) False. Consider the languages $L_1 = \{a\}$ and $L_2 = \{b\}$. Then $b \in (L_1^* \cdot L_2^*)^+$ but $b \notin (L_1^+ \cdot L_2^+)^*$.

1) b) True. Assume that $L_1 = \overline{L_2}$, i.e. $L_2 = \overline{L_1}$. If L_1 is not regular then so is L_2 (because, if L_2 would be regular then, by the closure properties of regular languages, L_1 would be regular too, thus leading to a contradiction). Since $L_1 \cup L_2 = \Sigma^*$ we have that the union of two non-regular languages can be regular.

1) c) True. Given that, for all $w \in L_1$, we also have that $w \in L_2$, the argument goes as follows. If $w' \in L_1^*$ then $w' = w_1 \cdot \ldots \cdot w_n$ for some $n \in \mathbb{N}$ and $w_i \in L_1$ $(1 \leq i \leq n)$. But then each w_i is also in L_2 and therefore $w' \in L_2^*$.

2) a) Assume that the language is regular.
Then, by the pumping lemma, we would have that:

there exists n such that
for all $w \in L$ such that $|w| \geq n$
there are three strings x, y, z such that $w = xyz$, $|xy| \leq n$, $|y| \geq 1$, and for all $k \geq 0$, $xy^kz \in L$.

Now, given some n, let $w = \overbrace{0\cdots0}^{n}\overbrace{1\cdots1}^{n}\overbrace{0\cdots0}^{2n} = 0^n 1^n 0^{2n}$. Since $|w| = 4n$ we have that $|w| \geq n$. In order to apply the pumping lemma we need to find strings x and y such that $|xy| \leq n$. The only possible choices are $x = 0^a$ and $y = 0^b$ where $b \geq 1$. But then we have that $xz = 0^{n-b} 1^n 0^{2n}$ and thus that $n - b + n \neq 2n$. Therefore, for $k = 0$, $xy^kz \notin L$. Since we assumed that the language is regular this is a contradiction. Hence the language cannot be regular.

2) b) Again, we use the pumping lemma.
Given some n, let $w = 0^n 1 0^n$.
If we consider x, y, z such that
a) $w = xyz$, b) $|xy| \leq n$, c) $|y| \geq 1$
then y can only be a non-empty string of 0's. Thus, for each $k > 1$, the string $xy^k z$ has more 0's on the left-hand side of 1 than on right-hand side. We can conclude that, for $k > 1$, $xy^k z \notin L$. Therefore we have that the language is not regular.