PRIMITIVE RECURSIVE FUNCTIONS

Exercise 1

Show that multiplication is a primitive recursive function.

Solution:

$$\begin{cases} mult(x,0) = g(x) = 0 \\ mult(x, y+1) = h(x, y, mult(x,y)) = mult(x,y) + x \end{cases}$$

where $g = z$ and $h = add \circ (p_3^{(3)}, p_1^{(3)})$

Exercise 2

Let $g(x,y)$ be a primitive recursive function. Then the following functions obtained from g are also PR.

a) $f(x, y, z_1, \dots, z_n) = g(x,y)$

b) $f(x,y) = g(y,x)$

c) $f(x) = g(x,x)$

Solution:

a) $f = g \circ (p_1^{(n+2)}, p_2^{(n+2)})$

b) $f = g \circ (p_2^{(2)}, p_1^{(2)})$

c) $f = g \circ (p_1^{(1)}, p_1^{(1)})$

Exercise 3

Let $p(x,z)$ be a primitive recursive predicate. Show that the following functions are primitive recursive.

a) $f_1(x, y_0, y)$ = the first value z in $[y_0, y]$ for which $p(x,z)$ is true

b) $f_2(x,y)$ = the second value z in $[0,y]$ for which $p(x,z)$ is true

c) $f_3(x,y)$ = the largest value z in $[0,y]$ for which $p(x,z)$ is true

If there is no value z in the range such that $p(x,z)$ is true, then f_i is $y+1$.

Solution:

a) $f_1(x, y_0, y) = \mu z \leq y\, [p(x,z) \cdot ge(z, y_0)]$

The PRF ge ("greater than or equal to") is used to enforce the lower bound; multiplication $\cdot$ works as "boolean and".

b) $f_2(x,y) = \mu z \leq y\, [p(x,z) \cdot gt(z, \mu z' \leq y\, [p(x,z')])]$

The PRF gt ("greater than") makes sure we skip the first value.

c) Let $f'(x,y) = y \dot{-} \underbrace{\mu z \leq y\, [p(x, y \dot{-} z)]}$

reverses the order of examination (i.e. we go from y down to 0)

Then:

$$f_3(x,y) = eq(y+1, \mu z \leq y\, [p(x,z)]) \cdot (y+1)$$
$$+\, neq(y+1, \mu z \leq y\, [p(x,z)]) \cdot f'(x,y)$$

It checks whether there is a z such that $z \leq y$ and $p(x,z) = \text{true}$, and outputs $f'(x,y)$ if it is the case and $y+1$ otherwise.

Exercise 4

Consider integer division $div(x,y)$: it's not defined for 0, hence not total and hence not PR. Let

$$quo(x,y) = \begin{cases} 0 & \text{if } y=0 \\ div(x,y) & \text{otherwise} \end{cases}$$

a) Define $quo(x,y)$ using bounded minimization.

b) Show that remainder, divides, number of divisors, and prime are primitive recursive.

Solution:

a) $quo(x,y) = sg(y) \cdot \mu z \le x\, [gt((z+1)\cdot y, x)]$

b) Remainder:

$$rem(x,y) = x \dot{-} (y \cdot quo(x,y))$$

Divides:

$$divides(x,y) = \begin{cases} 1 & \text{if } x>0, y>0, \text{ and } y \text{ is a divisor of } x \\ 0 & \text{otherwise} \end{cases}$$

$$divides(x,y) = eq(rem(x,y), 0) \cdot sg(x)$$

Number of divisors:

$$ndivisors(x,y) = \Sigma_{i=0}^{x}\ divides(x,y)$$

Prime:

$$prime(x) = \begin{cases} 1 & \text{if } x \text{ is prime} \\ 0 & \text{otherwise} \end{cases}$$

$$prime(x) = eq(ndivisors(x), 2)$$

Exercise 5

Show that the function $pn(i)$ computing the i-th prime is PR by exploiting the fact that $pn(x+1) \leq pn(x)! + 1$.

Solution:

$$\begin{cases} pn(0) = 2 \\ pn(x+1) = \mu z \leq (pn(x)! + 1)[\, prime(z) \cdot gt(z, pn(x))\,] \end{cases}$$

Exercise 6

Show that the Ackermann function

$$\begin{cases} A(0,y) = y+1 \\ A(x+1,0) = A(x,1) \\ A(x+1,y+1) = A(x, A(x+1,y)) \end{cases}$$

is defined for every pair $x, y \in \mathbb{N}$.

Solution:

By induction on x (main induction)

Base case: $A(0,y) = y+1$

Inductive step: By induction on y (secondary induction)
$A(x+1,y)$

Base case: $A(x+1,0) = A(x,1)$ and the
$A(x+1,0)$
main induction hypothesis applies

Inductive step: By the secondary induction
$A(x+1,y+1)$
hypothesis $A(x+1,y)$ is defined;
thus for $A(x+1,y+1) = A(x, A(x+1,y))$ the main induction hypothesis applies

Exercise 7

Define a primitive recursive function $f : \mathbb{N} \to \mathbb{N}$ that counts the number of occurrences of the digit 5 in a natural number.

Solution:

We need some auxiliary primitive recursive functions

- exponential $m^n : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

$$\exp(m,n) = \begin{cases} \exp(m,0) = 1 \\ \exp(m,n+1) = \exp(m,n) \cdot m \end{cases}$$

- length (number of digits) $: \mathbb{N} \to \mathbb{N}$

  $\text{length}(n) = (\mu z \leq n\, [\, gt(10^{z+1}, n)\,]) + 1$

  examples : $\text{length}(0) = \text{length}(1) = \ldots = \text{length}(9) = 1$, $\text{length}(10) = 2, \ldots$

$f : \mathbb{N} \to \mathbb{N}$ is then defined as follows

$$f(n) = \sum_{i=1}^{\text{length}(n)} eq(5, rem(quo(n, 10^{i-1}), 10))$$

Example :

$$\begin{aligned} f(253) = {} & eq(5, \underbrace{rem(\underbrace{quo(253,1)}_{253}, 10)}_{3}) \\ & + eq(5, \underbrace{rem(\underbrace{quo(253,10)}_{25}, 10)}_{5}) \\ & + eq(5, \underbrace{rem(\underbrace{quo(253,100)}_{2}, 10)}_{2}) \\ = {} & 0 + 1 + 0 = 1 \end{aligned}$$

$$\begin{array}{c} 0 \\ + \\ 1 \\ + \\ 0 \\ \hline 1 \end{array}$$

Exercise 8

Define a primitive recursive function $f: \mathbb{N} \to \mathbb{N}$ that reverses the digits of a natural number, i.e. $f(253) = 352$, $f(5524) = 4255$.

Solution:

$$f(n) = \sum_{i=1}^{\text{length}(n)} \left(\text{quo}\left(\text{rem}\left(n, 10^{\text{length}(n)-i+1}\right), 10^{\text{length}(n)-i}\right) \cdot 10^{i-1}\right.$$

Example:

$$\begin{aligned} f(5524) = \; & \underbrace{\text{quo}(\underbrace{\text{rem}(5524, 10000)}_{5524}, 1000)}_{5} \cdot 1 \\ & + \underbrace{\text{quo}(\underbrace{\text{rem}(5524, 1000)}_{524}, 100)}_{5} \cdot 10 \\ & + \underbrace{\text{quo}(\underbrace{\text{rem}(5524, 100)}_{24}, 10)}_{2} \cdot 100 \\ & + \underbrace{\text{quo}(\underbrace{\text{rem}(5524, 10)}_{4}, 1)}_{4} \cdot 1000 \\ = \; & 5 + 50 + 200 + 4000 = 4255 \end{aligned}$$