

Functions

Consider a binary relation $R \subseteq A \times B$.

R is a total function if for each $a \in A$ there exists a unique $b \in B$ such that $(a, b) \in R$.

R is a partial function if for each $a \in A$ there is at most one $b \in B$ such that $(a, b) \in R$.

Notation: functions $f \subseteq A \times B$ are also written as $f: A \rightarrow B$.

A function $f: A \rightarrow B$ is called

- injective if $f(a_1) = f(a_2)$ implies $a_1 = a_2$;
- surjective if for all $b \in B$ there exists $a \in A$ such that $f(a) = b$;
- bijective if it is both injective and surjective

EXERCISE 1

Which of the following functions are injective, surjective, bijective?

a) $f: \mathbb{N} \rightarrow \mathbb{N}$
 $x \mapsto x^2$

b) $f: \mathbb{N} \rightarrow \mathbb{N}$
 $x \mapsto x^2$

c) $f: \mathbb{N} \rightarrow \mathbb{N}$
 $x \mapsto x+1$

d) $f: A \times B \rightarrow B \times A$
 $(a, b) \mapsto (b, a)$

EXERCISE 2

30/10/2009
E1.2

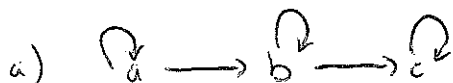
Assume that $(A, \leq_A)$ and $(B, \leq_B)$ are partially ordered sets.

a) Show that $\{((a_1, b_1), (a_2, b_2)) \mid b_1 =_B b_2\}$ is an equivalence relation on $S = A \times B$.

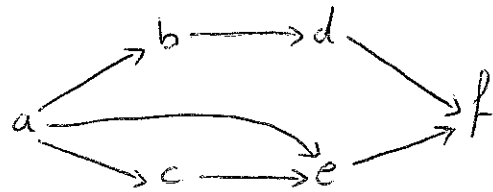
b) Show that $\{((a_1, b_1), (a_2, b_2)) \mid a_1 \leq_A a_2 \text{ or } (a_1 =_A a_2 \text{ and } b_1 \leq_B b_2)\}$ is a partial order on $S = A \times B$.

EXERCISE 3

Consider the following graphs representing relations. Which properties are satisfied by these relations? Provide the transitive closure of the relations which are not transitive.



d)

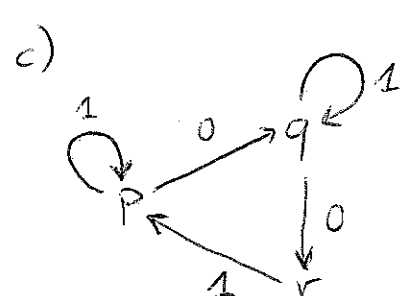
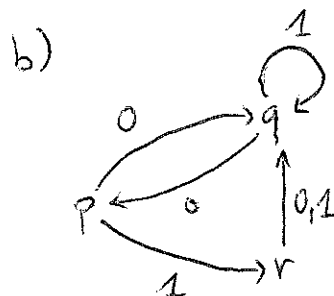
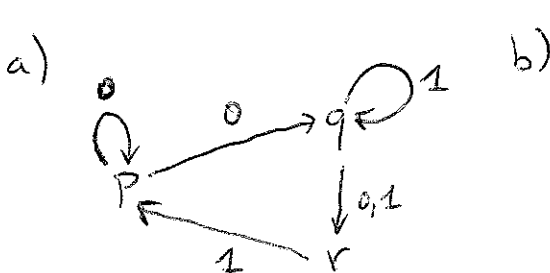


EXERCISE 4

Consider the following graphs representing relations

$R = \{((x, l), y) \mid x \xrightarrow{l} y \text{ is a labelled transition in the graph}\}$.

Determine which of these is a function? Is it total or partial?



SOLUTIONS (09/10/2008)

1) a) f is neither injective* nor surjective**

1) b) (*) $f(-1) = f(1)$ but $-1 \neq 1$

(**) there is no $x \in \mathbb{Z}$ s.t. $f(x) = 2$

1) c) f is injective* but not surjective**

(*) $f(x) = f(y) \Rightarrow x+1 = y+1 \Rightarrow x=y$

(**) there is no $x \in \mathbb{N}$ s.t. $f(x) = 0$

1) d) f is bijective, i.e. injective* and surjective**

(*) $f(a_1, b_1) = f(a_2, b_2) \Rightarrow (b_1, a_1) = (b_2, a_2) \Rightarrow b_1 = b_2$ and $a_1 = a_2$

(**) for each $(b, a) \in A \times B$, (b, a) is the image of (a, b)

2) a) Let's call the relation E

We have to show that E is reflexive, symmetric and transitive:

(reflexive) $(a, b) E (a, b)$ since $b \leq_B b$ and thus $b =_B b$

(symmetric) $(a_1, b_1) E (a_2, b_2)$ implies $(a_2, b_2) E (a_1, b_1)$ since $b_1 =_B b_2$

(transitive) $(a_1, b_1) E (a_2, b_2)$ and $(a_2, b_2) E (a_3, b_3)$ implies

$(a_1, b_1) E (a_3, b_3)$ since $b_1 =_B b_2 =_B b_3$

2) b) Let's call the relation P

We have to show that P is reflexive, transitive and antisymmetric:

(reflexive) $(a, b) P (a, b)$ since $a =_A a$ and $b =_B b$

(antisymmetric) $(a_1, b_1) P (a_2, b_2)$ and $(a_2, b_2) P (a_1, b_1)$ implies

$(a_1, b_1) = (a_2, b_2)$ since we can only have that $a_1 =_A a_2$

and, as a consequence thereof, that $b_1 =_B b_2$ (from $b_1 \leq_B b_2$ and $b_2 \leq_B b_1$)

2) b) con't

(transitive) $(a_1, b_1) P(a_2, b_2)$ and $(a_2, b_2) P(a_3, b_3)$ implies $(a_1, b_1) P(a_3, b_3)$ since: a) if either $a_1 < a_2$ or $a_1 < a_3$ or both then $a_1 < a_3$ b) if $a_1 = a_2 = a_3$ then $b_1 \leq b_3$ as a consequence of the fact that $b_1 \leq b_2$ and $b_2 \leq b_3$

3) a) The relation is reflexive and antisymmetric

3) b) The relation is transitive and antisymmetric

3) c) The relation is symmetric and antireflexive

3) d) The relation does not satisfy any particular property

REMARK: we omit the transitive closures of 3) a), 3) c) and 3) d)

4) We provide tables that, for each $x \in \{p, q, r\}$ and $l \in \{0, 1\}$, give us the number of y 's such that $x \xrightarrow{l} y$ is a labelled transition in the graph. Note that, if all numbers in the table are $= 1$ (≤ 1) then the relation is a total (partial) function.

a) no function

since:

	0	1
p	2	0
q	1	2
r	0	1

b) total function

since:

	0	1
p	1	1
q	1	1
r	1	1

c) partial function

since:

	0	1
p	1	1
q	1	1
r	0	1