<u>Exercises in preparation of Midterm Exam</u>

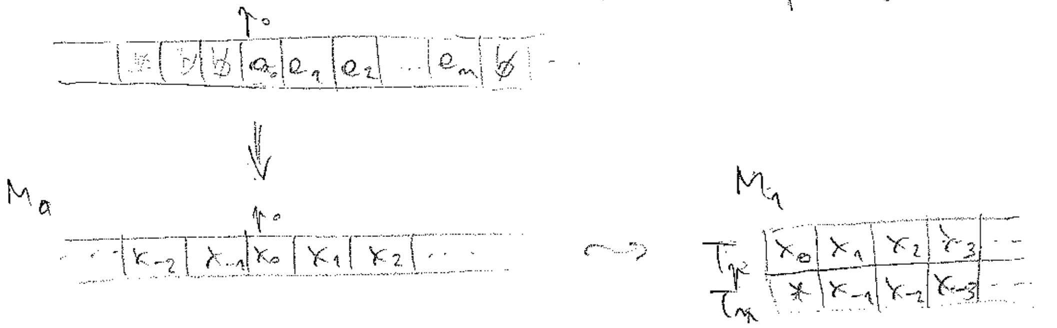
E7.1

<u>Exercise 1:</u> Consider a TM $M_0 = (Q_0, \Sigma, \Gamma_0, \delta_0, q_0, \not{b}, F_0)$.

Show that $\mathcal{L}(M)$ is also accepted by a TM M_1 that never moves left of its initial position (i.e., a by a TM with a semi-infinite tape).

Idea: M_1 is a two track TM: $M_1 = (Q_1, \Sigma, \Gamma_1, \delta_1, q_1, \not{b}, F_1)$

Let us call p_0 the initial tape position of M_0



- The states of M_1 are all the states of M_0, with an additional component P or N, indicating whether M_1 is currently working on the track representing the positive or negative portion of the tape of M_0: $Q_1 = Q_0 \times \{P, N\}$

- Γ_1 is the set of pairs of symbols of Γ_0, plus symbols with $*$ on T_n
$$\Gamma_1 = \Gamma_0 \times (\Gamma_0 \cup \{*\})$$

The $*$ on T_1 is used to detect when M_1 reaches the leftmost tape position

- Initially, Γ_1 writes $*$ on T_1 of the leftmost position (for this it actually needs two additional states).

- For the transitions of M_1, we need to distinguish 4 cases:
  1) M_0 is to the right of $p_0 \leadsto M_1$ works on track T_p
  2) " " left " " T_n
  3) M_0 is on $p_0 \leadsto M_1$ is on $[*]$

Let $\delta_0(q, X) = (q', Y, d)$ be a transition of M_0 (E7.2)

Then we have

1) $\delta_1([q, P], [\substack{X \\ Z}]) = ([q', P], [\substack{Y \\ Z}], d)$ for every $Z \in \Gamma_0$ (i.e. $Z \neq *$)

2) $\delta_1([q, N], [\substack{Z \\ X}]) = ([q', N], [\substack{Z \\ X}], \bar{d})$ for every $Z \in \Gamma_0$

where $\bar{d} = L$ if $d = R$
$\bar{d} = R$ if $d = L$

3) if M_0 moves right, i.e. $d = R$

$\delta_1([q, -], [\substack{X \\ *}]) = ([q', P], [\substack{Y \\ *}], R)$

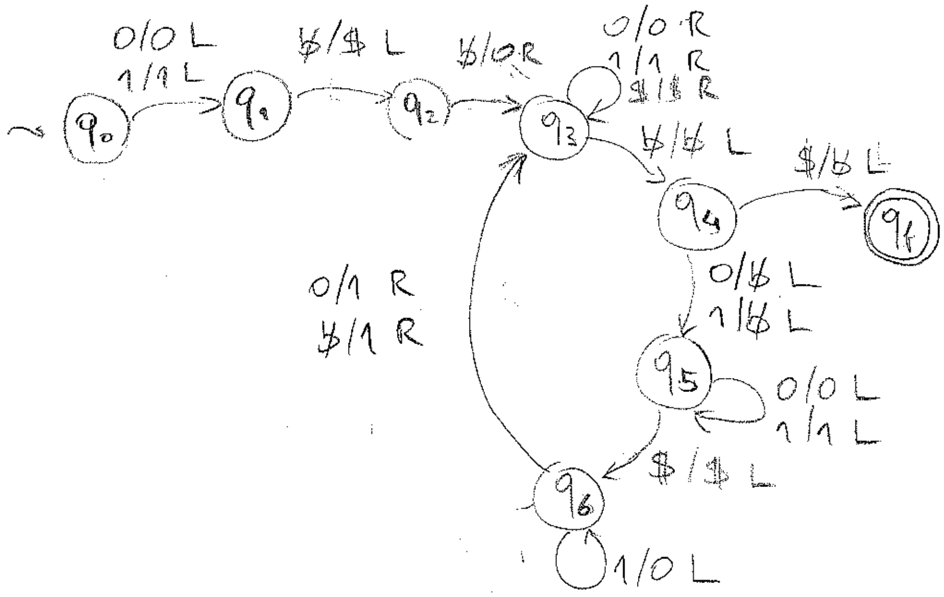
if M_0 moves left, i.e. $d = L$

$\delta_1([q, -], [\substack{X \\ *}]) = ([q', N], [\substack{Y \\ *}], R)$

- Final states of M_1 : $F_1 = F_0 \times \{P, N\}$

Exercise 2: Construct a TM that computes the length of its input string, represented as a binary number (with the least significant digit on the right). Assume $\Sigma = \{0,1\}$ (E7.3)

Idea: we write a counter to the left of the input separated by a \$.
We repeatedly move to the right of the input, delete the last symbol, come back and increment the counter



E 7.4

Exercise 3: For a TM M with input alphabet Σ, let $\langle M, w \rangle$ denote the encoding $\mathcal{E}(M)$ of M followed by input w.

Consider the language $L = \{ \langle M, w \rangle \mid$ M when started on an input string w, eventually does three consecutive transitions in which it moves the head in the same direction $\}$

a) Show that L is recursively enumerable

b) Show that L is not recursive

a) We reduce L to L_u.

The reduction R is a TM that takes as input $\langle M, w \rangle$ and produces as output $R(\langle M, w \rangle) = \langle M', w \rangle$ such that

$$\langle M, w \rangle \in L \quad \text{iff} \quad \langle M', w \rangle \in L_u .$$

We describe how R has to transform $\mathcal{E}(M)$ to obtain $\mathcal{E}(M')$:

- R has to add to the states of M a second component that counts how many consecutive transitions M has made in the same direction:

  the values of the counter component are $-3, -2, -1, 1, 2, 3$

- the transitions of M are modified to update the counter

  if M moves right then in M':
$$\begin{cases} c = -2 \rightsquigarrow c = 1 \\ c = -1 \rightsquigarrow c = 1 \\ c = 1 \rightsquigarrow c = 2 \\ c = 2 \rightsquigarrow c' = 3 \end{cases}$$

  if M moves left, then in M':
$$\begin{cases} c = -2 \rightarrow c = -3 \\ c = -1 \rightsquigarrow c = -2 \\ c = 1 \rightarrow c = -1 \\ c = 2 \rightarrow c = -1 \end{cases}$$

- the states with the counter 3 or -3 are the <u>only</u> final states

E7.5

b) We reduce the halting problem L_H to L

The reduction R is a TM that takes as input $\langle M, w\rangle$ and produces as output $R(\langle M, w\rangle) = \langle M', w\rangle$ such that

$$\langle M, w\rangle \in L_H \quad \text{iff} \quad \langle M', w\rangle \in L$$

We describe how R has to transform $\mathcal{E}(M)$ to obtain $\mathcal{E}(M')$:

- the final states of M are made non-final in M'
- from a final or blocking state of M we add to M' a transition to a new state from which M' makes 3 transitions to the right
- we have to make sure that M' never does 3 consecutive transitions in the same direction (except the ones above):

Hence:

if M does an R-move, then
  M' does an R-L-R move

if M does an L move, then
  M' does an L-R-L move

- the tape symbol is changed only in the first of the three moves, while the other two leave the tape unchanged
- for the dummy moves, additional states are needed, and these need to be distinct for each state of M.

Exercise 4: Let $g(x)$ be a PRF.

E7.6

a) Show that the following predicate is a PRF:

$$f(x,y) = \begin{cases} 1 & \text{if } g(i) < g(x) \text{ for all } 0 \le i \le y \\ 0 & \text{otherwise} \end{cases}$$

$$f(x,y) = \prod_{i=0}^{y} lt(g(i), g(x))$$

b) Let f be defined by

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 2 & \text{if } x = 1 \\ 3 & \text{if } x = 2 \\ f(x-3) + f(x-1) & \text{if } x \ge 3 \end{cases}$$

- Give the values $f(4)$, $f(5)$, $f(6)$.

$$f(3) = f(0) + f(2) = 1 + 3 = 4$$
$$f(4) = f(1) + f(3) = 2 + 4 = 6$$
$$f(5) = f(2) + f(4) = 3 + 6 = 9$$
$$f(6) = f(3) + f(5) = 4 + 9 = 13$$

Show that f is a PRF.

We have that $f(y+1) = f(y-2) + f(y)$.

We introduce an auxiliary function h with

$$h(y) = [f(y), f(y+1), f(y+2)] = gn_2(f(y), f(y+1), f(y+2))$$

$$\begin{cases} h(0) = gn_2(f(0), f(1), f(2)) = gn_2(1,2,3) = 2^2 \cdot 3^3 \cdot 5^4 \\ h(y+1) = [f(y+1), f(y+2), f(y+3)] = \end{cases}$$

$$= [f(y+1), f(y+2), f(y) + f(y+2)] =$$

$$= [dec(1, h(y)), dec(2, h(y)), dec(0, h(y)) + dec(2, h(y))]$$

$$= gn_2(\dots)$$

Hence h is PR. Then $f(y) = dec(0, h(y))$ is also PR.