FUNCTIONS & RELATIONS

09/10/2008

E1.1

Functions

Consider a binary relation $R \subseteq A \times B$.
R is a <u>total function</u> if for each $a \in A$ there exists a unique $b \in B$ such that $(a,b) \in R$.
R is a <u>partial function</u> if for each $a \in A$ there is a most one $b \in B$ such that $(a,b) \in R$.

<u>Notation</u>: functions $f \subseteq A \times B$ are also written as $f : A \longrightarrow B$.

A function $f : A \to B$ is called

- <u>injective</u> if $f(a_1) = f(a_2)$ implies $a_1 = a_2$;
- <u>surjective</u> if for all $b \in B$ there exists $a \in A$ such that $f(a) = b$;
- <u>bijective</u> if it is both injective and surjective

EXERCISE 1

Which of the following functions are injective, surjective, bijective?

a) $f : \mathbb{Z} \longrightarrow \mathbb{Z}$, $x \longmapsto x^2$

b) $f : \mathbb{Z} \longrightarrow \mathbb{N}$, $x \longmapsto x^2$

c) $f : \mathbb{N} \longrightarrow \mathbb{N}$, $x \longmapsto x+1$

d) $f : A \times B \longrightarrow B \times A$, $(a,b) \longmapsto (b,a)$

EXERCISE 2

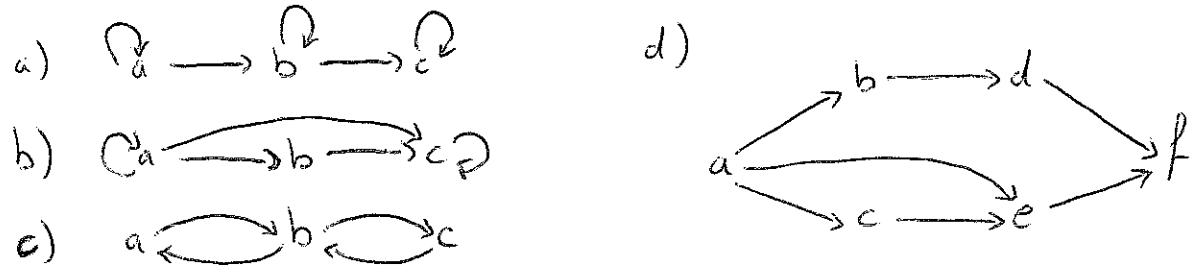
Assume that $(A, \leq_A)$ and $(B, \leq_B)$ are partially ordered sets.

a) Show that $\{((a_1,b_1),(a_2,b_2)) \mid b_1 =_B b_2\}$ is an equivalence relation on $S = A \times B$.

b) Show that $\{((a_1,b_1),(a_2,b_2)) \mid a_1 <_A a_2 \text{ or } (a_1 =_A a_2 \text{ and } b_1 \leq_B b_2)\}$ is a partial order on $S = A \times B$.

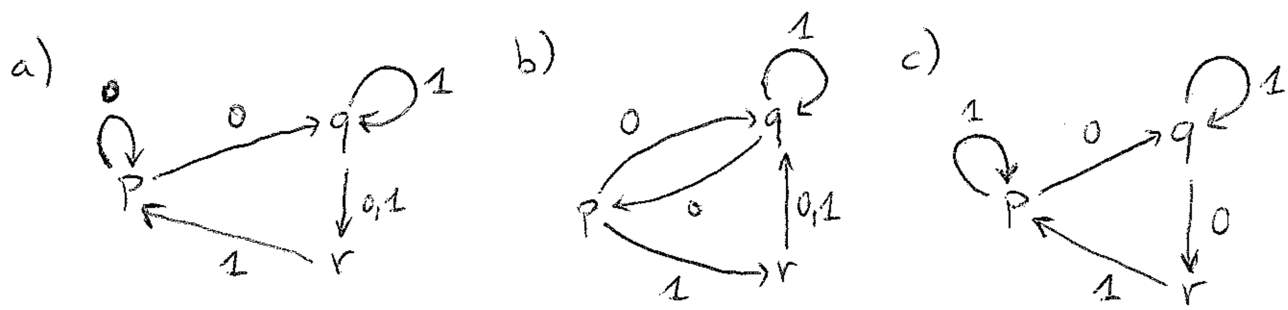
EXERCISE 3

Consider the following graphs representing relations. Which properties are satisfied by these relations? Provide the transitive closure of the relations which are not transitive.



EXERCISE 4

Consider the following graphs representing relations $R = \{((x,\ell),y) \mid x \xrightarrow{\ell} y \text{ is a labelled transition in the graph}\}$. Determine which of these is a function? Is it total or partial?



SOLUTIONS (09/10/2008)

E 1.3

1) a) f is neither injective* nor surjective**

1) b) (*) $f(-1) = f(1)$ but $-1 \neq 1$

(**) there is no $x \in \mathbb{Z}$ s.t. $f(x) = 2$

1) c) f is injective* but not surjective**

(*) $f(x) = f(y) \Rightarrow x+1 = y+1 \Rightarrow x = y$

(**) there is no $x \in \mathbb{N}$ s.t. $f(x) = 0$

1) d) f is bijective, i.e. injective* and surjective**

(*) $f(a_1, b_1) = f(a_2, b_2) \Rightarrow (b_1, a_1) = (b_2, a_2) \Rightarrow b_1 = b_2$ and $a_1 = a_2$

(**) for each $(b, a) \in A \times B$, (b, a) is the image of (a, b)

2) a) Let's call the relation E

We have to show that E is riflexive, symmetric and transitive:

(riflexive) $(a,b) E (a,b)$ since $b \leq_B b$ and thus $b =_B b$

(symmetric) $(a_1, b_1) E (a_2, b_2)$ implies $(a_2, b_2) E (a_1, b_1)$ since $b_1 =_B b_2$

(transitive) $(a_1, b_1) E (a_2, b_2)$ and $(a_2, b_2) E (a_3, b_3)$ implies $(a_1, b_1) E (a_3, b_3)$ since $b_1 =_B b_2 =_B b_3$

2) b) Let's call the relation P

We have to show that P is riflexive, transitive and antisymmetric:

(reflexive) $(a,b) P (a,b)$ since $a =_A a$ and $b =_B b$

(antisymmetric) $(a_1, b_1) P (a_2, b_2)$ and $(a_2, b_2) P (a_1, b_1)$ implies $(a_1, b_1) = (a_2, b_2)$ since we can only have that $a_1 =_A a_2$ and, as a consequence thereof, that $b_1 =_B b_2$ (from $b_1 \leq_B b_2$ and $b_2 \leq_B b_1$)

E1.4

2) b) con't

(transitive) $(a_1,b_1) P (a_2,b_2)$ and $(a_2,b_2) P (a_3,b_3)$ implies $(a_1,b_1) P (a_3,b_3)$

since : a) if either $a_1 <_A a_2$ or $a_2 <_A a_3$ or both then $a_1 <_A a_3$

b) if $a_1 =_A a_2 =_A a_3$ then $b_1 \leq_B b_3$ as a consequence of the fact that $b_1 \leq_B b_2$ and $b_2 \leq_B b_3$

3) a) The relation is reflexive and antisymmetric

3) b) The relation is transitive and antisymmetric

3) c) The relation is symmetric and antireflexive

3) d) The relation does not satisfy any particular property

<u>REMARK</u> : we omit the transitive closures of 3) a), 3) c) and 3) d)

4) We provide tables that, for each $x \in \{p,q,r\}$ and $\ell \in \{0,1\}$, give us the number of y's such that $x \xrightarrow{\ell} y$ is a labelled transition in the graph. Note that, if all numbers in the table are $=1$ (≤ 1) then the relation is a total (partial) function.

a) no function

since :

	0	1
p	2	0
q	2	1
r	0	1

b) total function

since :

	0	1
p	1	1
q	1	1
r	1	1

c) partial function

since :

	0	1
p	1	1
q	1	1
r	0	1