

Knowledge Bases and Databases

Part 1: First-Order Queries

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Overview of Part 1: First-order queries

- ① First-order logic
 - ① Syntax of first-order logic
 - ② Semantics of first-order logic
 - ③ First-order logic queries
- ② First-order query evaluation
 - ① Query evaluation problem
 - ② Complexity of query evaluation
- ③ Conjunctive queries
 - ① Evaluation of conjunctive queries
 - ② Containment of conjunctive queries
 - ③ Unions of conjunctive queries

Chapter I

First-Order Logic



Outline

- 1 Syntax of first-order logic
- 2 Semantics of first-order logic
- 3 First-order logic queries



Outline

- 1 Syntax of first-order logic
- 2 Semantics of first-order logic
- 3 First-order logic queries



First-order logic

- First-order logic (FOL) is the logic to speak about **objects**, which are the domain of discourse or universe.
- FOL is concerned about **properties** of these objects and **relations** over objects (resp., unary and n -ary **predicates**).
- FOL also has **functions** including **constants** that denote objects.



FOL syntax – Terms

We first introduce:

- A set $Vars = \{x_1, \dots, x_n\}$ of **individual variables** (i.e., variables that denote single objects).
- A set of **functions symbols**, each of given arity ≥ 0 . Functions of arity 0 are called **constants**.

Def.: The set of **Terms** is defined inductively as follows:

- $Vars \subseteq Terms$;
- If $t_1, \dots, t_k \in Terms$ and f^k is a k -ary function symbol, then $f^k(t_1, \dots, t_k) \in Terms$;
- Nothing else is in $Terms$.



FOL syntax – Formulas

Def.: The set of **Formulas** is defined inductively as follows:

- If $t_1, \dots, t_k \in Terms$ and P^k is a k -ary predicate, then $P^k(t_1, \dots, t_k) \in Formulas$ (atomic formulas).
- If $t_1, t_2 \in Terms$, then $t_1 = t_2 \in Formulas$.
- If $\varphi \in Formulas$ and $\psi \in Formulas$ then
 - $\neg\varphi \in Formulas$
 - $\varphi \wedge \psi \in Formulas$
 - $\varphi \vee \psi \in Formulas$
 - $\varphi \rightarrow \psi \in Formulas$
- If $\varphi \in Formulas$ and $x \in Vars$ then
 - $\exists x.\varphi \in Formulas$
 - $\forall x.\varphi \in Formulas$
- Nothing else is in $Formulas$.

Note: a predicate of arity 0 is a proposition of propositional logic.



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Interpretations

Given an **alphabet** of predicates P_1, P_2, \dots and functions f_1, f_2, \dots , each with an associated arity, a FOL **interpretation** is:

$$\mathcal{I} = (\Delta^{\mathcal{I}}, P_1^{\mathcal{I}}, P_2^{\mathcal{I}}, \dots, f_1^{\mathcal{I}}, f_2^{\mathcal{I}}, \dots)$$

where:

- $\Delta^{\mathcal{I}}$ is the domain (a set of objects)
- if P_i is a k -ary predicate, then $P_i^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \dots \times \Delta^{\mathcal{I}}$ (k times)
- if f_i is a k -ary function, then $f_i^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \dots \times \Delta^{\mathcal{I}} \rightarrow \Delta^{\mathcal{I}}$ (k times)
- if f_i is a constant (i.e., a 0-ary function), then $f_i^{\mathcal{I}} : () \rightarrow \Delta^{\mathcal{I}}$ (i.e., f_i denotes exactly one object of the domain)



Assignment

Let $Vars$ be a set of (individual) variables.

Def.: Given an interpretation \mathcal{I} , an **assignment** is a function

$$\alpha : Vars \longrightarrow \Delta^{\mathcal{I}}$$

that assigns to each variable $x \in Vars$ an object $\alpha(x) \in \Delta^{\mathcal{I}}$.

It is convenient to extend the notion of assignment to terms. We can do so by defining a function $\hat{\alpha} : Terms \longrightarrow \Delta^{\mathcal{I}}$ inductively as follows:

- $\hat{\alpha}(x) = \alpha(x)$, if $x \in Vars$
- $\hat{\alpha}(f(t_1, \dots, t_k)) = f^{\mathcal{I}}(\hat{\alpha}(t_1), \dots, \hat{\alpha}(t_k))$

Note: for constants $\hat{\alpha}(c) = c^{\mathcal{I}}$.



Truth in an interpretation wrt an assignment

We define when a FOL formula φ is **true** in an interpretation \mathcal{I} wrt an assignment α , written $\mathcal{I}, \alpha \models \varphi$:

- $\mathcal{I}, \alpha \models P(t_1, \dots, t_k)$ if $(\hat{\alpha}(t_1), \dots, \hat{\alpha}(t_k)) \in P^{\mathcal{I}}$
- $\mathcal{I}, \alpha \models t_1 = t_2$ if $\hat{\alpha}(t_1) = \hat{\alpha}(t_2)$
- $\mathcal{I}, \alpha \models \neg\varphi$ if $\mathcal{I}, \alpha \not\models \varphi$
- $\mathcal{I}, \alpha \models \varphi \wedge \psi$ if $\mathcal{I}, \alpha \models \varphi$ and $\mathcal{I}, \alpha \models \psi$
- $\mathcal{I}, \alpha \models \varphi \vee \psi$ if $\mathcal{I}, \alpha \models \varphi$ or $\mathcal{I}, \alpha \models \psi$
- $\mathcal{I}, \alpha \models \varphi \rightarrow \psi$ if $\mathcal{I}, \alpha \models \varphi$ implies $\mathcal{I}, \alpha \models \psi$
- $\mathcal{I}, \alpha \models \exists x.\varphi$ if for some $a \in \Delta^{\mathcal{I}}$ we have $\mathcal{I}, \alpha[x \mapsto a] \models \varphi$
- $\mathcal{I}, \alpha \models \forall x.\varphi$ if for every $a \in \Delta^{\mathcal{I}}$ we have $\mathcal{I}, \alpha[x \mapsto a] \models \varphi$

Here, $\alpha[x \mapsto a]$ stands for the new assignment obtained from α as follows:

$$\begin{aligned} \alpha[x \mapsto a](x) &= a \\ \alpha[x \mapsto a](y) &= \alpha(y) \quad \text{for } y \neq x \end{aligned}$$



Open vs. closed formulas

Definitions

- A variable x in a formula φ is **free** if x does not occur in the scope of any quantifier, otherwise it is **bounded**.
- An **open formula** is a formula that has some free variable.
- A **closed formula**, also called **sentence**, is a formula that has no free variables.

For **closed formulas** (but not for open formulas) we can define what it means to be **true in an interpretation**, written $\mathcal{I} \models \varphi$, without mentioning the assignment, since the assignment α does not play any role in verifying $\mathcal{I}, \alpha \models \varphi$.

Instead, open formulas are strongly related to **queries** — cf. relational databases.



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FOL queries

Def.: A **FOL query** is an (open) FOL formula.

When φ is a FOL query with free variables (x_1, \dots, x_k) , then we sometimes write it as $\varphi(x_1, \dots, x_k)$, and say that φ has **arity** k .

Given an interpretation \mathcal{I} , we are interested in those assignments that map the variables x_1, \dots, x_k (and only those). We write an assignment α s.t. $\alpha(x_i) = a_i$, for $i = 1, \dots, k$, as $\langle a_1, \dots, a_k \rangle$.

Def.: Given an interpretation \mathcal{I} , the **answer to a query** $\varphi(x_1, \dots, x_k)$ is

$$\varphi(x_1, \dots, x_k)^{\mathcal{I}} = \{ \langle a_1, \dots, a_k \rangle \mid \mathcal{I}, \langle a_1, \dots, a_k \rangle \models \varphi(x_1, \dots, x_k) \}$$

Note: We will also use the notation $\varphi^{\mathcal{I}}$, which keeps the free variables implicit, and $\varphi(\mathcal{I})$ making apparent that φ becomes a functions from interpretations to set of tuples.



FOL boolean queries

Def.: A **FOL boolean query** is a FOL query without free variables.

Hence, the answer to a boolean query $\varphi()$ is defined as follows:

$$\varphi()^{\mathcal{I}} = \{ () \mid \mathcal{I}, \langle \rangle \models \varphi() \}$$

Such an answer is

- $()$, if $\mathcal{I} \models \varphi$
- \emptyset , if $\mathcal{I} \not\models \varphi$.

As an obvious convention we read $()$ as “true” and \emptyset as “false”.



FOL formulas: logical tasks

Definitions

- **Validity**: φ is **valid** iff for all \mathcal{I} and α we have that $\mathcal{I}, \alpha \models \varphi$.
- **Satisfiability**: φ is **satisfiable** iff there exists an \mathcal{I} and α such that $\mathcal{I}, \alpha \models \varphi$, and **unsatisfiable** otherwise.
- **Logical implication**: φ **logically implies** ψ , written $\varphi \models \psi$ iff for all \mathcal{I} and α , if $\mathcal{I}, \alpha \models \varphi$ then $\mathcal{I}, \alpha \models \psi$.
- **Logical equivalence**: φ is **logically equivalent** to ψ , iff for all \mathcal{I} and α , we have that $\mathcal{I}, \alpha \models \varphi$ iff $\mathcal{I}, \alpha \models \psi$ (i.e., $\varphi \models \psi$ and $\psi \models \varphi$).



FOL queries – Logical tasks

- **Validity**: if φ is valid, then $\varphi^{\mathcal{I}} = \Delta^{\mathcal{I}} \times \dots \times \Delta^{\mathcal{I}}$ for all \mathcal{I} , i.e., the query always returns all the tuples of \mathcal{I} .
- **Satisfiability**: if φ is satisfiable, then $\varphi^{\mathcal{I}} \neq \emptyset$ for some \mathcal{I} , i.e., the query returns at least one tuple.
- **Logical implication**: if φ logically implies ψ , then $\varphi^{\mathcal{I}} \subseteq \psi^{\mathcal{I}}$ for all \mathcal{I} , written $\varphi \subseteq \psi$, i.e., the answer to φ is contained in that of ψ in every interpretation. This is called **query containment**.
- **Logical equivalence**: if φ is logically equivalent to ψ , then $\varphi^{\mathcal{I}} = \psi^{\mathcal{I}}$ for all \mathcal{I} , written $\varphi \equiv \psi$, i.e., the answer to the two queries is the same in every interpretation. This is called **query equivalence** and corresponds to query containment in both directions.

Note: These definitions can be extended to the case where we have **axioms**, i.e., **constraints** on the admissible interpretations.



Chapter II

First-Order Query Evaluation



Outline

- 4 Query evaluation problem
- 5 Complexity of query evaluation



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- 4 Query evaluation problem
- 5 Complexity of query evaluation



Query evaluation

Let us consider:

- a **finite alphabet**, i.e., we have a finite number of predicates and functions, and
- a **finite interpretation** \mathcal{I} , i.e., an interpretation (over the finite alphabet) for which $\Delta^{\mathcal{I}}$ is finite.

Then we can consider query evaluation as an algorithmic problem, and study its computational properties.

Note: To study the **computational complexity** of the problem, we need to define a corresponding decision problem.



Query evaluation problem

Definitions

- **Query answering problem:** given a finite interpretation \mathcal{I} and a FOL query $\varphi(x_1, \dots, x_k)$, compute

$$\varphi^{\mathcal{I}} = \{(a_1, \dots, a_k) \mid \mathcal{I}, \langle a_1, \dots, a_k \rangle \models \varphi(x_1, \dots, x_k)\}$$

- **Recognition problem (for query answering):** given a finite interpretation \mathcal{I} , a FOL query $\varphi(x_1, \dots, x_k)$, and a tuple (a_1, \dots, a_k) , with $a_i \in \Delta^{\mathcal{I}}$, check whether $(a_1, \dots, a_k) \in \varphi^{\mathcal{I}}$, i.e., whether

$$\mathcal{I}, \langle a_1, \dots, a_k \rangle \models \varphi(x_1, \dots, x_k)$$

Note: The recognition problem for query answering is the decision problem corresponding to the query answering problem.



Query evaluation algorithm

We define now an algorithm that computes the function $\text{Truth}(\mathcal{I}, \alpha, \varphi)$ in such a way that $\text{Truth}(\mathcal{I}, \alpha, \varphi) = \text{true}$ iff $\mathcal{I}, \alpha \models \varphi$.

We make use of an auxiliary function $\text{TermEval}(\mathcal{I}, \alpha, t)$ that, given an interpretation \mathcal{I} and an assignment α , evaluates a term t returning an object $o \in \Delta^{\mathcal{I}}$:

```

 $\Delta^{\mathcal{I}}$  TermEval( $\mathcal{I}, \alpha, t$ ) {
  if ( $t$  is  $x \in \text{Vars}$ )
    return  $\alpha(x)$ ;
  if ( $t$  is  $f(t_1, \dots, t_k)$ )
    return  $f^{\mathcal{I}}(\text{TermEval}(\mathcal{I}, \alpha, t_1), \dots, \text{TermEval}(\mathcal{I}, \alpha, t_k))$ ;
}
```

Then, $\text{Truth}(\mathcal{I}, \alpha, \varphi)$ can be defined by structural recursion on φ .



Query evaluation algorithm (cont'd)

```
boolean Truth( $\mathcal{I}, \alpha, \varphi$ ) {  
  if ( $\varphi$  is  $t_1 = t_2$ )  
    return TermEval( $\mathcal{I}, \alpha, t_1$ ) = TermEval( $\mathcal{I}, \alpha, t_2$ );  
  if ( $\varphi$  is  $P(t_1, \dots, t_k)$ )  
    return  $P^{\mathcal{I}}$ (TermEval( $\mathcal{I}, \alpha, t_1$ ), ..., TermEval( $\mathcal{I}, \alpha, t_k$ ));  
  if ( $\varphi$  is  $\neg\psi$ )  
    return  $\neg$ Truth( $\mathcal{I}, \alpha, \psi$ );  
  if ( $\varphi$  is  $\psi \circ \psi'$ )  
    return Truth( $\mathcal{I}, \alpha, \psi$ )  $\circ$  Truth( $\mathcal{I}, \alpha, \psi'$ );  
  if ( $\varphi$  is  $\exists x.\psi$ ) {  
    boolean b = false;  
    for all ( $a \in \Delta^{\mathcal{I}}$ )  
      b = b  $\vee$  Truth( $\mathcal{I}, \alpha[x \mapsto a], \psi$ );  
    return b;  
  }  
  if ( $\varphi$  is  $\forall x.\psi$ ) {  
    boolean b = true;  
    for all ( $a \in \Delta^{\mathcal{I}}$ )  
      b = b  $\wedge$  Truth( $\mathcal{I}, \alpha[x \mapsto a], \psi$ );  
    return b;  
  }  
}
```



Query evaluation – Results

Theorem (Termination of Truth($\mathcal{I}, \alpha, \varphi$))

The algorithm Truth terminates.

Proof. Immediate.

Theorem (Correctness)

The algorithm Truth is sound and complete, i.e., $\mathcal{I}, \alpha \models \varphi$ if and only if Truth($\mathcal{I}, \alpha, \varphi$) = true.

Proof. Easy, since the algorithm is very close to the semantic definition of $\mathcal{I}, \alpha \models \varphi$.



Outline

- 4 Query evaluation problem
- 5 Complexity of query evaluation



Query evaluation – Time complexity I

Theorem (Time complexity of $\text{Truth}(\mathcal{I}, \alpha, \varphi)$)

The time complexity of $\text{Truth}(\mathcal{I}, \alpha, \varphi)$ is $(|\mathcal{I}| + |\alpha| + |\varphi|)^{|\varphi|}$, i.e., polynomial in the size of \mathcal{I} and exponential in the size of φ .

Proof.

- $f^{\mathcal{I}}$ (of arity k) can be represented as k -dimensional array, hence accessing the required element can be done in time linear in $|\mathcal{I}|$.
- $\text{TermEval}(\dots)$ visits the term, so it generates a polynomial number of recursive calls, hence is time polynomial in $(|\mathcal{I}| + |\alpha| + |\varphi|)$.



Query evaluation – Time complexity II

- $P^{\mathcal{I}}$ (of arity k) can be represented as k -dimensional boolean array, hence accessing the required element can be done in time linear in $|\mathcal{I}|$.
- $\text{Truth}(\dots)$ for the boolean cases simply visits the formula, so generates either one or two recursive calls.
- $\text{Truth}(\dots)$ for the quantified cases $\exists x.\varphi$ and $\forall x.\psi$ involves looping for all elements in $\Delta^{\mathcal{I}}$ and testing the resulting assignments.
- The total number of such testings is $O(|\mathcal{I}|^{\#Vars})$.

Hence the claim holds. □



Query evaluation – Space complexity I

Theorem (Space complexity of $\text{Truth}(\mathcal{I}, \alpha, \varphi)$)

The space complexity of $\text{Truth}(\mathcal{I}, \alpha, \varphi)$ is $|\varphi| \cdot (|\varphi| \cdot \log |\mathcal{I}|)$, i.e., logarithmic in the size of \mathcal{I} and polynomial in the size of φ .

Proof.

- $f^{\mathcal{I}}(\dots)$ can be represented as k -dimensional array, hence accessing the required element requires $O(\log |\mathcal{I}|)$;
- $\text{TermEval}(\dots)$ simply visits the term, so it generates a polynomial number of recursive calls. Each activation record has a constant size, and we need $O(|\varphi|)$ activation records;
- $P^{\mathcal{I}}(\dots)$ can be represented as k -dimensional boolean array, hence accessing the required element requires $O(\log |\mathcal{I}|)$;



Query evaluation – Space complexity II

- $\text{Truth}(\dots)$ for the boolean cases simply visits the formula, so generates either one or two recursive calls, each requiring constant size;
- $\text{Truth}(\dots)$ for the quantified cases $\exists x.\varphi$ and $\forall x.\psi$ involves looping for all elements in $\Delta^{\mathcal{I}}$ and testing the resulting assignments;
- The total number of activation records that need to be at the same time on the stack is $O(\#Vars) \leq O(|\varphi|)$.

Hence the claim holds. □

Note: the worst case form for the formula is

$$\forall x_1.\exists x_2.\dots.\forall x_{n-1}.\exists x_n.P(x_1, x_2, \dots, x_{n-1}, x_n).$$



Query evaluation – Complexity measures [Var82]

Definition (Combined complexity)

The **combined complexity** is the complexity of $\{\langle \mathcal{I}, \alpha, \varphi \rangle \mid \mathcal{I}, \alpha \models \varphi\}$, i.e., interpretation, tuple, and query are all considered part of the input.

Definition (Data complexity)

The **data complexity** is the complexity of $\{\langle \mathcal{I}, \alpha \rangle \mid \mathcal{I}, \alpha \models \varphi\}$, i.e., the query φ is fixed (and hence not considered part of the input).

Definition (Query complexity)

The **query complexity** is the complexity of $\{\langle \alpha, \varphi \rangle \mid \mathcal{I}, \alpha \models \varphi\}$, i.e., the interpretation \mathcal{I} is fixed (and hence not considered part of the input).



Query evaluation – Combined, data, query complexity

Theorem (Combined complexity of query evaluation)

The complexity of $\{\langle \mathcal{I}, \alpha, \varphi \rangle \mid \mathcal{I}, \alpha \models \varphi\}$ is:

- time: exponential
- space: PSPACE-complete — see [Var82] for hardness

Theorem (Data complexity of query evaluation)

The complexity of $\{\langle \mathcal{I}, \alpha \rangle \mid \mathcal{I}, \alpha \models \varphi\}$ is:

- time: polynomial
- space: in LOGSPACE

Theorem (Query complexity of query evaluation)

The complexity of $\{\langle \alpha, \varphi \rangle \mid \mathcal{I}, \alpha \models \varphi\}$ is:

- time: exponential
- space: PSPACE-complete — see [Var82] for hardness

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Chapter III

Conjunctive Queries

Outline

- 6 Evaluation of conjunctive queries
- 7 Containment of conjunctive queries
- 8 Unions of conjunctive queries



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Conjunctive queries (CQs)

Def.: A **conjunctive query (CQ)** is a FOL query of the form

$$\exists \vec{y}. conj(\vec{x}, \vec{y})$$

where $conj(\vec{x}, \vec{y})$ is a conjunction (i.e., an “and”) of atoms and equalities, over the free variables \vec{x} , the existentially quantified variables \vec{y} , and possibly constants.

Note:

- CQs contain no disjunction, no negation, no universal quantification, and no function symbols besides constants.
- Hence, they correspond to relational algebra **select-project-join (SPJ) queries**.
- CQs are the most frequently asked queries.



Conjunctive queries and SQL – Example

Relational alphabet:

Person(name, age), Lives(person, city), Manages(boss, employee)

Query: return name and age of all persons that live in the same city as their boss.

Expressed in SQL:

```
SELECT P.name, P.age
FROM Person P, Manages M, Lives L1, Lives L2
WHERE P.name = L1.person AND P.name = M.employee AND
      M.boss = L2.person AND L1.city = L2.city
```

Expressed as a CQ:

$$\exists b, e, p_1, c_1, p_2, c_2. Person(n, a) \wedge Manages(b, e) \wedge Lives(p_1, c_1) \wedge Lives(p_2, c_2) \wedge n = p_1 \wedge n = e \wedge b = p_2 \wedge c_1 = c_2$$

Or simpler: $\exists b, c. Person(n, a) \wedge Manages(b, n) \wedge Lives(n, c) \wedge Lives(b, c)$



Datalog notation for CQs

A CQ $q = \exists \vec{y}. \text{conj}(\vec{x}, \vec{y})$ can also be written using **datalog notation** as

$$q(\vec{x}_1) \leftarrow \text{conj}'(\vec{x}_1, \vec{y}_1)$$

where $\text{conj}'(\vec{x}_1, \vec{y}_1)$ is the list of atoms in $\text{conj}(\vec{x}, \vec{y})$ obtained by equating the variables \vec{x} , \vec{y} according to the equalities in $\text{conj}(\vec{x}, \vec{y})$.

As a result of such an equality elimination, we have that \vec{x}_1 and \vec{y}_1 can contain constants and multiple occurrences of the same variable.

Def.: In the above query q , we call:

- $q(\vec{x}_1)$ the **head**;
- $\text{conj}'(\vec{x}_1, \vec{y}_1)$ the **body**;
- the variables in \vec{x}_1 the **distinguished variables**;
- the variables in \vec{y}_1 the **non-distinguished variables**.

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Conjunctive queries – Example

- Consider an **interpretation** $\mathcal{I} = (\Delta^{\mathcal{I}}, E^{\mathcal{I}})$, where $E^{\mathcal{I}}$ is a binary relation – *note that such interpretation is a (directed) graph*.
- The following **CQ** q returns all nodes that participate to a triangle in the graph:

$$\exists y, z. E(x, y) \wedge E(y, z) \wedge E(z, x)$$

- The query q in **datalog notation** becomes:

$$q(x) \leftarrow E(x, y), E(y, z), E(z, x)$$

- The query q in **SQL** is (we use $\text{Edge}(f, s)$ for $E(x, y)$):

```
SELECT E1.f
FROM Edge E1, Edge E2, Edge E3
WHERE E1.s = E2.f AND E2.s = E3.f AND E3.s = E1.f
```



Nondeterministic evaluation of CQs

Since a CQ contains only existential quantifications, we can evaluate it by:

- ① **guessing a truth assignment** for the non-distinguished variables;
- ② **evaluating** the resulting formula (that has no quantifications).

```
boolean ConjTruth( $\mathcal{I}, \alpha, \exists \vec{y}. conj(\vec{x}, \vec{y})$ ) {
  GUESS assignment  $\alpha[\vec{y} \mapsto \vec{a}]$  {
    return Truth( $\mathcal{I}, \alpha[\vec{y} \mapsto \vec{a}], conj(\vec{x}, \vec{y})$ );
  }
}
```

where $\text{Truth}(\mathcal{I}, \alpha, \varphi)$ is defined as for FOL queries, considering only the required cases.



Nondeterministic CQ evaluation algorithm

```
boolean Truth( $\mathcal{I}, \alpha, \varphi$ ) {
  if ( $\varphi$  is  $t_1 = t_2$ )
    return TermEval( $\mathcal{I}, \alpha, t_1$ ) = TermEval( $\mathcal{I}, \alpha, t_2$ );
  if ( $\varphi$  is  $P(t_1, \dots, t_k)$ )
    return  $P^{\mathcal{I}}(\text{TermEval}(\mathcal{I}, \alpha, t_1), \dots, \text{TermEval}(\mathcal{I}, \alpha, t_k))$ ;
  if ( $\varphi$  is  $\psi \wedge \psi'$ )
    return Truth( $\mathcal{I}, \alpha, \psi$ )  $\wedge$  Truth( $\mathcal{I}, \alpha, \psi'$ );
}
```

```
 $\Delta^{\mathcal{I}}$  TermEval( $\mathcal{I}, \alpha, t$ ) {
  if ( $t$  is a variable  $x$ ) return  $\alpha(x)$ ;
  if ( $t$  is a constant  $c$ ) return  $c^{\mathcal{I}}$ ;
}
```



CQ evaluation – Combined, data, and query complexity

Theorem (Combined complexity of CQ evaluation)

$\{\langle \mathcal{I}, \alpha, q \rangle \mid \mathcal{I}, \alpha \models q\}$ is **NP-complete** — see below for hardness.

- time: exponential
- space: polynomial

Theorem (Data complexity of CQ evaluation)

$\{\langle \mathcal{I}, \alpha \rangle \mid \mathcal{I}, \alpha \models q\}$ is in **LOGSPACE**

- time: polynomial
- space: logarithmic

Theorem (Query complexity of CQ evaluation)

$\{\langle \alpha, q \rangle \mid \mathcal{I}, \alpha \models q\}$ is **NP-complete** — see below for hardness.

- time: exponential
- space: polynomial

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3-colorability

A graph is **k -colorable** if it is possible to assign to each node one of k colors in such a way that every two nodes connected by an edge have different colors.

Def.: **3-colorability** is the following decision problem

Given a graph $G = (V, E)$, is it 3-colorable?

Theorem

3-colorability is NP-complete.

We exploit 3-colorability to show NP-hardness of conjunctive query evaluation.

Reduction from 3-colorability to CQ evaluation

Let $G = (V, E)$ be a graph. We define:

- An **Interpretation**: $\mathcal{I} = (\Delta^{\mathcal{I}}, E^{\mathcal{I}})$ where:
 - $\Delta^{\mathcal{I}} = \{r, g, b\}$
 - $E^{\mathcal{I}} = \{(r, g), (g, r), (r, b), (b, r), (g, b), (b, g)\}$
- A **conjunctive query**: Let $V = \{x_1, \dots, x_n\}$, then consider the boolean conjunctive query defined as:

$$q_G = \exists x_1, \dots, x_n. \bigwedge_{(x_i, x_j) \in E} E(x_i, x_j) \wedge E(x_j, x_i)$$

Theorem

G is 3-colorable iff $\mathcal{I} \models q_G$.



NP-hardness of CQ evaluation

The previous reduction immediately gives us the hardness for combined complexity.

Theorem

CQ evaluation is NP-hard in combined complexity.

Note: in the previous reduction, the interpretation does not depend on the actual graph. Hence, the reduction provides also the lower-bound for query complexity.

Theorem

CQ evaluation is NP-hard in query (and combined) complexity.



Outline

- 6 Evaluation of conjunctive queries
- 7 Containment of conjunctive queries**
- 8 Unions of conjunctive queries



Homomorphism

Let $\mathcal{I} = (\Delta^{\mathcal{I}}, P^{\mathcal{I}}, \dots, c^{\mathcal{I}}, \dots)$ and $\mathcal{J} = (\Delta^{\mathcal{J}}, P^{\mathcal{J}}, \dots, c^{\mathcal{J}}, \dots)$ be two interpretations over the same alphabet (for simplicity, we consider only constants as functions).

Def.: A **homomorphism** from \mathcal{I} to \mathcal{J}

is a mapping $h : \Delta^{\mathcal{I}} \rightarrow \Delta^{\mathcal{J}}$ such that:

- $h(c^{\mathcal{I}}) = c^{\mathcal{J}}$
- $h(P^{\mathcal{I}}(a_1, \dots, a_k)) = P^{\mathcal{J}}(h(a_1), \dots, h(a_k))$

Note: An **isomorphism** is a homomorphism that is one-to-one and onto.

Theorem

FOL is unable to distinguish between interpretations that are isomorphic.

Proof. See any standard book on logic. \square



Recognition problem and boolean query evaluation

Consider the recognition problem associated to the evaluation of a query q of arity k . Then

$$\mathcal{I}, \alpha \models q(x_1, \dots, x_k) \quad \text{iff} \quad \mathcal{I}_{\alpha, \vec{c}} \models q(c_1, \dots, c_k)$$

where $\mathcal{I}_{\alpha, \vec{c}}$ is identical to \mathcal{I} but includes new constants c_1, \dots, c_k that are interpreted as $c_i^{\mathcal{I}_{\alpha, \vec{c}}} = \alpha(x_i)$.

That is, we can **reduce the recognition problem to the evaluation of a boolean query**.



Canonical interpretation of a (boolean) CQ

Let q be a conjunctive query $\exists x_1, \dots, x_n \cdot \text{conj}$

Def.: The **canonical interpretation** \mathcal{I}_q associated with q

is the interpretation $\mathcal{I}_q = (\Delta^{\mathcal{I}_q}, P^{\mathcal{I}_q}, \dots, c^{\mathcal{I}_q}, \dots)$, where

- $\Delta^{\mathcal{I}_q} = \{x_1, \dots, x_n\} \cup \{c \mid c \text{ constant occurring in } q\}$,
i.e., all the variables and constants in q ;
- $c^{\mathcal{I}_q} = c$, for each constant c in q ;
- $(t_1, \dots, t_k) \in P^{\mathcal{I}_q}$ iff the atom $P(t_1, \dots, t_k)$ occurs in q .

Sometimes the procedure for obtaining the canonical interpretation is called **freezing** of q .



Canonical interpretation of a (boolean) CQ – Example

Consider the boolean query q

$$q(c) \leftarrow E(c, y), E(y, z), E(z, c)$$

Then, the canonical interpretation \mathcal{I}_q is defined as

$$\mathcal{I}_q = (\Delta^{\mathcal{I}_q}, E^{\mathcal{I}_q}, c^{\mathcal{I}_q})$$

where

- $\Delta^{\mathcal{I}_q} = \{y, z, c\}$
- $E^{\mathcal{I}_q} = \{(c, y), (y, z), (z, c)\}$
- $c^{\mathcal{I}_q} = c$



Canonical interpretation and (boolean) CQ evaluation

Theorem ([CM77])

For boolean CQs, $\mathcal{I} \models q$ iff there exists a homomorphism from \mathcal{I}_q to \mathcal{I} .

Proof.

“ \Rightarrow ” Let $\mathcal{I} \models q$, let α be an assignment to the existential variables that makes q true in \mathcal{I} , and let $\hat{\alpha}$ be its extension to constants. Then $\hat{\alpha}$ is a homomorphism from \mathcal{I}_q to \mathcal{I} .

“ \Leftarrow ” Let h be a homomorphism from \mathcal{I}_q to \mathcal{I} . Then restricting h to the variables only we obtain an assignment to the existential variables that makes q true in \mathcal{I} . □



Canonical interpretation and (boolean) CQ evaluation

The previous result can be rephrased as follows:

(The recognition problem associated to) **query evaluation can be reduced to finding a homomorphism.**

Finding a homomorphism between two interpretations (aka relational structures) is also known as solving a **Constraint Satisfaction Problem (CSP)**, a problem well-studied in AI – see also [KV98].



Query containment

Def.: Query containment

Given two FOL queries φ and ψ of the same arity, φ is contained in ψ , denoted $\varphi \subseteq \psi$, if for all interpretations \mathcal{I} and all assignments α we have that

$$\mathcal{I}, \alpha \models \varphi \text{ implies } \mathcal{I}, \alpha \models \psi$$

(In logical terms: $\varphi \models \psi$.)

Note: Query containment is of special interest in query optimization.

Theorem

For FOL queries, query containment is undecidable.

Proof.: Reduction from FOL logical implication. \square



Query containment for CQs

For CQs, query containment $q_1(\vec{x}) \subseteq q_2(\vec{x})$ can be reduced to query evaluation.

- 1 **Freeze the free variables**, i.e., consider them as constants.
This is possible, since $q_1(\vec{x}) \subseteq q_2(\vec{x})$ iff
 - $\mathcal{I}, \alpha \models q_1(\vec{x})$ implies $\mathcal{I}, \alpha \models q_2(\vec{x})$, for all \mathcal{I} and α ; or equivalently
 - $\mathcal{I}_{\alpha, \vec{c}} \models q_1(\vec{c})$ implies $\mathcal{I}_{\alpha, \vec{c}} \models q_2(\vec{c})$, for all $\mathcal{I}_{\alpha, \vec{c}}$, where \vec{c} are new constants, and $\mathcal{I}_{\alpha, \vec{c}}$ extends \mathcal{I} to the new constants with $c^{\mathcal{I}_{\alpha, \vec{c}}} = \alpha(x)$.
- 2 **Construct the canonical interpretation $\mathcal{I}_{q_1(\vec{c})}$ of the CQ $q_1(\vec{c})$ on the left hand side ...**
- 3 **... and evaluate on $\mathcal{I}_{q_1(\vec{c})}$ the CQ $q_2(\vec{c})$ on the right hand side, i.e., check whether $\mathcal{I}_{q_1(\vec{c})} \models q_2(\vec{c})$.**



Reducing containment of CQs to CQ evaluation

Theorem ([CM77])

For CQs, $q_1(\vec{x}) \subseteq q_2(\vec{x})$ iff $\mathcal{I}_{q_1(\vec{c})} \models q_2(\vec{c})$, where \vec{c} are new constants.

Proof.

“ \Rightarrow ” Assume that $q_1(\vec{x}) \subseteq q_2(\vec{x})$.

- Since $\mathcal{I}_{q_1(\vec{c})} \models q_1(\vec{c})$ it follows that $\mathcal{I}_{q_1(\vec{c})} \models q_2(\vec{c})$.

“ \Leftarrow ” Assume that $\mathcal{I}_{q_1(\vec{c})} \models q_2(\vec{c})$.

- By [CM77] on hom., for every \mathcal{I} such that $\mathcal{I} \models q_1(\vec{c})$ there exists a homomorphism h from $\mathcal{I}_{q_1(\vec{c})}$ to \mathcal{I} .
- On the other hand, since $\mathcal{I}_{q_1(\vec{c})} \models q_2(\vec{c})$, again by [CM77] on hom., there exists a homomorphism h' from $\mathcal{I}_{q_2(\vec{c})}$ to $\mathcal{I}_{q_1(\vec{c})}$.
- The mapping $h \circ h'$ (obtained by composing h and h') is a homomorphism from $\mathcal{I}_{q_2(\vec{c})}$ to \mathcal{I} . Hence, once again by [CM77] on hom., $\mathcal{I} \models q_2(\vec{c})$.

So we can conclude that $q_1(\vec{c}) \subseteq q_2(\vec{c})$, and hence $q_1(\vec{x}) \subseteq q_2(\vec{x})$. \square



Query containment for CQs

For CQs, we also have that (boolean) query evaluation $\mathcal{I} \models q$ can be reduced to query containment.

Let $\mathcal{I} = (\Delta^{\mathcal{I}}, P^{\mathcal{I}}, \dots, c^{\mathcal{I}}, \dots)$.

We construct the (boolean) CQ $q_{\mathcal{I}}$ as follows:

- $q_{\mathcal{I}}$ has no existential variables (hence no variables at all);
- the constants in $q_{\mathcal{I}}$ are the elements of $\Delta^{\mathcal{I}}$;
- for each relation P interpreted in \mathcal{I} and for each fact $(a_1, \dots, a_k) \in P^{\mathcal{I}}$, $q_{\mathcal{I}}$ contains one atom $P(a_1, \dots, a_k)$ (note that each $a_i \in \Delta^{\mathcal{I}}$ is a constant in $q_{\mathcal{I}}$).

Theorem

For CQs, $\mathcal{I} \models q$ iff $q_{\mathcal{I}} \subseteq q$.



Query containment for CQs – Complexity

From the previous results and NP-completeness of combined complexity of CQ evaluation, we immediately get:

Theorem

Containment of CQs is NP-complete.

Since CQ evaluation is NP-complete even in query complexity, the above result can be strengthened:

Theorem

Containment $q_1(\vec{x}) \subseteq q_2(\vec{x})$ of CQs is NP-complete, even when q_1 is considered fixed.



Outline

- 6 Evaluation of conjunctive queries
- 7 Containment of conjunctive queries
- 8 Unions of conjunctive queries**



Union of conjunctive queries (UCQs)

Def.: A **union of conjunctive queries (UCQ)** is a FOL query of the form

$$\bigvee_{i=1, \dots, n} \exists \vec{y}_i. \text{conj}_i(\vec{x}, \vec{y}_i)$$

where each $\text{conj}_i(\vec{x}, \vec{y}_i)$ is a conjunction of atoms and equalities with free variables \vec{x} and \vec{y}_i , and possibly constants.

Note: Obviously, each conjunctive query is also a union of conjunctive queries.



Datalog notation for UCQs

A union of conjunctive queries

$$q = \bigvee_{i=1, \dots, n} \exists \vec{y}_i. \text{conj}_i(\vec{x}, \vec{y}_i)$$

is written in **datalog notation** as

$$\left\{ \begin{array}{l} q(\vec{x}) \leftarrow \text{conj}'_1(\vec{x}, \vec{y}'_1) \\ \vdots \\ q(\vec{x}) \leftarrow \text{conj}'_n(\vec{x}, \vec{y}'_n) \end{array} \right\}$$

where each element of the set is the datalog expression corresponding to the conjunctive query $q_i = \exists \vec{y}_i. \text{conj}_i(\vec{x}, \vec{y}_i)$.

Note: in general, we omit the set brackets.



Evaluation of UCQs

From the definition of FOL query we have that:

$$\mathcal{I}, \alpha \models \bigvee_{i=1, \dots, n} \exists \vec{y}_i. \text{conj}_i(\vec{x}, \vec{y}_i)$$

if and only if

$$\mathcal{I}, \alpha \models \exists \vec{y}_i. \text{conj}_i(\vec{x}, \vec{y}_i) \quad \text{for some } i \in \{1, \dots, n\}.$$

Hence to evaluate a UCQ q , we simply evaluate a number (linear in the size of q) of conjunctive queries in isolation.

Hence, **evaluating UCQs has the same complexity as evaluating CQs.**



UCQ evaluation – Combined, data, and query complexity

Theorem (Combined complexity of UCQ evaluation)

$\{\langle \mathcal{I}, \alpha, q \rangle \mid \mathcal{I}, \alpha \models q\}$ is **NP-complete**.

- time: exponential
- space: polynomial

Theorem (Data complexity of UCQ evaluation)

$\{\langle \mathcal{I}, q \rangle \mid \mathcal{I}, \alpha \models q\}$ is in **LOGSPACE** (query q fixed).

- time: polynomial
- space: logarithmic

Theorem (Query complexity of UCQ evaluation)

$\{\langle \alpha, q \rangle \mid \mathcal{I}, \alpha \models q\}$ is **NP-complete** (interpretation \mathcal{I} fixed).

- time: exponential
- space: polynomial

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Query containment for UCQs

Theorem

For UCQs, $\{q_1, \dots, q_k\} \subseteq \{q'_1, \dots, q'_n\}$ iff for each q_i there is a q'_j such that $q_i \subseteq q'_j$.

Proof.

“ \Leftarrow ” Obvious.

“ \Rightarrow ” If the containment holds, then we have

$\{q_1(\vec{c}), \dots, q_k(\vec{c})\} \subseteq \{q'_1(\vec{c}), \dots, q'_n(\vec{c})\}$, where \vec{c} are new constants:

- Now consider $\mathcal{I}_{q_i(\vec{c})}$. We have $\mathcal{I}_{q_i(\vec{c})} \models q_i(\vec{c})$, and hence $\mathcal{I}_{q_i(\vec{c})} \models \{q_1(\vec{c}), \dots, q_k(\vec{c})\}$.
- By the containment, we have that $\mathcal{I}_{q_i(\vec{c})} \models \{q'_1(\vec{c}), \dots, q'_n(\vec{c})\}$. I.e., there exists a $q'_j(\vec{c})$ such that $\mathcal{I}_{q_i(\vec{c})} \models q'_j(\vec{c})$.
- Hence, by [CM77] on containment of CQs, we have that $q_i \subseteq q'_j$.

□



Query containment for UCQs – Complexity

From the previous result, we have that we can check

$\{q_1, \dots, q_k\} \subseteq \{q'_1, \dots, q'_n\}$ by at most $k \cdot n$ CQ containment checks.

We immediately get:

Theorem

Containment of UCQs is NP-complete.



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