E 6.1

Exercise

Write a grammar for the language

$$\{a^{n^2} \mid n \geq 1\}$$

solution

$S \rightarrow IT$

$T \rightarrow aTM \mid a$

$aM \rightarrow MAa$
$aA \rightarrow Aa$
$AM \rightarrow MA$
$IM \rightarrow I$

$IA \rightarrow aI$

$I \rightarrow \epsilon$

Comments: I is a marker at the beginning of the strings; the second production generates n symbols a and $n-1$ symbols M. The idea is that M is a sort of "multiplier", adding a symbol A for every symbol a. At the end, the marker I "travels" from left to right, "killing" symbols M and turning A into a. The number of symbols a at the end, being n the number of a generated in the beginning, is $n + (n-1) \cdot n = n^2$.

example of derivation:

$S \Rightarrow IT \Rightarrow IaTM \Rightarrow^* IaaaMM \Rightarrow IaaMAaM \Rightarrow$

$\Rightarrow IaMAaAaM \Rightarrow IMAaAaAaM \Rightarrow^* IMAAAaaaM \Rightarrow^*$

$\Rightarrow^* IMMAAAAAAaaa \Rightarrow^* IAAAAAAaaa \Rightarrow^* aaaaaaIaaa \Rightarrow$

$\Rightarrow aaaaaaaaa$

E 6.2

Exercise

Prove that the language $L = \{ a^k \mid k \text{ is not prime} \}$ is not regular.

solution

We have already proved that the language

$$M = \{ a^k \mid k \text{ is prime} \}$$

is not regular (see page 4.3). By contradiction, assume that L is regular. Since the class of regular languages is closed under complementation, we have that $\bar{L}$ is regular. Since $\bar{L} = M$, then M is regular too, which is a contradiction. Therefore L cannot be regular.

E 6.3

Exercise

Give a context-free grammar for the language over $\Sigma = \{0,1\}$ defined as follows: $\{0^i 1^j \mid i \leq j \leq 2i \text{ and } i \geq 0\}$.

solution

The grammar is

$$S \to 0S11 \mid 0S1 \mid \epsilon$$

Exercise

Prove that the language

$$L = \{a^{k^3} \mid k \geq 1\}$$

is not regular.

solution

We apply the pumping lemma, assuming that L is regular. We choose w such that $|w| \geq m$ by choosing

$w = 0^{m^3}$; in fact here $|w| = m^3 > m$.

By the pumping lemma $w = xyz$, $|xy| \leq m$, $xy^iz \in L$ for all $i \geq 0$. We choose $i = 2$, so

$$w' = xy^2z \in L.$$

Observe that $|w'| = |w| + |y| = m^3 + |y|$. Since $|xy| \leq m$, a fortiori $|y| \leq m$, so

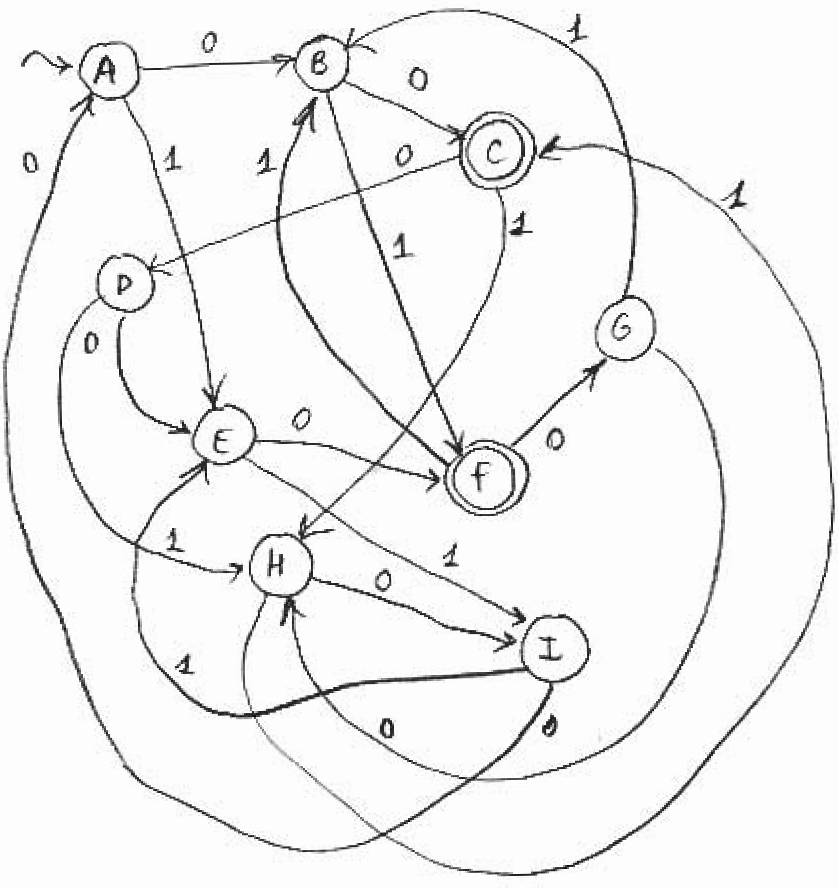
$$|w'| = m^3 + |y| \leq m^3 + m < m^3 + 3m^2 + 3m + 1 = (m+1)^3$$

Therefore $|w| = m^3 < |w'| < (m+1)^3$

so $|w'|$ cannot be a perfect cube.

Exercise (4.4.2 from textbook)

Minimise the following DFA:



	0	1
→ A	B	E
B	C	F
* C	D	H
D	E	H
E	F	I
* F	G	B
G	H	B
H	I	C
I	A	E

solution

We construct the table of distinguishabilities. Notice that, in the table, pairs marked with i are those that are not equivalent with respect to $\equiv_i$ and are equivalent w.r.t. $\equiv_{i-1}$. In the beginning, we mark with 0 the pairs (q_h, q_k) where q_h is final and q_k is not or vice-versa; in fact, we are determining classes of equivalence w.r.t. $\equiv_0$. At the next step we somehow refine the previous partitioning by determining pairs of states that are not equivalent w.r.t. $\equiv_1$ (we recall that if $q_h \not\equiv_i q_k$ then $q_h \not\equiv_{i-1} q_k$). We proceed in this way until the partitioning cannot be further refined.

E 6.5

B	1							
C	0	0						
D	2	1	0					
E	1	1	0	1				
F	0	0	2	0	0			
G	2	1	0	2	1	0		
H	1	1	0	1	1	0	1	
I	2	1	0	2	1	0	2	1
	A	B	C	D	E	F	G	H

At the first step the marks "0" partition the set of states w.r.t. $\equiv_0$

$$\{C, F\} \ , \ \{A, B, D, E, G, H, I\}$$

The partition w.r.t. $\equiv_1$ is the following:

$$\{C, F\} \ , \ \{A, I, D, G\}, \{B\} \ , \{H\}$$

Finally, the partition w.r.t. $\equiv_2$, obtained by considering marks "0", "1", and "2", is

$$\{A\}, \{B\}, \{C\}, \{D\}, \{E\}, \{F\}, \{G\}, \{H\}, \{I\}.$$

Therefore the automaton is already minimal.