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Capturing model-based ontology evolution at the instance level: The case of *DL-Lite* ☆

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ABSTRACT

Evolution of Knowledge Bases (KBs) expressed in Description Logics (DLs) has gained a lot of attention lately. Recent studies on the topic have mostly focused on so-called model-based approaches (MBAs), where the evolution of a KB results in a set of models. For KBs expressed in tractable DLs, such as those of the *DL-Lite* family, which we consider here, it has been shown that one faces inexpressibility of evolution, i.e., the result of evolution of a *DL-Lite* KB in general cannot be *expressed* in *DL-Lite*, in other words, *DL-Lite* is *not closed* under evolution. What is still missing in these studies is a thorough *understanding* of various important aspects of the evolution problem for *DL-Lite* KBs: Which fragments of *DL-Lite* are closed under evolution? What causes the inexpressibility? Can one approximate evolution in *DL-Lite*, and if yes, how? This work provides some understanding of these issues for an important class of MBAs, which cover the cases of both update and revision. We describe what causes inexpressibility, and we propose techniques (based on what we call prototypes) that help to approximate evolution under the well-known approach by Winslett, which is inexpressible in *DL-Lite*. We also identify a fragment of *DL-Lite* closed under evolution, and for this fragment we provide polynomial-time algorithms to compute or approximate evolution results for various MBAs.

1. Introduction

Description Logics (DLs) provide excellent mechanisms for representing structured knowledge. In DLs, a Knowledge Base (KB) consists of two components. The first component is a TBox, which describes general knowledge about an application domain in terms of classes of objects with common properties, so-called *concepts*, and binary relationships between objects, so-called *roles*. The second component of a KB is an ABox, which describes facts about individual objects. DLs constitute the foundations for various dialects of the Web Ontology Language (OWL) [1], which is the language standardized by the World Wide Web Consortium (W3C) for representing ontologies in the Semantic Web.

Traditionally DLs have been used for modeling at the intensional level the *static* and structural aspects of an application domain [2]. Recently, however, the scope of KBs has broadened, and they are now used also for providing support in the maintenance and *evolution* phase of information systems. This makes it necessary to study *evolution of KBs* [3], where the goal is to incorporate new knowledge $\mathcal{N}$ into an existing KB $\mathcal{K}$ so as to take into account changes that occur in the underlying application domain. In general, $\mathcal{N}$ is a set of formulas that represent properties that should be true after $\mathcal{K}$ evolves, and the result of evolution is also intended to be a set of formulas. In the case where $\mathcal{N}$ interacts with $\mathcal{K}$ in an undesirable way,

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e.g., by causing the KB or relevant parts of it to become unsatisfiable, the new knowledge $\mathcal{N}$ cannot be simply added to the KB. Instead, suitable changes need to be made in $\mathcal{K}$ so as to avoid this undesirable interaction, e.g., by deleting parts of $\mathcal{K}$ conflicting with $\mathcal{N}$. Different choices for changes are possible, corresponding to different approaches to the semantics of KB evolution [4–6].

An important group of approaches to evolution semantics, on which we focus in this paper, is called *model-based*. Under model-based approaches (MBAs), the result of evolution $\mathcal{K} \diamond \mathcal{N}$ is a *set of models* of $\mathcal{N}$, minimally distant from the models of $\mathcal{K}$. Depending on how the distance between models is specified, different MBAs can be defined. In this paper we refer to eight different MBAs presented in [7], which we will define and discuss in detail in Section 2.3. Since the result of evolution $\mathcal{K} \diamond \mathcal{N}$ is a set of models, while $\mathcal{K}$ and $\mathcal{N}$ are logical theories, it is desirable to represent $\mathcal{K} \diamond \mathcal{N}$ as a logical theory using the same language as for $\mathcal{K}$ and $\mathcal{N}$. Thus, looking for representations of $\mathcal{K} \diamond \mathcal{N}$ is one of the main challenges in studies of evolution under MBAs. When $\mathcal{K}$ and $\mathcal{N}$ are propositional theories, representing $\mathcal{K} \diamond \mathcal{N}$ as a propositional theory is always possible [6], while the situation becomes dramatically more complicated as soon as $\mathcal{K}$ and $\mathcal{N}$ are first-order, e.g., DL KBs [7,8].

Model-based evolution of KBs where both $\mathcal{K}$ and $\mathcal{N}$ are expressed in a language of the *DL-Lite* family [9] has recently received a lot of attention [8,10–13]. The focus on *DL-Lite* is not surprising since this family has been specifically designed to capture the fundamental constructs of widely used conceptual modeling formalisms, such as UML Class Diagrams and the Entity-Relationship model [14,15]. *DL-Lite* is also at the basis of OWL 2 QL, one of the tractable fragments (or profiles) of the OWL 2 standard [1,16]. It has been shown that for each of the eight quite natural model-based semantics presented in [7], one can find *DL-Lite* KBs $\mathcal{K}$ and $\mathcal{N}$ such that $\mathcal{K} \diamond \mathcal{N}$ cannot be expressed in *DL-Lite* [7,17], that is, *DL-Lite* is *not closed* under MBAs to evolution. This phenomenon was also noted in [8] for other DLs. What is missing in all these studies of evolution for *DL-Lite* is a thorough *understanding* of:

(1) *DL-Lite w.r.t. evolution*: Which fragments of *DL-Lite* are closed under model-based semantics of evolution? Which *DL-Lite* formulas are responsible for inexpressibility of model-based semantics of evolution?
(2) *Evolution w.r.t. DL-Lite*: Is it possible to capture evolution of *DL-Lite* KBs in richer logics and how can it be done? Which are these logics?
(3) *Approximation of evolution results*: For *DL-Lite* KBs $\mathcal{K}$ and $\mathcal{N}$, is it possible to obtain "good" approximations of $\mathcal{K} \diamond \mathcal{N}$ in *DL-Lite* and how can it be done?

In this paper we study Problems (1) and (3)[1] for evolution that affects the ABox-level only, so-called *ABox evolution*, where $\mathcal{N}$ is an ABox, and the TBox of $\mathcal{K}$ should stay the same before and after the evolution. ABox evolution is relevant in those settings where the structural knowledge (TBox) is well crafted and stable, while (ABox) facts about specific individuals may get changed, or new facts may be inserted in the ABox. These ABox changes should be reflected in the resulting KB without affecting the TBox. Our study covers both cases of ABox updates and ABox revision [5]. One significant area where ABox evolution is particularly relevant is Semantic Web services, where one is interested in studying the effects of services that perform operations over the instance data. Such data are inherently incomplete and thus can be effectively represented by means of an ABox. Moreover, the services have to obey the semantics of the domain of interest, which is modeled through a TBox, which is assumed not to change over time.

We now describe the contributions of this work and how it is organized. In Section 2.1, we review relevant notions from Description Logics and present the DL *DL-Lite*$_{core}$, which subsumes the DLs considered in this paper for which we study evolution; in Section 2.3, we introduce the notion of evolution, eight evolution semantics, and the two main problems related to evolution. Our contributions are the following:

- In Section 2.2, we introduce *DL-Lite*^{pr}, a restriction of *DL-Lite*$_{core}$, which should be interesting in practice because, on the one hand, it extends the first-order fragment of the RDFS ontology language [19], and, on the other hand, we prove that it is closed under most of MBAs and for the other MBAs "good" approximations of evolution can be efficiently computed.
- In Section 3, we study evolution of *DL-Lite*^{pr} KBs under three model-based semantics. More precisely, we prove that
  - *DL-Lite*^{pr} is closed under two of them, and for this case we present two corresponding polynomial-time algorithms to compute evolution results (in Sections 3.1 and 3.2);
  - *DL-Lite*^{pr} is not closed under the third semantics, and for this case we present a polynomial-time approximation algorithm.
- In Section 4, we introduce the notion of *subsumption relation* between model-based evolution semantics and prove this relation between some of them, first, for the case of arbitrary DLs, and then for *DL-Lite*^{pr}. In particular, we show that for *DL-Lite*^{pr} all the eight MBAs considered in this work collapse into three different equivalence classes w.r.t. the subsumption relation. Moreover, the three MBAs for which we study evolution in Section 3 are representatives of these

[1] We focus on these two problems because they are of higher practical value than Problem (2). We refer the reader to [18] for more details on Problem (2).

equivalence classes. Thus, the results we present in Section 3 carry over to all the other model-based semantics of this work.

- In Section 5, we study evolution beyond *DL-Lite*^{pr} under an important MBA corresponding to the well accepted *Winslett's semantics* [20], which is one of the eight MBAs considered in this work.
  - For this semantics we show which combination of TBox and ABox assertions in $\mathcal{K}$ together with ABox assertions of $\mathcal{N}$ leads to inexpressibility of evolution (Section 5.1).
  - We introduce *prototypes*, which are a generalization of canonical models.
- In Section 6 we continue the study of Section 5 and show how an approximation of evolution under this semantics can be efficiently computed.

Note. Due to space limitations, some proofs are sketched or omitted from the main body of the paper. Full proofs can be found in Appendices A–H.

2. The *DL-Lite* family of Description Logics and knowledge evolution

We first present the Description Logic *DL-Lite*$_{core}$ and its fragment *DL-Lite*^{pr}. We then define the problem of knowledge evolution, model-based approaches to the evolution problem, and the main challenges with these approaches.

2.1. The Description Logic DL-Lite$_{core}$: basic definitions

In DLs [2], the domain of interest is modeled by means of *concepts*, denoting sets of objects, *roles*, denoting binary relations between objects, and *constants*, denoting objects. Complex concepts and roles are obtained from atomic ones by applying suitable constructs. We consider here the logic *DL-Lite*$_{core}$, a member of the *DL-Lite* family of lightweight DLs [9]. We observe that all other members of the family extend *DL-Lite*$_{core}$ with additional constructs (see [9,21] for details). Hence, all negative results about expressibility that we establish in this work extend immediately to the other members of the *DL-Lite* family. In *DL-Lite*$_{core}$, (complex) concepts and roles are constructed according to the following syntax:

$$B ::= A \mid \exists R, \qquad C ::= B \mid \neg B, \qquad R ::= P \mid P^-,$$

where A denotes an *atomic concept*, B a *basic concept*, and C a *general* concept; P denotes an *atomic role* and R a *basic* role, i.e., either a direct or inverse role. Note that in the following, when we write R^- for a basic role R, it will denote (i) P^- if $R = P$, and (ii) P if $R = P^-$.

In *DL-Lite*$_{core}$ the knowledge about the domain of interest is represented by means of a *knowledge base* (KB) $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, constituted by a set of assertions partitioned into a TBox $\mathcal{T}$ and an ABox $\mathcal{A}$. A TBox consists of *positive* and *negative* (*concept*) *inclusion assertions* (PIs and NIs for short), respectively of the form

$$B_1 \sqsubseteq B_2, \qquad B_1 \sqsubseteq \neg B_2,$$

which are used to assert intensional domain knowledge. When $B_1 \sqsubseteq \neg B_2 \in \mathcal{T}$, we will say that B_1 and B_2 are *disjoint*. An ABox consists of *membership assertions* (MAs) of the form

$$P(c,d), \qquad A(c), \qquad \exists R(c), \qquad \neg A(c),$$

where c and d are constants. MAs of the form $\neg A(c)$ are called *negative*, while those of the other forms are called *positive*. A membership assertion of the form $A(c)$ or $P(c,d)$ is also called an *atom*. A *literal* is an atom or the negation of an atom. With a slight abuse of notation, whenever needed we will use $R(a,b)$ to denote the atom $P(a,b)$ when $R = P$, and the atom $P(b,a)$ when $R = P^-$.

Example 2.1. Consider a KB $\mathcal{K}_0 = \mathcal{T}_0 \cup \mathcal{A}_0$, where

$$\mathcal{T}_0 = \{Card \sqsubseteq Priest, \exists HasHusb^- \sqsubseteq \neg Priest, Husb \sqsubseteq \exists HasHusb^-, Wife \sqsubseteq \exists HasHusb\}.$$

Intuitively, $\mathcal{T}_0$ says that cardinals are priests, husbands are not priests, and husbands and wives participate in the *HasHusb* relationship. Let now

$$\mathcal{A}_0 = \{Priest(pedro), Priest(ivan), Husb(john), Wife(mary), Wife(chloe), HasHusb(mary, john)\}.$$

Intuitively $\mathcal{A}_0$ says that there are two priests *pedro* and *ivan*, one husband *john*, two wives *mary* and *chloe*, and that *mary* and *john* are married. □

A *signature* Σ is a finite set of concept names, role names, and constants. The *signature* $\Sigma(F)$ *of an assertion* F is the set of concept names, role names, and constants occurring in F, and the signature of a KB $\mathcal{K}$ is $\Sigma(\mathcal{K}) = \bigcup_{F \in \mathcal{K}} \Sigma(F)$. The *size* $|\mathcal{K}|$ of a KB $\mathcal{K}$ is the cardinality $|\Sigma(\mathcal{K})|$ of its signature. We say that $\mathcal{K}$ *is over a signature* Σ_0 if $\Sigma(\mathcal{K}) \subseteq \Sigma_0$. The *active domain* of $\mathcal{K}$, denoted $\mathsf{adom}(\mathcal{K})$, is the set of all constants occurring in $\mathcal{K}$. It clearly holds that $\mathsf{adom}(\mathcal{K}) \subseteq \Sigma(\mathcal{K})$.

The semantics of *DL-Lite$_{core}$* KBs is given in the standard way, using first-order interpretations, which we assume to be all over the same countably infinite domain Δ. An *interpretation* $\mathcal{I}$ *over a signature* Σ_0 (or just *interpretation* when the signature is clear from the context or not important) is a function $\cdot^{\mathcal{I}}$ that assigns to each constant a an element $a^{\mathcal{I}} \in \Delta$, to each concept C a subset $C^{\mathcal{I}}$ of Δ, and to each role R a binary relation $R^{\mathcal{I}}$ over Δ in such a way that $(P^-)^{\mathcal{I}} = \{(a_2, a_1) \mid (a_1, a_2) \in P^{\mathcal{I}}\}$, $(\exists R)^{\mathcal{I}} = \{a \mid \exists a'.(a, a') \in R^{\mathcal{I}}\}$, and $(\neg B)^{\mathcal{I}} = \Delta \setminus B^{\mathcal{I}}$. Following a common practice, we adopt so-called *standard names* [9,21], that is, we assume that Δ contains the constants and for every interpretation $\mathcal{I}$ and every constant c, it holds that $c^{\mathcal{I}} = c$. It remains to be investigated how the results of this work can be extended to the case of *DL-Lite$_{core}$* without standard names, though we conjecture that dropping standard names is harmless.

An interpretation $\mathcal{I}$ is a *model* of an inclusion assertion $C_1 \sqsubseteq C_2$ if $C_1^{\mathcal{I}} \subseteq C_2^{\mathcal{I}}$, of a membership assertion $C(a)$ if $a \in C^{\mathcal{I}}$, and of a membership assertion $P(a, b)$ if $(a, b) \in P^{\mathcal{I}}$. It is often convenient to view interpretations as sets of atoms and say that $A(a) \in \mathcal{I}$ if and only if $a \in A^{\mathcal{I}}$, and $P(a, b) \in \mathcal{I}$ if and only if $(a, b) \in P^{\mathcal{I}}$. Under this view, if $\mathcal{I}' \subseteq \mathcal{I}$, we say that $\mathcal{I}'$ is a *submodel* of $\mathcal{I}$. As usual, we use $\mathcal{I} \models F$ to denote that $\mathcal{I}$ is a *model* of an assertion F, and $\mathcal{I} \models \mathcal{K}$ to denote that $\mathcal{I}$ is a *model* of a KB $\mathcal{K}$, i.e., an interpretation over $\Sigma(\mathcal{K})$ that is a model of each assertion in $\mathcal{K}$. We use $\mathsf{Mod}(\mathcal{K})$ to denote the set of all models of $\mathcal{K}$. A KB is *satisfiable* if it has at least one model.

In the following, we consider *only* satisfiable KBs. Notice that this assumption does not affect the complexity results of this work due to the nice computational properties of the *DL-Lite* family.

For example, for *DL-Lite$_{core}$*, checking KB satisfiability can be done in polynomial time in the size of the TBox and with logarithmic-space[2] in the size of the ABox [21,22].

We use *entailment between KBs*, denoted $\mathcal{K} \models \mathcal{K}'$, in the standard sense, i.e., every model of $\mathcal{K}$ is also a model of $\mathcal{K}'$ (similarly, we use entailment between TBoxes and between ABoxes). An ABox $\mathcal{A}$ *$\mathcal{T}$-entails* an ABox $\mathcal{A}'$, denoted $\mathcal{A} \models_{\mathcal{T}} \mathcal{A}'$, if $\mathcal{T} \cup \mathcal{A} \models \mathcal{A}'$, and $\mathcal{A}$ is *$\mathcal{T}$-equivalent* to $\mathcal{A}'$, denoted $\mathcal{A} \equiv_{\mathcal{T}} \mathcal{A}'$, if $\mathcal{A} \models_{\mathcal{T}} \mathcal{A}'$ and $\mathcal{A}' \models_{\mathcal{T}} \mathcal{A}$. We use $\perp$ to denote falsehood, and $\mathcal{A} \not\models_{\mathcal{T}} \perp$ to denote that the KB $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ is satisfiable (i.e., $\mathcal{A}$ does not $\mathcal{T}$-entail falsehood). We also say that $\mathcal{A}$ and $\mathcal{A}'$ are *$\mathcal{T}$-satisfiable* if $\mathcal{A} \cup \mathcal{A}' \not\models_{\mathcal{T}} \perp$.

The deductive *closure of a TBox* $\mathcal{T}$, denoted $\mathsf{cl}(\mathcal{T})$, is the set of all TBox assertions F such that $\mathcal{T} \models F$. For a satisfiable KB $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, the *closure of the ABox* $\mathcal{A}$ (w.r.t. $\mathcal{T}$), denoted $\mathsf{cl}_{\mathcal{T}}(\mathcal{A})$, is the set of all membership assertions f (both positive and negative) over $\mathsf{adom}(\mathcal{K})$ such that $\mathcal{A} \models_{\mathcal{T}} f$. In *DL-Lite$_{core}$*, both $\mathsf{cl}(\mathcal{T})$ and $\mathsf{cl}_{\mathcal{T}}(\mathcal{A})$ are computable in time quadratic in the number of assertions of $\mathcal{T}$ and $\mathcal{T} \cup \mathcal{A}$, respectively. Whenever needed, we will assume w.l.o.g. that all TBoxes and ABoxes are closed.

A *homomorphism* μ from an interpretation $\mathcal{I}$ to an interpretation $\mathcal{J}$ over the same signature Σ_0, is a structure-preserving mapping from Δ to Δ satisfying: (i) $\mu(a) = a$ for every constant $a \in \Sigma_0$; (ii) if $x \in A^{\mathcal{I}}$, then $\mu(x) \in A^{\mathcal{J}}$ for every atomic concept A; (iii) if $(x, y) \in P^{\mathcal{I}}$, then $(\mu(x), \mu(y)) \in P^{\mathcal{J}}$ for every atomic role P. We write $\mathcal{I} \hookrightarrow \mathcal{J}$ if there is a homomorphism from $\mathcal{I}$ to $\mathcal{J}$.

Now we provide the definition of *chase*[3] of an ABox as introduced in [9], which is an adaptation of the notion of *restricted chase* from [24]. Let $\mathcal{T}$ and $\mathcal{A}$ be a *DL-Lite$_{core}$* TBox and ABox, respectively. Then, the *chase* of $\mathcal{A}$ w.r.t. $\mathcal{T}$, denoted $\mathsf{chase}_{\mathcal{T}}(\mathcal{A})$, is an interpretation of $\mathcal{T} \cup \mathcal{A}$ that can be defined procedurally as follows: take $\mathsf{chase}_{\mathcal{T}}(\mathcal{A}) := \{g \mid g \in \mathcal{A}$ and g has form $A(a)$ or $P(a, b)\}$, and exhaustively apply the following rules.

- if $A_1(x) \in \mathsf{chase}_{\mathcal{T}}(\mathcal{A})$, $A_1 \sqsubseteq A_2 \in \mathcal{T}$, and $A_2(x) \notin \mathsf{chase}_{\mathcal{T}}(\mathcal{A})$, then $\mathsf{chase}_{\mathcal{T}}(\mathcal{A}) := \mathsf{chase}_{\mathcal{T}}(\mathcal{A}) \cup \{A_2(x)\}$;
- if $A(x) \in \mathsf{chase}_{\mathcal{T}}(\mathcal{A})$, $A \sqsubseteq \exists R \in \mathcal{T}$, and there is no y such that $R(x, y) \in \mathsf{chase}_{\mathcal{T}}(\mathcal{A})$, then $\mathsf{chase}_{\mathcal{T}}(\mathcal{A}) := \mathsf{chase}_{\mathcal{T}}(\mathcal{A}) \cup \{R(x, y_{new})\}$, where $y_{new} \in \Delta \setminus \mathsf{adom}(\mathcal{K})$ is a fresh element that has not appeared in $\mathsf{chase}_{\mathcal{T}}(\mathcal{A})$ before;
- if $R(x, y) \in \mathsf{chase}_{\mathcal{T}}(\mathcal{A})$, $\exists R \sqsubseteq A \in \mathcal{T}$, and $A(x) \notin \mathsf{chase}_{\mathcal{T}}(\mathcal{A})$, then $\mathsf{chase}_{\mathcal{T}}(\mathcal{A}) := \mathsf{chase}_{\mathcal{T}}(\mathcal{A}) \cup \{A(x)\}$;
- if $R_1(x, z) \in \mathsf{chase}_{\mathcal{T}}(\mathcal{A})$, $\exists R_1 \sqsubseteq \exists R_2 \in \mathcal{T}$, and there is no y such that $R_2(x, y) \in \mathsf{chase}_{\mathcal{T}}(\mathcal{A})$, then $\mathsf{chase}_{\mathcal{T}}(\mathcal{A}) := \mathsf{chase}_{\mathcal{T}}(\mathcal{A}) \cup \{R_2(x, y_{new})\}$, where $y_{new} \in \Delta \setminus \mathsf{adom}(\mathcal{K})$ is a fresh element that has not appeared in $\mathsf{chase}_{\mathcal{T}}(\mathcal{A})$ before.

Note that the procedure does not terminate in general and $\mathsf{chase}_{\mathcal{T}}(\mathcal{A})$ can be an infinite interpretation. It was shown in [9] that $\mathsf{chase}_{\mathcal{T}}(\mathcal{A})$ is a *canonical model* for $\mathcal{T} \cup \mathcal{A}$, that is, it can be homomorphically embedded into every model of $\mathcal{T} \cup \mathcal{A}$.[4] In the following, we will refer to $\mathsf{chase}_{\mathcal{T}}(\mathcal{A})$ as $\mathcal{I}_{\mathsf{can}}$, assuming that $\mathcal{K}$ is clear from the context. We also can extend naturally the notion of chase $\mathsf{chase}_{\mathcal{T}}(\mathcal{A})$ from ABoxes $\mathcal{A}$ to interpretations $\mathcal{I}$ i.e., to $\mathsf{chase}_{\mathcal{T}}(\mathcal{I})$, since an interpretation can be seen as an infinite ABox.

Finally, we recall that *DL-Lite$_{core}$* has the *finite model property*, i.e., every satisfiable *DL-Lite$_{core}$* KB has at least one finite model (this follows from results in [26]).

[2] Actually, the data complexity of satisfiability and of other inference tasks that involve the ABox is AC^0.

[3] The notion of chase was originally introduced in [23] to reason about data dependencies.

[4] Note that our canonical models are similar to *core solutions* of data-exchange settings [25] in the sense that they cannot be homomorphically embedded in their proper sub-models, but they are different because canonical models are in general infinite, while core solutions are always finite.

2.2. The Description Logic *DL-Lite*^{pr}

In this section we introduce a restriction of *DL-Lite*$_{core}$, which we call *DL-Lite*^{pr} (where *pr* stands for *positive role* interaction). Intuitively, in *DL-Lite*^{pr} "negative" information that involves roles is forbidden both at the ABox and at the TBox level. More precisely:

Definition 2.2 *(DL-Lite*^{pr}*).* A *DL-Lite*$_{core}$ knowledge base $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ is in *DL-Lite*^{pr} if for every basic role R, every constant a, and every basic concept B, *neither* of the following two entailments holds:

$$\text{(i)}\quad \mathcal{K} \models \neg\exists R(a) \quad \text{and} \quad \text{(ii)}\quad \mathcal{T} \models \exists R \sqsubseteq \neg B. \qquad \square$$

Example 2.3. Consider again $\mathcal{K}_0$ of Example 2.1. To see that $\mathcal{K}_0$ is not in *DL-Lite*^{pr}, observe that it violates both Condition (i) and Condition (ii) of Definition 2.2. Indeed, $\mathcal{K}_0 \models \neg\exists HasHusb^-(a)$ for $a \in \{pedro, ivan\}$, violating Condition (i). Moreover, $\mathcal{T}_0 \models \exists HasHusb^- \sqsubseteq \neg Priest$, violating Condition (ii). Consider the following subset of $\mathsf{cl}(\mathcal{T}_0)$:

$$\mathcal{T}_1 = \{Card \sqsubseteq Priest, Husb \sqsubseteq \neg Priest\}.$$

Clearly $(\mathcal{T}_1, \mathcal{A}_0)$ is in *DL-Lite*^{pr}. $\square$

DL-Lite^{pr} is defined semantically, but one can effectively check in polynomial time whether a given *DL-Lite*$_{core}$ KB $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ is in *DL-Lite*^{pr}. Both conditions of Definition 2.2 can be checked by computing (in time polynomial in $|\mathcal{K}|$) the closure $\mathsf{cl}(\mathcal{K})$ of $\mathcal{K}$ (i.e., $\mathsf{cl}(\mathcal{T}) \cup \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$) and verifying whether an assertion of the form $\neg\exists R(a)$ or of the form $\exists R \sqsubseteq \neg B$ is in the closure.

We see *DL-Lite*^{pr} as an important language to study because it is an extension of the RDFS ontology language [19] (more precisely, of the first-order logic fragment of RDFS) that adds to RDFS the ability of expressing disjointness of concepts ($B_1 \sqsubseteq \neg B_2$) and mandatory participation to roles ($A \sqsubseteq \exists R$).

2.3. ABox evolution of knowledge bases

We now formally introduce the problem of ABox evolution of DL knowledge bases, concentrating on model-based approaches. We discuss different semantics for the problem and put them into relationship with each other. Specifically, we focus on the eight semantics that have been presented first in [7] (see Fig. 2.1, right). Notice that the notions we introduce do not depend on any specific DL, although we will apply them later to the cases of *DL-Lite*$_{core}$ and *DL-Lite*^{pr}.

2.3.1. Model-based evolution

Let $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ be a DL KB and $\mathcal{N}$ a "new" ABox. Intuitively, $\mathcal{N}$ represents information that is considered to be true and we study how to incorporate assertions of $\mathcal{N}$ into $\mathcal{K}$, that is, how $\mathcal{K}$ *evolves* w.r.t. $\mathcal{N}$ [3]. More specifically, we study *evolution operators* $\diamond$ that take $\mathcal{K}$ and $\mathcal{N}$ as input and return, preferably in *polynomial time*, a DL KB $\mathcal{K}' = \mathcal{T} \cup \mathcal{A}'$ (with the same TBox as that of $\mathcal{K}$) that captures the evolution, and that we call *the* (*ABox*) *evolution of* $\mathcal{K}$ *w.r.t.* $\mathcal{N}$. Based on the evolution principles of [7,12], which are natural and usually adopted (see, e.g., [12]), we require $\mathcal{K}$ and $\mathcal{K}'$ to be satisfiable and *coherent*, where the latter means that for every concept and role name occurring in $\mathcal{K}$ (resp. $\mathcal{K}'$) there is a model $\mathcal{I}$ of $\mathcal{K}$ (resp. $\mathcal{K}'$) that interprets this name as a non-empty set. Note that coherency of a KB is determined only by the KB's TBox. Since we study ABox evolution of KBs $\mathcal{K}$, which does not affect the TBox of $\mathcal{K}$, coherency of $\mathcal{K}$ implies coherency of $\mathcal{K}'$. Now we define formally evolution setting we adopt.

Definition 2.4 *(Evolution setting).* Let $\mathcal{D}$ be a DL. A $\mathcal{D}$*-evolution setting* (or just *evolution setting*, when $\mathcal{D}$ is clear) is a pair $\mathcal{E} = (\mathcal{K}, \mathcal{N})$, where $\mathcal{N}$ is a $\mathcal{D}$ ABox containing only positive MAs, and both $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ and $\mathcal{T} \cup \mathcal{N}$ are satisfiable and coherent $\mathcal{D}$ KBs. $\square$

Example 2.5. Consider the KB $\mathcal{K}_0$ from Example 2.1 and the two ABoxes:

$$\mathcal{N}_1 = \{Husb(pedro), Wife(tanya)\} \quad \text{and} \quad \mathcal{N}_2 = \{Priest(john)\}.$$

The pairs $(\mathcal{K}_0, \mathcal{N}_1)$ and $(\mathcal{K}_0, \mathcal{N}_2)$ are *DL-Lite*$_{core}$-evolution settings. $\square$

Note that we require $\mathcal{N}$ to contain only positive MAs for the ease of exposition only. This does not restrict the applicability of our results since in *DL-Lite*$_{core}$-evolution settings $(\mathcal{T} \cup \mathcal{A}, \mathcal{N})$, negative MAs in $\mathcal{N}$ can be simulated by a slight extension of the TBox $\mathcal{T}$.

An *evolution* for a $\mathcal{D}$-evolution setting $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ can now be defined as a $\mathcal{D}$ KB $\mathcal{K}'$ that (i) preserves $\mathcal{N}$ and (ii) changes the semantics of $\mathcal{K}$ "as little as possible", due to the *principle of minimal change* [6]. Under *model-based approaches* (MBAs), these two conditions on $\mathcal{K}'$ are reflected as follows: the set $\mathsf{Mod}(\mathcal{K}')$ of models of $\mathcal{K}'$ is precisely the set of interpretations $\mathcal{J}$ such that (i) $\mathcal{J} \models \mathcal{T} \cup \mathcal{N}$ and (ii) $\mathcal{J}$ is "minimally distant" from the models of $\mathcal{K}$.

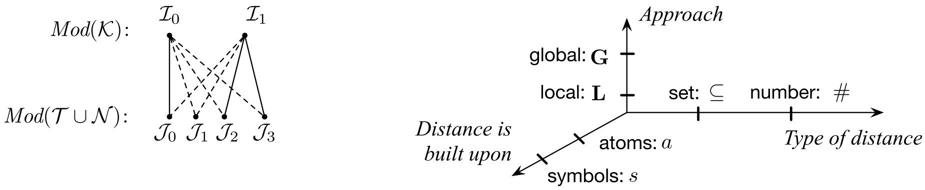

Fig. 2.1. Left: measuring distances between models and finding local and global minima (Examples 2.7 and 2.9). Right: three-dimensional space of approaches to model-based evolution semantics.

Katsuno and Mendelzon [5] considered two ways, so-called *local* and *global*, of deciding which models are minimally distant from the models of $\mathcal{K}$ (w.r.t. some notion of distance), where the former choice corresponds to *knowledge update* and the latter one to *knowledge revision*. Now we discuss these two ways in more detail.

The idea of the local approaches is to consider all models $\mathcal{I}$ of $\mathcal{K}$ one by one, and for each such $\mathcal{I}$ to take those models $\mathcal{J}$ of $\mathcal{T} \cup \mathcal{N}$ that are minimally distant from $\mathcal{I}$. Formally,

Definition 2.6 *(Local MBA).* Let $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ be a $\mathcal{D}$-evolution setting, where $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, and let $\mathsf{dist}(\cdot, \cdot)$ be a distance function between interpretations. For an interpretation $\mathcal{I}$, let $\mathsf{loc\_min}(\mathcal{I}, \mathcal{T}, \mathcal{N})$ be the set of interpretations $\mathcal{J}$ such that $\mathcal{J} \models \mathcal{T} \cup \mathcal{N}$ and among all such interpretations the value of $\mathsf{dist}(\mathcal{I}, \mathcal{J})$ is minimal on $\mathcal{J}$. Then, $\mathcal{K}'$ in $\mathcal{D}$ is an **L**-*evolution* for $\mathcal{E}$ if $\mathsf{Mod}(\mathcal{K}') = \mathcal{K} \diamond_{\mathbf{L}} \mathcal{N}$, where

$$\mathcal{K} \diamond_{\mathbf{L}} \mathcal{N} = \bigcup_{\mathcal{I} \in \mathsf{Mod}(\mathcal{K})} \mathsf{loc\_min}(\mathcal{I}, \mathcal{T}, \mathcal{N}). \qquad \square$$

The distance function dist varies from approach to approach and commonly takes as values either a number or a set. We will discuss distance functions later in this section.

Example 2.7. To get a better intuition of the local semantics, consider Fig. 2.1, left, where two models $\mathcal{I}_0$ and $\mathcal{I}_1$ of a KB $\mathcal{K}$ and four models $\mathcal{J}_0, \ldots, \mathcal{J}_3$ of $\mathcal{T} \cup \mathcal{N}$ are presented. The distance between a model of $\mathcal{K}$ and a model of $\mathcal{T} \cup \mathcal{N}$ is represented by the length of the line connecting them. Solid lines correspond to minimal distances, while dashed ones to distances that are not minimal. In this figure, $\mathsf{loc\_min}(\mathcal{I}_0, \mathcal{T}, \mathcal{N}) = \{\mathcal{J}_0\}$ and $\mathsf{loc\_min}(\mathcal{I}_1, \mathcal{T}, \mathcal{N}) = \{\mathcal{J}_2, \mathcal{J}_3\}$. Hence, $\mathcal{K} \diamond_{\mathbf{L}} \mathcal{N} = \{\mathcal{J}_0, \mathcal{J}_2, \mathcal{J}_3\}$. $\square$

Under a global approach, one chooses models of $\mathcal{T} \cup \mathcal{N}$ that are minimally distant from the whole set $\mathsf{Mod}(\mathcal{K})$:

Definition 2.8 *(Global MBA).* Let $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ be a $\mathcal{D}$-evolution setting and $\mathsf{dist}(\cdot, \cdot)$ a distance function between interpretations. For an interpretation $\mathcal{J}$, let the distance between $\mathsf{Mod}(\mathcal{K})$ and $\mathcal{J}$ be defined as follows: $\mathsf{dist}(\mathsf{Mod}(\mathcal{K}), \mathcal{J}) = \min_{\mathcal{I} \in \mathsf{Mod}(\mathcal{K})} \mathsf{dist}(\mathcal{I}, \mathcal{J})$. Furthermore, let $\mathsf{glob\_min}(\mathcal{K}, \mathcal{N})$ be the set of interpretations $\mathcal{J}$ such that $\mathcal{J} \models \mathcal{T} \cup \mathcal{N}$ and among all such interpretations the value of $\mathsf{dist}(\mathcal{I}, \mathcal{J})$ is minimal on $\mathcal{J}$. Then, $\mathcal{K}'$ in $\mathcal{D}$ is a **G**-*evolution* for $\mathcal{E}$ if $\mathsf{Mod}(\mathcal{K}') = \mathcal{K} \diamond_{\mathbf{G}} \mathcal{N}$, where

$$\mathcal{K} \diamond_{\mathbf{G}} \mathcal{N} = \mathsf{glob\_min}(\mathcal{K}, \mathcal{N}). \qquad \square$$

Example 2.9. Consider again Fig. 2.1, left, and assume that the distance between $\mathcal{I}_0$ and $\mathcal{J}_0$ is the global minimum. Thus, we obtain that $\mathcal{K} \diamond_{\mathbf{G}} \mathcal{N} = \mathsf{glob\_min}(\mathcal{K}, \mathcal{N}) = \{\mathcal{J}_0\}$. $\square$

2.3.2. Measuring distance between interpretations

The classical MBAs were developed for propositional theories [6], where interpretations can be seen as finite sets of propositional symbols. In that case, two distance functions have been introduced, respectively based on symmetric difference "$\ominus$" and on the cardinality of symmetric difference:

$$\mathsf{dist}_{\subseteq}(\mathcal{I}, \mathcal{J}) = \mathcal{I} \ominus \mathcal{J} \quad \text{and} \quad \mathsf{dist}_{\#}(\mathcal{I}, \mathcal{J}) = |\mathcal{I} \ominus \mathcal{J}|, \tag{1}$$

where $\mathcal{I} \ominus \mathcal{J} = (\mathcal{I} \setminus \mathcal{J}) \cup (\mathcal{J} \setminus \mathcal{I})$. Distances under $\mathsf{dist}_{\subseteq}$ are sets and are compared by set inclusion, that is, $\mathsf{dist}_{\subseteq}(\mathcal{I}_1, \mathcal{J}_1) \leqslant \mathsf{dist}_{\subseteq}(\mathcal{I}_2, \mathcal{J}_2)$ if and only if $\mathsf{dist}_{\subseteq}(\mathcal{I}_1, \mathcal{J}_1) \subseteq \mathsf{dist}_{\subseteq}(\mathcal{I}_2, \mathcal{J}_2)$. Finite distances under $\mathsf{dist}_{\#}$ are natural numbers and are compared in the standard way.

These distances can be extended to DL interpretations in two ways. First, one can consider interpretations $\mathcal{I}$ and $\mathcal{J}$ as sets of atoms, in which case the symmetric difference $\mathcal{I} \ominus \mathcal{J}$ and the corresponding distances are defined as in the propositional case. We denote the distances in Eq. (1) extended in this way as $\mathsf{dist}^a_{\subseteq}(\mathcal{I}, \mathcal{J})$ and $\mathsf{dist}^a_{\#}(\mathcal{I}, \mathcal{J})$, respectively. Note that in contrast to the propositional case, $\mathcal{I} \ominus \mathcal{J}$ (and hence also distances $\mathsf{dist}^a_{\subseteq}(\mathcal{I}, \mathcal{J})$ and $\mathsf{dist}^a_{\#}(\mathcal{I}, \mathcal{J})$) can be infinite.

Finally, one can also define distances at the level of the concept and role *symbols* in the signature Σ underlying the interpretations:

$$\mathsf{dist}^s_{\subseteq}(\mathcal{I},\mathcal{J}) = \{S \in \Sigma \mid S^{\mathcal{I}} \neq S^{\mathcal{J}}\} \quad \text{and} \quad \mathsf{dist}^s_{\#}(\mathcal{I},\mathcal{J}) = \left|\{S \in \Sigma \mid S^{\mathcal{I}} \neq S^{\mathcal{J}}\}\right|.$$

Summing up across the different possibilities, we have three dimensions with two values each: (1) *local* vs. *global* approach, (2) *atom*-based vs. *symbol*-based for defining distances, and (3) *set inclusion* vs. *cardinality* to compare symmetric differences. This gives eight evolution semantics, as shown in Fig. 2.1, right. We denote each of the resulting eight semantics by using a combination of three symbols, indicating the choice in each dimension, e.g., $\mathbf{L}^a_{\#}$ denotes the local semantics where the distances are expressed in terms of cardinality of sets of atoms. We can then define $\mathsf{loc\_min}^y_x$ and $\mathbf{L}^y_x$-evolution as in Definition 2.6, and $\mathsf{glob\_min}^y_x$ and $\mathbf{G}^y_x$-evolution as in Definition 2.8, by using the specific distances determined by the values of $x \in \{\subseteq, \#\}$ and $y \in \{a, s\}$.

Recall that in Definitions 2.6 and 2.8, when we define the set $\mathsf{Mod}(\mathcal{K}')$ of models, we indicate the specific evolution semantics S as a subscript of the evolution operator $\diamond$, i.e., as in $\mathcal{K} \diamond_S \mathcal{N}$. In terms of the introduced notation for the eight semantics, for, say, $\mathbf{G}^a_{\#}$ semantics, the set of models $\mathsf{Mod}(\mathcal{K}')$ should be denoted as $\mathcal{K} \diamond_{\mathbf{G}^a_{\#}} \mathcal{N}$. To avoid this overloaded notation we will write "$\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{G}^a_{\#}$" instead of $\mathcal{K} \diamond_{\mathbf{G}^a_{\#}} \mathcal{N}$.

2.3.3. Closure under evolution and approximation

In the propositional case, each set of interpretations over finitely many symbols can be captured by a suitable formula, that is, a formula whose models are exactly those interpretations. In the case of DLs, this is no more necessarily the case, since, on the one hand, interpretations can be infinite, and on the other hand, logics may miss some connectives, e.g., disjunction, that would be necessary to capture those interpretations.

Let $\mathcal{D}$ be a DL and $\mathcal{M}$ a set of models. We say that $\mathcal{M}$ is *axiomatizable* in $\mathcal{D}$ if there is a KB $\mathcal{K}$ in $\mathcal{D}$ such that $\mathsf{Mod}(\mathcal{K}) = \mathcal{M}$.

Definition 2.10 *(Closure under evolution).* Let S be an MBA. We say that a DL $\mathcal{D}$ is *closed under* S (or, evolution under S is *expressible* in $\mathcal{D}$) if for every $\mathcal{D}$-evolution setting $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ there is a KB $\mathcal{K}'$ in $\mathcal{D}$ such that $\mathcal{K}'$ is an S-evolution for $\mathcal{E}$, i.e., $\mathsf{Mod}(\mathcal{K}') = \mathcal{K} \diamond_S \mathcal{N}$. □

The notion of expressibility immediately suggests the following *expressibility problem*.

[EXPRESS]: Does an S-evolution always exist for a given $\mathcal{D}$-evolution setting and an MBA S?

It has been shown in [7,17] that *DL-Lite* is not closed under any of the eight model-based semantics presented above. The observation underlying these results is that, on the one hand, the principle of minimal change intrinsically introduces implicit disjunction in the evolved KB. On the other hand, since *DL-Lite* is a slight extension of Horn logic [27], it does not allow to express genuine disjunction (see Lemma 1 in [7] for details).

The negative answer to the **EXPRESS** problem for *DL-Lite* suggests the following *approximation problem*:

[APPROXIMATE]: If S-evolution does not exists for a given $\mathcal{D}$-evolution setting $\mathcal{E} = (\mathcal{T} \cup \mathcal{A}, \mathcal{N})$ and an MBA S, is there a KB $\widetilde{\mathcal{K}} = \mathcal{T} \cup \widetilde{\mathcal{A}}$ that is a "good" approximation of $(\mathcal{T} \cup \mathcal{A}) \diamond_S \mathcal{N}$?

There are two commonly used notions of approximation for knowledge evolution [28]: *sound* and *complete* approximations. In this paper we will address sound approximations only and leave the study of complete ones as future work. We now define the notion of sound approximation formally.

Definition 2.11 *(Sound approximation).* Let $\mathcal{M}$ be a set of models and $\mathcal{D}$ a DL. We say that a $\mathcal{D}$ KB $\widetilde{\mathcal{K}}$ is a *sound* $\mathcal{D}$-*approximation* of $\mathcal{M}$ if $\mathcal{M} \subseteq \mathsf{Mod}(\widetilde{\mathcal{K}})$. Moreover, we say that a sound $\mathcal{D}$-approximation $\widetilde{\mathcal{K}}$ of $\mathcal{M}$ is *maximal* if for every sound $\mathcal{D}$-approximation $\widetilde{\mathcal{K}}_1$ of $\mathcal{M}$ it holds that $\mathsf{Mod}(\widetilde{\mathcal{K}}_1) \not\subset \mathsf{Mod}(\widetilde{\mathcal{K}})$. □

Summary of Section 2. We presented basic definitions from Description Logics and presented two DLs: *DL-Lite*$_{core}$ and its sublanguage *DL-Lite*^{pr}. Finally, we defined the notion of knowledge evolution and two main evolution problems: express and approximate. We now study these evolution problems for *DL-Lite*^{pr}.

3. Evolution of *DL-Lite*^{pr} KBs

In this section, we first consider how to capture evolution under $\mathbf{L}^a_{\subseteq}$ and $\mathbf{G}^s_{\subseteq}$ in *DL-Lite*^{pr}. Further, we show that evolution under $\mathbf{L}^s_{\subseteq}$ is inexpressible in *DL-Lite*^{pr}, and we present an algorithm to compute the maximal sound *DL-Lite*^{pr}-approximation of an evolution under this semantics. In Section 4 we will discuss how the results obtained in the current section can be applied to the remaining semantics of the eight ones that we have introduced above.

Algorithm 3.1: $\mathsf{AtAlg}(\mathcal{E})$.

INPUT : *DL-Lite*^{pr}-evolution setting $\mathcal{E} = (\mathcal{T} \cup \mathcal{A}, \mathcal{N})$
OUTPUT: The maximal set $\mathcal{A}' \subseteq \mathsf{cl}_\mathcal{T}(\mathcal{A})$ of ABox assertions that is $\mathcal{T}$-satisfiable with $\mathcal{N}$

```
1 A' := ∅; X := cl_T(A);
2 repeat
3 |  choose some g ∈ X; X := X \ {g};
4 |  if {g} ∪ N ⊭_T ⊥ then A' := A' ∪ {g};
5 until X = ∅;
6 return A'.
```

3.1. Capturing $\mathbf{L}^a_\subseteq$-evolution

Consider the algorithm AtAlg presented in Algorithm 3.1, which takes as input a *DL-Lite*^{pr}-evolution setting $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ and returns the maximal subset of $\mathsf{cl}_\mathcal{T}(\mathcal{A})$ that is $\mathcal{T}$-satisfiable with $\mathcal{N}$. We illustrate how AtAlg works on the following example.

Example 3.1. Continuing with Examples 2.1, 2.3, and 2.5, we now illustrate how the algorithm AtAlg works on the following two evolution settings: $\mathcal{E}_1 = (\mathcal{T}_1 \cup \mathcal{A}_0, \mathcal{N}_1)$ and $\mathcal{E}_2 = (\mathcal{T}_1 \cup \mathcal{A}_0, \mathcal{N}_2)$, We remind the reader our notations: $\mathcal{T}_1 = \{Card \sqsubseteq Priest, Husb \sqsubseteq \neg Priest\}$, $\mathcal{N}_1 = \{Husb(pedro), Wife(tanya)\}$, $\mathcal{N}_2 = \{Priest(john)\}$, and

$$\mathcal{A}_0 = \big\{Priest(pedro), Priest(ivan), Husb(john), Wife(mary), Wife(chloe), HasHusb(mary, john)\big\}.$$

The computation of both $\mathsf{AtAlg}(\mathcal{E}_1)$ and $\mathsf{AtAlg}(\mathcal{E}_2)$ relies on the computation of $\mathsf{cl}_{\mathcal{T}_1}(\mathcal{A}_0)$, which is equal to:

$$\mathsf{cl}_{\mathcal{T}_1}(\mathcal{A}_0) = \mathcal{A}_0 \cup \big\{\neg Husb(pedro), \neg Husb(ivan), \neg Priest(john), \neg Card(john)\big\}.$$

By dropping from $\mathsf{cl}_{\mathcal{T}_1}(\mathcal{A}_0)$ the atoms that conflict with $\mathcal{N}_1$ and $\mathcal{N}_2$ we obtain, respectively:

$$\mathsf{AtAlg}(\mathcal{E}_1) = \mathsf{cl}_\mathcal{T}(\mathcal{A}_0) \setminus \big\{Priest(pedro), \neg Husb(pedro)\big\},$$
$$\mathsf{AtAlg}(\mathcal{E}_2) = \mathsf{cl}_\mathcal{T}(\mathcal{A}_0) \setminus \big\{Husb(john), \neg Priest(john)\big\}. \qquad \square$$

We are going to prove that using AtAlg one can efficiently compute $\mathbf{L}^a_\subseteq$-evolutions in *DL-Lite*^{pr}. Before doing that, we will present the following definitions, auxiliary propositions, and a lemma. Detailed proofs or proof-sketches can be found in Appendix A.

Observe an important property of *DL-Lite*^{pr} KBs, which shows that the source of inconsistency in these KBs comes from the interaction between *unary* atoms only. As we will see in Section 5, inconsistency of KBs that are not in *DL-Lite*^{pr} can also come from the interaction between binary atoms, which immediately leads to expressibility issues with evolution (see Section 5.1 for details).

Proposition 3.2. *For a DL-Lite*^{pr} *KB* $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ *and an assertion* g *of the form* $R(a, b)$ *or* $\exists R(a)$, *if* $\mathcal{A} \models_\mathcal{T} g$, *then also* $\mathsf{AtAlg}(\mathcal{K}, \mathcal{N}) \cup \mathcal{N} \models_\mathcal{T} g$.

Proof (Sketch). The proof is based on the facts that (i) a *DL-Lite*^{pr} TBox $\mathcal{T}$ does not entail NIs of the form $B \sqsubseteq \neg\exists R$, for a basic concept B and a basic role R, and (ii) $\mathcal{N}$ does not contain negative MAs (due to Definition 2.4). Therefore, assertions of the form $R(a, b)$ or $\exists R(a)$ cannot $\mathcal{T}$-contradict $\mathcal{N}$. $\square$

The following proposition shows the cases when the union of models of $\mathcal{T}$ is also a model of $\mathcal{T}$.

Proposition 3.3. *Let* $\mathcal{T}$ *be a DL-Lite*$_{core}$ *TBox, and let* $\mathcal{I}_1$, $\mathcal{I}_2$ *be models of* $\mathcal{T}$. *Then,* $\mathcal{I}_1 \cup \mathcal{I}_2 \models \mathcal{T}$ *if and only if for every* $f_1 \in \mathcal{I}_1$ *and* $f_2 \in \mathcal{I}_2$, *it holds that* $\{f_1, f_2\} \not\models_\mathcal{T} \bot$.

Proof (Sketch). The "only if" direction is trivial. Regarding the "if" direction, for the PIs of $\mathcal{T}$ it holds since they are satisfied in both $\mathcal{I}_1$ and $\mathcal{I}_2$; for the NIs of $\mathcal{T}$ it holds since each such assertion involves at most two concepts, and hence its violation involves at most two atoms. $\square$

We now define how to *uproot* an atom f from a model w.r.t. a TBox, i.e., how to delete the atoms g of the model from which f can be "deduced" using the TBox in the logic programming sense. We denote the set of these atoms g to be uprooted as $\mathsf{root}_\mathcal{T}(f)$. Note that in the following, with a slight abuse of notation, we treat sequences of elements as sets when needed, e.g., by applying union or set inclusion to sequences. Formally:

Definition 3.4 *(*$\mathsf{root}_{\mathcal{T}}$*).* Let $\mathcal{T}$ be a *DL-Lite$_{core}$* TBox and $V^n_{\mathcal{T}}$ a sequence $\langle f_1, \ldots, f_n, L\rangle$, where $f_1, \ldots, f_n$ are ground atoms and L is a ground literal, such that there is a sequence of TBox assertions $\alpha_1, \ldots, \alpha_n$ in $\mathsf{cl}(\mathcal{T})$, with *no* α_i of the form $\exists R \sqsubseteq \exists R$, $f_i \to f_{i+1}$ is an instantiation of the first-order interpretation of α_i[5] for $1 \leqslant i \leqslant n-1$, and $f_n \to L$ is an instantiation of the first-order interpretation of α_n. Note that α_n can be either a PI (if L is positive) or an NI (if L is negative), while all the other α_is are PIs. Then,

$$\mathsf{root}_{\mathcal{T}}\big(C(a)\big) = \bigcup_{V^n_{\mathcal{T}}(C(a)):\ n\in\mathbb{N}} V^n_{\mathcal{T}}\big(C(a)\big),$$

$$\mathsf{root}_{\mathcal{T}}\big(R(a,b)\big) = \bigcup_{V^n_{\mathcal{T}}(R(a,d)):\ n\in\mathbb{N},\, d\in\Delta} V^n_{\mathcal{T}}\big(R(a,d)\big) \cup \bigcup_{V^n_{\mathcal{T}}(R(d,b)):\ n\in\mathbb{N},\, d\in\Delta} V^n_{\mathcal{T}}\big(R(d,b)\big).$$

If $\mathcal{I}$ is an interpretation, then $\mathsf{root}^{\mathcal{I}}_{\mathcal{T}}(C(a))$ denotes the restriction of $\mathsf{root}_{\mathcal{T}}(C(a))$ to $\mathcal{I}$, i.e., the subset of $\mathsf{root}_{\mathcal{T}}(C(a))$ for which each $V^n_{\mathcal{T}}(C(a))$ in the union is contained in $\mathcal{I}$. In the following, whenever we write $\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g)$ for an MA g and a model $\mathcal{I}$, we always mean $\mathcal{I} \setminus \mathsf{root}^{\mathcal{I}}_{\mathcal{T}}(g)$. $\square$

Example 3.5. Consider a TBox $\mathcal{T} = \{B \sqsubseteq \exists R^-, \exists R \sqsubseteq C\}$ and an interpretation $\mathcal{I} = \{B(b), R(a,b), C(a)\}$. Note that $\mathcal{I} \models \mathcal{T}$. Let us see how the interpretation $\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(C(a))$ (that is, $\mathcal{I} \setminus \mathsf{root}^{\mathcal{I}}_{\mathcal{T}}(C(a))$) looks like. Note that the restriction of $\mathsf{root}_{\mathcal{T}}(C(a))$ on $\mathcal{I}$ includes only one sequence of atoms of length three, $V^2_{\mathcal{T}} = \langle f_1, f_2, C(a)\rangle$ (the sequences of smaller lengths are "subsumed" by $V^2_{\mathcal{T}}(C(a))$), where $f_1 = B(b)$, and $f_2 = R(a,b)$; moreover, $B(b) \to R(a,b)$ instantiates $\alpha_1 = B \sqsubseteq \exists R^-$ and $R(a,b) \to C(a)$ instantiates $\alpha_2 = \exists R \sqsubseteq C$. Thus, $\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(C(a)) = \emptyset$. $\square$

The following proposition shows that $\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(C(a))$ is a model of $\mathcal{T}$ if $\mathcal{I}$ is a model of $\mathcal{T}$.

Proposition 3.6. *Let $\mathcal{T}$ be a DL-Lite$_{core}$ TBox, and $\mathcal{I} \models \mathcal{T}$. Then, $\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g) \models \mathcal{T}$ for every MA g.*

Proof (Sketch). Let α be a TBox assertion such that $\mathcal{T} \models \alpha$. If α is an NI, then we conclude that $\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g) \models \alpha$ since $\mathcal{I} \models \alpha$ and $\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g) \subseteq \mathcal{I}$. If α is a PI of the form $B_1 \sqsubseteq B_2$, then assume that $\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g) \not\models \alpha$. This is the case if there are atoms f_1 and f_2, that instantiate B_1 and B_2, respectively, such that $f_1 \in \mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g)$ and $f_2 \notin \mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g)$. Since $\mathcal{I} \models \mathcal{T}$, we have that $f_2 \in \mathcal{I}$ and hence $f_2 \in \mathsf{root}_{\mathcal{T}}(g)$; by the definition of $\mathsf{root}_{\mathcal{T}}$, we imply that $f_1 \in \mathsf{root}_{\mathcal{T}}(g)$, and therefore $f_1 \notin \mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g)$. This contradicts the assumption above and concludes the proof. $\square$

Now we present a lemma that will help us for a given model $\mathcal{J} \in \mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_{\subseteq}$ to construct a model $\mathcal{I} \models \mathcal{K}$ such that $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$. This method of constructing $\mathcal{I}$ is the key for proving a number of results in this work. Intuitively, such $\mathcal{I}$ can be constructed in two steps: (i) drop from $\mathcal{J}$ all unary atomic MAs (i.e., unary atoms) that are *not* $\mathcal{T}$-satisfiable with $\mathcal{A}$, and then (ii) add unary atomic MAs that are $\mathcal{T}$-entailed from $\mathcal{A}$. The following notions will be used in the proof of this lemma below.

Definition 3.7. Let $\mathcal{T}$ be a TBox and $\mathcal{A}$ an ABox satisfiable with $\mathcal{T}$. Then the *unary closure* of $\mathcal{A}$ w.r.t. $\mathcal{T}$ is

$$\mathsf{ucl}_{\mathcal{T}}(\mathcal{A}) = \big\{A(c) \bigm| A \text{ is an atomic concept, } c \text{ is a constant, and } \mathcal{A} \models_{\mathcal{T}} A(c)\big\}.$$

Moreover, let $\mathcal{J}$ be an interpretation. Then the set of atoms of $\mathcal{J}$ that are in *conflict* with $\mathcal{A}$ w.r.t. $\mathcal{T}$ is

$$\mathsf{conf}_{\mathcal{T}}(\mathcal{J}, \mathcal{A}) = \big\{A(c) \in \mathcal{J} \bigm| A \text{ is an atomic concept, } c \text{ is a constant, and } \mathcal{A} \cup \big\{A(c)\big\} \models_{\mathcal{T}} \bot\big\}. \qquad \square$$

Lemma 3.8. *Let $(\mathcal{K}, \mathcal{N})$, where $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, be a DL-Lite^{pr}-evolution setting and let $\mathcal{J}$ be a model of $\mathcal{T} \cup \mathsf{AtAlg}(\mathcal{K}, \mathcal{N}) \cup \mathcal{N}$. Then, the following interpretation is a model of $\mathcal{K}$:*

$$\mathcal{I} = \big(\mathcal{J} \setminus \mathsf{conf}_{\mathcal{T}}(\mathcal{J}, \mathcal{A})\big) \cup \mathsf{ucl}_{\mathcal{T}}(\mathcal{A}). \tag{2}$$

Moreover, $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$ and $\mathsf{dist}^a_{\subseteq}(\mathcal{I}, \mathcal{J})$ is finite.

Proof. Finiteness of $\mathsf{dist}^a_{\subseteq}(\mathcal{I}, \mathcal{J})$ follows from finiteness of $\mathsf{conf}_{\mathcal{T}}(\mathcal{J}, \mathcal{A})$ and $\mathsf{ucl}_{\mathcal{T}}(\mathcal{A})$. Indeed, since $\mathcal{A} \models_{\mathcal{T}} A(c)$ for every $A(c) \in \mathsf{ucl}_{\mathcal{T}}(\mathcal{A})$, we conclude that $c \in \mathsf{adom}(\mathcal{K})$; since $\mathcal{A} \cup \{A(c)\} \models_{\mathcal{T}} \bot$ for every $A(c) \in \mathsf{conf}_{\mathcal{T}}(\mathcal{J}, \mathcal{A})$, we conclude that $\mathcal{A} \models_{\mathcal{T}} \neg A(c)$, and again $c \in \mathsf{adom}(\mathcal{K})$. Due to the finiteness of $\Sigma(\mathcal{K})$, it holds that $|\mathsf{adom}(\mathcal{K})| \leqslant n$ and $|\{A \mid A \in \Sigma(\mathcal{K})\}| \leqslant m$ for some $n, m \in \mathbb{N}$. Hence, $|\mathsf{ucl}_{\mathcal{T}}(\mathcal{A})| \leqslant n \times m$ and $|\mathsf{conf}_{\mathcal{T}}(\mathcal{J}, \mathcal{A})| \leqslant n \times m$ (see details in Appendix A).

[5] If α_i is of the form $A_1 \sqsubseteq A_2$, $A_1 \sqsubseteq \exists R$, or $\exists R \sqsubseteq A_2$, then the first-order interpretation of α_i is respectively the implication $A_1(x) \to A_2(x)$, $A_1(x) \to R(x,y)$, or $R(x,y) \to A_2(x)$, where x and y are some variables, and this interpretation can be instantiated with, e.g., atoms $A_1(a)$, $A_2(a)$ and $R(a,b)$ as follows: $A_1(a) \to A_2(a)$, $A_1(a) \to R(a,b)$, or $R(a,b) \to A_2(a)$.

Now we prove that $\mathcal{I} \models \mathcal{K}$ by showing that $\mathcal{I} \models \mathcal{A}$ and $\mathcal{I} \models \mathcal{T}$. Afterwards we will prove that $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$. Let $\mathcal{A}' = \mathsf{AtAlg}(\mathcal{K}, \mathcal{N}) \cup \mathcal{N}$.

$\mathcal{I} \models \mathcal{A}$: Let $g \in \mathcal{A}$ be an MA; we show that $\mathcal{I} \models g$. We have the following cases:

(i) g is of the form $A(c)$; then $g \in \mathsf{ucl}_{\mathcal{T}}(\mathcal{A})$ and we conclude that $g \in \mathcal{I}$ by the definition of $\mathcal{I}$, so $\mathcal{I} \models g$.
(ii) g is of the form $R(a,b)$; then, by Proposition 3.2, it holds that $g \in \mathcal{J}$, i.e., $\mathcal{J} \models g$. Since $\mathcal{I}$ and $\mathcal{J}$ differ only on unary atoms by the definition of $\mathcal{I}$, we conclude that $g \in \mathcal{I}$, so $\mathcal{I} \models g$.
(iii) g is of the form $\exists R(a)$; then, by Proposition 3.2, it holds that $R(a,x) \in \mathcal{J}$ for some $x \in \Delta$. Since $\mathcal{I}$ and $\mathcal{J}$ differ only on unary atoms by the definition of $\mathcal{I}$, we conclude that $R(a,x) \in \mathcal{I}$, so $\mathcal{I} \models g$.
(iv) g is of the form $\neg A(c)$; from $\neg A(c) \in \mathcal{A}$ and the definition of $\mathsf{conf}_{\mathcal{T}}(\mathcal{J}, \mathcal{A})$, we conclude that $A(c) \in \mathsf{conf}_{\mathcal{T}}(\mathcal{J}, \mathcal{A})$. Thus, $A(c) \notin \mathcal{I}$ by the construction of $\mathcal{I}$ and therefore $\mathcal{I} \models g$.

Hence we can conclude that $\mathcal{I} \models \mathcal{A}$.

$\mathcal{I} \models \mathcal{T}$: We now show that $\mathcal{I} \models \mathcal{T}$ in two steps. First, observe that $\mathcal{J} \setminus \mathsf{conf}_{\mathcal{T}}(\mathcal{J}, \mathcal{A}) \models \mathcal{T}$. Indeed, since $\mathcal{J} \models \mathcal{T}$, it is enough to show that for every $f \in \mathsf{conf}_{\mathcal{T}}(\mathcal{J}, \mathcal{A})$, if $\{f'\} \models_{\mathcal{T}} f$ for some $f' \in \mathcal{J}$, then $f' \in \mathsf{conf}_{\mathcal{T}}(\mathcal{J}, \mathcal{A})$. This is clearly the case because $\{f'\} \models_{\mathcal{T}} f$ and $\mathcal{A} \cup \{f\} \models_{\mathcal{T}} \bot$ imply $\mathcal{A} \cup \{f'\} \models_{\mathcal{T}} \bot$, and consequently $f' \in \mathsf{conf}_{\mathcal{T}}(\mathcal{J}, \mathcal{A})$. Now we show that adding $\mathsf{ucl}_{\mathcal{T}}(\mathcal{A})$ to $\mathcal{J} \setminus \mathsf{conf}_{\mathcal{T}}(\mathcal{J}, \mathcal{A})$ does not violate $\mathcal{T}$. Indeed, let $f \in \mathsf{ucl}_{\mathcal{T}}(\mathcal{A})$, we need to show that for every MA g such that $\{f\} \models_{\mathcal{T}} g$ it holds $\mathcal{I} \models g$. Clearly, g can only be of the form either $A(c)$ or $\exists R(a)$. If $g = A(c)$, then $g \in \mathsf{ucl}_{\mathcal{T}}(\mathcal{A})$ and obviously $\mathcal{I} \models g$. If $g = \exists R(a)$, then observe that $\{f\} \models_{\mathcal{T}} g$ implies $\mathcal{A} \models_{\mathcal{T}} g$; thus, due to Proposition 3.2, $\mathcal{A}' \models g$ and, as we showed above, $\mathcal{I} \models g$.

$\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$: By the definition of $\mathbf{L}^a_{\subseteq}$-evolution, we need to show that there is *no* $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ such that $\mathcal{I} \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J}$. Assume there exists such $\mathcal{J}'$. Thus, there is an atom f such that $f \notin \mathcal{I} \ominus \mathcal{J}'$ while $f \in \mathcal{I} \ominus \mathcal{J}$. By the definition of $\mathcal{I}$, interpretations $\mathcal{I}$ and $\mathcal{J}$ differ only on atoms of the form $A(c)$; hence, f is of the form $A(c)$ (it cannot be of the form $R(a,b)$). We have two cases: (i) $A(c) \in \mathcal{I}$, $A(c) \notin \mathcal{J}$, and $A(c) \in \mathcal{J}'$, and (ii) $A(c) \notin \mathcal{I}$, $A(c) \in \mathcal{J}$, and $A(c) \notin \mathcal{J}'$. In either case, a contradiction can be obtained (see details in Appendix A).

Thus, $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$ and we conclude the proof. □

Finally, consider the following definition of an auxiliary model $\mathcal{I}[g]$.

Definition 3.9 *(Submodel $\mathcal{I}[g]$).* Let $\mathcal{T}$ be a *DL-Lite^{pr}*-TBox, g a positive MA, and $\mathcal{I}$ a model of $\mathcal{T}$. Then $\mathcal{I}[g]$ is a sub-interpretation of $\mathcal{I}$ satisfying both g and $\mathcal{T}$ that is minimal w.r.t. set inclusion. □

We are now ready to state our result formally.

Theorem 3.10. *Let $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ with $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ be a DL-Lite^{pr}-evolution setting. Then,*

$$\mathcal{K}' = \mathcal{T} \cup \mathsf{AtAlg}(\mathcal{E}) \cup \mathcal{N} \tag{3}$$

is a DL-Lite^{pr} KB and it is an $\mathbf{L}^a_{\subseteq}$-evolution for $\mathcal{E}$. Moreover, $\mathcal{K}'$ is computable in time polynomial in $|\mathcal{E}|$.

Proof. Let $\mathcal{M}$ denote the set of models $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_{\subseteq}$. For simplicity we denote: $\mathcal{A}'' = \mathsf{AtAlg}(\mathcal{E})$ and $\mathcal{A}' = \mathcal{A}'' \cup \mathcal{N}$. Thus, $\mathcal{K}' = \mathcal{T} \cup \mathcal{A}'$. Also, let Σ_0 be a signature such that $\Sigma(\mathcal{T}) \cup \Sigma(\mathcal{A}) \cup \Sigma(\mathcal{N}) \subseteq \Sigma_0$.

Polynomiality of AtAlg in $|\mathcal{K} \cup \mathcal{N}|$ follows from the worst case quadratic size of $\mathsf{cl}_{\mathcal{T}}(\mathcal{A})$ in $|\mathcal{K}|$ and polynomiality in $|\mathcal{K} \cup \mathcal{N}|$ of the test $\{\phi\} \cup \mathcal{N} \not\models_{\mathcal{T}} \bot$.

Observe that $\mathcal{K}'$ is a *DL-Lite^{pr}* KB. Indeed,

(i) Since $\mathcal{T} \cup \mathcal{A}$ is in *DL-Lite^{pr}* and $\mathcal{A}'' \subseteq \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$, we conclude that $\mathcal{A}'' \not\models_{\mathcal{T}} \neg\exists R(a)$ for every constant $a \in \Sigma_0$ and every role $R \in \Sigma_0$; since $\mathcal{T} \cup \mathcal{N}$ is in *DL-Lite^{pr}*, we conclude that $\mathcal{N} \not\models_{\mathcal{T}} \neg\exists R(a)$ for every constant $a \in \Sigma_0$ and every role $R \in \Sigma_0$. Now, assume that $\mathcal{A}'' \cup \mathcal{N} \models_{\mathcal{T}} \neg\exists Q(b)$ for some constant $b \in \Sigma_0$ and some role $Q \in \Sigma_0$. Then we conclude that there is an MA $g \in \mathcal{A}'' \cup \mathcal{N}$ such that $\{g\} \models_{\mathcal{T}} \neg\exists Q(b)$ (see Proposition A.1 in Appendix A). Obviously, $g \in \mathcal{A}''$ or $g \in \mathcal{N}$, which contradicts the fact that neither $\mathcal{A}'' \models_{\mathcal{T}} \neg\exists Q(b)$ nor $\mathcal{N} \models_{\mathcal{T}} \neg\exists Q(b)$. The obtained contradiction yields $\mathcal{A}'' \cup \mathcal{N} \not\models_{\mathcal{T}} \neg\exists Q(b)$ for every constant $b \in \Sigma_0$ and every role $Q \in \Sigma_0$.
(ii) Since $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ is in *DL-Lite^{pr}*, we conclude that $\mathcal{T} \not\models \exists R \sqsubseteq \neg B$ for every concept $B \in \Sigma_0$ and every role $R \in \Sigma_0$. Note that $\mathcal{K}' = \mathcal{T} \cup \mathcal{A}'$, i.e., it has the same TBox as $\mathcal{K}$.

Combining the observations above, by Definition 2.2, we conclude that $\mathcal{K}'$ is in *DL-Lite^{pr}*.

To prove that $\mathcal{K}'$ is an $\mathbf{L}^a_{\subseteq}$-evolution, we are going to show that $\mathcal{M} \subseteq \mathsf{Mod}(\mathcal{K}')$ and $\mathsf{Mod}(\mathcal{K}') \subseteq \mathcal{M}$.

$\mathcal{M} \subseteq \mathsf{Mod}(\mathcal{K}')$: Let $\mathcal{J} \in \mathcal{M}$, we show $\mathcal{J} \in \mathsf{Mod}(\mathcal{K}')$, i.e., $\mathcal{J} \in \mathsf{Mod}(\mathcal{T})$ and $\mathcal{J} \in \mathsf{Mod}(\mathcal{A}')$. By the definition of $\mathbf{L}^a_{\subseteq}$-evolution, $\mathcal{J} \in \mathcal{M}$ implies $\mathcal{J} \in \mathsf{Mod}(\mathcal{T})$.

Assume $\mathcal{J} \notin \mathsf{Mod}(\mathcal{A}')$. Since $\mathcal{J} \in \mathcal{M}$ we have $\mathcal{J} \models \mathcal{N}$. Since also $\mathcal{J} \notin \mathsf{Mod}(\mathcal{A}')$ we have $\mathcal{J} \not\models \mathcal{A}''$. Thus, there is an MA $g \in \mathcal{A}''$ such that $\mathcal{J} \not\models g$, where the assertion g can be either a positive MA or a negative one. From $\mathcal{J} \not\models g$, we will deduce a contradiction by showing $\mathcal{J} \notin \mathcal{M}$, that is, by showing that for every $\mathcal{I} \in \mathsf{Mod}(\mathcal{K})$ there is $\mathcal{J}' \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$ such that

$$\mathcal{I} \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J}. \tag{4}$$

(i) Assume g is a positive MA. Consider an arbitrary $\mathcal{I} \in \mathsf{Mod}(\mathcal{K})$. Clearly, $\mathcal{I} \models g$ (since $\mathcal{I} \models \mathcal{T} \cup \mathcal{A}$, $\mathcal{A} \models_{\mathcal{T}} \mathcal{A}''$, and $g \in \mathcal{A}''$). Now let $\mathcal{J}' = \mathcal{J} \cup \mathcal{I}[g]$ (recall that $\mathcal{I}[g]$ is a minimal (w.r.t. set-inclusion) submodel of $\mathcal{I}$ satisfying both g and $\mathcal{T}$). Clearly, such $\mathcal{I}[g]$ exists while it may be not unique. If $\mathcal{I}[g]$ is not unique, then any such $\mathcal{I}[g]$ can be used in the construction of $\mathcal{J}'$.
Observe that $\mathcal{J}' \models \mathcal{N}$ and $\mathcal{J}' \models \mathcal{T}$. The former entailment holds since $\mathcal{J} \models \mathcal{N}$. Now we show the latter entailment. Assume $\mathcal{J}' \not\models \mathcal{T}$. Since $\mathcal{J} \models \mathcal{T}$, $\mathcal{I}[g] \models \mathcal{T}$, then, due to Proposition 3.3, there are $f_1 \in \mathcal{J}$ and $f_2 \in \mathcal{I}[g]$ such that $\{f_1, f_2\} \models_{\mathcal{T}} \bot$. Note that, as a consequence of Proposition 3.2, both atoms f_1 and f_2 are unary. From $f_2 \in \mathcal{I}[g]$ and $\mathcal{I}[g] \subseteq \mathcal{I}$ we conclude that $f_2 \in \mathcal{I}$; combining $f_2 \in \mathcal{I}$ with $\mathcal{I} \models \mathcal{T}$ and $\{f_1, f_2\} \models_{\mathcal{T}} \bot$, we conclude that $f_1 \notin \mathcal{I}$.
Now we show that the conclusion $f_1 \notin \mathcal{I}$ leads to the contradiction with the fact that $\mathcal{J} \notin \mathcal{M}$, which will prove that $\mathcal{J}' \models \mathcal{T}$. To this effect we need to define another interpretation $\mathcal{J}_1$ in the following way: $\mathcal{J}_1 = \mathcal{J} \setminus \mathsf{root}_{\mathcal{T}}(f_1)$. We will show that $\mathcal{J}_1 \models \mathcal{T} \cup \mathcal{N}$ and $\mathcal{I} \ominus \mathcal{J}_1 \subsetneq \mathcal{I} \ominus \mathcal{J}$, thus $\mathcal{J} \notin \mathcal{M}$, which will give us a contradiction. The entailment $\mathcal{J}_1 \models \mathcal{T}$ holds by Proposition 3.6. To see that $\mathcal{J}_1 \models \mathcal{N}$, observe that $\mathcal{N} \not\models_{\mathcal{T}} f_1$. Indeed, if $\mathcal{N} \models_{\mathcal{T}} f_1$, then from $g \models_{\mathcal{T}} f_2$ and $\{f_1, f_2\} \models_{\mathcal{T}} \bot$ we can derive that $\mathcal{N} \cup \{g\} \models \bot$ which contradicts the assumption $g \in \mathcal{A}''$ (such g should have been dropped by AtAlg from $\mathcal{A}''$, see Line 4 of Algorithm 3.1). Using the definition of $\ominus$ and the facts that $\mathsf{root}^{J}_{\mathcal{T}}(f_1) \cap \mathcal{I} = \emptyset$ and $(\mathcal{J} \setminus \mathsf{root}_{\mathcal{T}}(f_1)) \subseteq \mathcal{J}$, it is easy to check that $\mathcal{I} \ominus \mathcal{J}_1 \subseteq \mathcal{I} \ominus \mathcal{J}$. Inequality $\mathcal{I} \ominus \mathcal{J}_1 \neq \mathcal{I} \ominus \mathcal{J}$ follows from the fact that $f_1 \notin \mathcal{I}$, $f_1 \notin \mathcal{J}_1$, and $f_1 \in \mathcal{J}$. This finishes the proof of $\mathcal{J}' \models \mathcal{T}$.
It remains to show that Eq. (4) holds for the constructed $\mathcal{J}'$. Since $\mathcal{I}[g] \subseteq \mathcal{I}$ one can apply the definition of $\ominus$ to conclude that $\mathcal{I} \ominus \mathcal{J}' = \mathcal{I} \ominus (\mathcal{J} \cup \mathcal{I}[g]) \subseteq \mathcal{I} \ominus \mathcal{J}$. The inequality $\mathcal{I} \ominus \mathcal{J}' \neq \mathcal{I} \ominus \mathcal{J}$ follows from the fact that $g \in \mathcal{I}$, $g \in \mathcal{J}'$, and $g \notin \mathcal{J}$. Thus, $\mathcal{J} \notin \mathcal{M}$ and we obtain a contradiction.

(ii) Assume g is a negative MA, i.e., $g = \neg h$ for some positive MA h. Consider an arbitrary $\mathcal{I} \in \mathsf{Mod}(\mathcal{K})$. Clearly, $\mathcal{J} \models h$ (by the assumption that $\mathcal{J} \not\models g$) and $\mathcal{I} \not\models h$ (since $\mathcal{I} \models \mathcal{T} \cup \mathcal{A}$, $\mathcal{A} \models_{\mathcal{T}} \mathcal{A}''$, and $\neg h \in \mathcal{A}''$). Now let $\mathcal{J}' := \mathcal{J} \setminus \mathsf{root}_{\mathcal{T}}(h)$. Observe that $\mathcal{J}' \models \mathcal{N}$ and $\mathcal{J}' \models \mathcal{T}$. The former entailment holds since $\neg h \in \mathcal{A}''$ and consequently $\{\neg h\} \cup \mathcal{N} \not\models_{\mathcal{T}} \bot$, i.e., $\mathcal{N} \not\models_{\mathcal{T}} h$; thus, $\mathsf{root}_{\mathcal{T}}(h) \cap \mathcal{N} = \emptyset$. The latter entailment holds due to Proposition 3.6.
Using the fact that $\mathcal{J}' \subseteq \mathcal{J}$, $\mathsf{root}_{\mathcal{T}}(h) \nsubseteq \mathcal{J}'$, $\mathsf{root}_{\mathcal{T}}(h) \nsubseteq \mathcal{I}$, and the definition of $\ominus$, it is easy to check that $\mathcal{I} \ominus \mathcal{J}' \subseteq \mathcal{I} \ominus \mathcal{J}$. From the facts that $h \in \mathcal{J}$, $h \notin \mathcal{J}'$ and $h \notin \mathcal{I}$ we conclude that $\mathcal{I} \ominus \mathcal{J}' \neq \mathcal{I} \ominus \mathcal{J}$, and therefore $\mathcal{I} \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J}$. We conclude that Eq. (4) holds, $\mathcal{J} \notin \mathcal{M}$, and obtain a contradiction.

$\mathcal{M} \supseteq \mathsf{Mod}(\mathcal{K}')$: Let $\mathcal{J} \models \mathcal{K}'$. To prove that $\mathcal{J} \in \mathcal{M}$ we need to show that $\mathcal{J} \models \mathcal{T} \cup \mathcal{N}$ and there exists a model $\mathcal{I}$ of $\mathcal{K}$ such that $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$. The former follows from the fact that $\mathcal{J} \models \mathcal{K}'$, while the latter from Lemma 3.8. Thus, $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$ and therefore $\mathcal{M} \supseteq \mathsf{Mod}(\mathcal{K}')$ holds, which concludes the proof. □

We conclude this section with an example.

Example 3.11. In the notations of Example 3.1, due to Theorem 3.10 we have that $\mathbf{L}^a_{\subseteq}$-evolution for $(\mathcal{T}_1 \cup \mathcal{A}_0, \mathcal{N}_1)$ is the following KB:

$$\mathcal{K}' = \mathcal{T}_1 \cup \mathsf{AtAlg}(\mathcal{T}_1 \cup \mathcal{A}_0, \mathcal{N}_1) \cup \{Priest(john)\}.$$

This result is quite intuitive and expected: the new knowledge $\mathcal{N}_1$ asserts that *john* is a priest now, while the TBox $\mathcal{T}_1$ forbids to be a priest and a husband at the same time; thus, we have to drop from the old knowledge that *john* is a husband and that he is not a priest. Also note that $\mathcal{K}'$ contains $\neg Card(john)$, that is, the fact that *john* became a priest does not make him a cardinal.

$\mathbf{L}^a_{\subseteq}$-evolution for $(\mathcal{T}_1 \cup \mathcal{A}_0, \mathcal{N}_2)$ is the following KB:

$$\mathcal{K}' = \mathcal{T}_1 \cup \mathsf{AtAlg}(\mathcal{T}_1 \cup \mathcal{A}_0, \mathcal{N}_2) \cup \{Husb(pedro), Wife(tanya)\}.$$

This result is again expected: *pedro* becomes a husband and we have to drop the old knowledge that he is a priest and not a husband. Moreover, *tanya* becomes a wife and, since this fact does not conflict with anything in the old knowledge, we just add it. □

3.2. Capturing $\mathbf{G}^s_{\subseteq}$-evolution

Consider the algorithm GSymbAlg in Algorithm 3.2. It takes a *DL-Lite*^{pr}-evolution setting $\mathcal{E} = (\mathcal{T} \cup \mathcal{A}, \mathcal{N})$ as input. Then, it computes the set $\mathsf{AtAlg}(\mathcal{E})$ and for every atom ϕ in $\mathcal{N}$ it checks whether $\neg\phi \in \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$ (Line 4). If it is the case, GSymbAlg deletes from $\mathsf{AtAlg}(\mathcal{E})$ all literals ϕ' that share the concept name with ϕ. Finally, GSymbAlg returns what remains from $\mathsf{AtAlg}(\mathcal{E})$. We will illustrate GSymbAlg with the following example.

Algorithm 3.2: $\mathsf{GSymbAlg}(\mathcal{E})$.

INPUT : *DL-Lite^{pr}*-evolution setting $\mathcal{E} = (\mathcal{T} \cup \mathcal{A}, \mathcal{N})$
OUTPUT: A set $\mathcal{A}' \subseteq \mathsf{cl}_{\mathcal{T}}(\mathcal{A}) \cup \mathsf{cl}_{\mathcal{T}}(\mathcal{N})$ of ABox assertions

1 $\mathcal{A}' := \emptyset$; $X := \mathsf{AtAlg}(\mathcal{E})$; $Y := \mathsf{cl}_{\mathcal{T}}(\mathcal{N})$;
2 **repeat**
3   **choose some** $\phi \in Y$; $Y := Y \setminus \{\phi\}$;
4   **if** $\neg\phi \in \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$ **then** $X := X \setminus \{\phi' \in \mathsf{cl}_{\mathcal{T}}(\mathcal{A}) \mid \phi$ and ϕ' have the same concept name$\}$
5 **until** $Y = \emptyset$;
6 $\mathcal{A}' := X \cup \mathcal{N}$;
7 **return** $\mathcal{A}'$.

(T1) $\mathcal{N} \models_{\mathcal{T}} A(c)$ and $\mathcal{K} \models A(c)$;	(T6) $\mathcal{N} \models_{\mathcal{T}} \neg A(c)$ and $\mathcal{K} \parallel A(c)$;
(T2) $\mathcal{N} \models_{\mathcal{T}} \neg A(c)$ and $\mathcal{K} \models \neg A(c)$;	(T7) $\mathcal{N} \parallel_{\mathcal{T}} A(c)$ and $\mathcal{K} \parallel A(c)$;
(T3) $\mathcal{N} \models_{\mathcal{T}} A(c)$ and $\mathcal{K} \models \neg A(c)$;	(T8) $\mathcal{N} \parallel_{\mathcal{T}} A(c)$ and $\mathcal{K} \models A(c)$;
(T4) $\mathcal{N} \models_{\mathcal{T}} \neg A(c)$ and $\mathcal{K} \models A(c)$;	(T9) $\mathcal{N} \parallel_{\mathcal{T}} A(c)$ and $\mathcal{K} \models \neg A(c)$.
(T5) $\mathcal{N} \models_{\mathcal{T}} A(c)$ and $\mathcal{K} \parallel A(c)$;	

Fig. 3.1. Classification of atom $A(c)$ w.r.t. $\mathcal{K}$ and $\mathcal{N}$.

Example 3.12. Consider a *DL-Lite^{pr}*-evolution setting $\mathcal{E} = (\mathcal{T} \cup \mathcal{A}, \mathcal{N})$ with

$$\mathcal{T} = \{Priest \sqsubseteq \neg Husb\}, \qquad \mathcal{A} = \big\{Priest(pedro), Priest(ivan)\big\}, \quad \text{and} \quad \mathcal{N} = \big\{Husb(pedro)\big\}.$$

Observe that $\mathsf{GSymbAlg}(\mathcal{E}) = \{Husb(pedro)\}$. Indeed,

$$Y = \mathsf{cl}_{\mathcal{T}}(\mathcal{N}) = \big\{Husb(pedro), \neg Priest(pedro)\big\};$$
$$\mathsf{cl}_{\mathcal{T}}(\mathcal{A}) = \big\{Priest(pedro), Priest(ivan), \neg Husb(pedro), \neg Husb(ivan)\big\};$$

therefore, $X = \mathsf{AtAlg}(\mathcal{E}) = \{Priest(ivan), \neg Husb(ivan)\}$. Now, the assertions $Husb(pedro)$ and $\neg Priest(pedro)$ satisfy the condition of Line 4 of GSymbAlg, and therefore the atoms of X should be deleted, that is, GSymbAlg returns $\mathcal{A}' = \emptyset \cup \mathcal{N}$. $\square$

We will show that GSymbAlg computes precisely $\mathbf{G}^s_{\subseteq}$-evolutions for *DL-Lite^{pr}*-evolution settings. Intuitively, GSymbAlg does so by tracing all assertions of the form $B(c)$ or $\neg B(c)$ entailed by $\mathcal{A}$ that should be deleted from (the $\mathcal{T}$-closure of) $\mathcal{A}$ due to $\mathcal{N}$. For such B's the change of interpretation is inevitable, i.e., if in some model $\mathcal{I}$ of $\mathcal{K}$ we had $b \in B^{\mathcal{I}}$, then in every $\mathcal{J} \in \mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{G}^s_{\subseteq}$ we have $b \notin B^{\mathcal{I}}$. Since symbol-based semantics trace changes on symbols only, and the interpretation of the symbol B is to be changed, one should drop from (the $\mathcal{T}$-closure of) $\mathcal{A}$ all the assertions over the symbol B, that is, of the form $B(d)$ and $\neg B(d)$ for some d. We will illustrate this phenomenon for $B = Priest$, $c = pedro$, and $d = ivan$ in the following example.

Example 3.13. Continuing with Example 3.12, observe that for every model $\mathcal{I} \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{A})$, it holds $\mathcal{I} \models Priest(pedro)$, and for every model $\mathcal{J} \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$, it holds $\mathcal{J} \models \neg Priest(pedro)$. Hence, for every pair of models $\mathcal{I} \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{A})$ and $\mathcal{J} \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$, it holds that $\{Priest\} \subseteq \mathsf{dist}^s_{\subseteq}(\mathcal{I}, \mathcal{J})$, and therefore $\{Priest\} \subseteq \mathsf{dist}^s_{\subseteq}(\mathsf{Mod}(\mathcal{T} \cup \mathcal{A}), \mathcal{J})$. Consider the following models $\mathcal{J}_1, \mathcal{J}_2 \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$:

$$\mathcal{J}_1 = \big\{Husb(pedro), Priest(ivan)\big\}, \qquad \mathcal{J}_2 = \big\{Husb(pedro)\big\}.$$

It is easy to see that for $\mathcal{I}_1 = \{Priest(pedro), Priest(ivan)\}$, we have: $\mathsf{dist}^s_{\subseteq}(\mathsf{Mod}(\mathcal{T} \cup \mathcal{A}), \mathcal{J}_i) \subseteq \mathsf{dist}^s_{\subseteq}(\mathcal{I}_1, \mathcal{J}_i) = \{Priest\}$ for $i \in \{1, 2\}$. Hence, $\mathsf{dist}^s_{\subseteq}(\mathsf{Mod}(\mathcal{T} \cup \mathcal{A}), \mathcal{J}_i) = \{Priest\}$, so we conclude that $\mathcal{J}_i \in (\mathcal{T} \cup \mathcal{A}) \diamond_S \mathcal{N}$ with $S = \mathbf{G}^s_{\subseteq}$ and for $i \in \{1, 2\}$. Observe that $\mathcal{J}_1 \models Priest(ivan)$ and $\mathcal{J}_2 \not\models Priest(ivan)$; thus, for $\mathcal{K}' = \mathcal{T} \cup \mathcal{A}'$, which is a $\mathbf{G}^s_{\subseteq}$-evolution for $\mathcal{E}$, it holds that $\mathcal{K}' \not\models Priest(ivan)$.

We emphasize that the behavior of $\mathbf{G}^s_{\subseteq}$-evolution is quite counterintuitive: as soon as we declare that a specific object is no longer in a concept, say A, (by asserting that it is in the complement to A, e.g., when we declared that *pedro* is no longer in *Priest* by asserting that he is in *Husb*), the old information about *all* the other objects in A should be erased (all old members of *Priest* should be erased). $\square$

Before proceeding to a formal proof of correctness for GSymbAlg, we present the following notations and a proposition. With $\mathcal{A} \parallel_{\mathcal{T}} \phi$ we denote the fact that neither $\mathcal{A} \models_{\mathcal{T}} \phi$ nor $\mathcal{A} \models_{\mathcal{T}} \neg\phi$ holds. The definition of $\mathcal{K} \parallel \phi$ is analogous. Observe that for every KB $\mathcal{K}$ and atom $A(c)$, there are three possible relations between them: $\mathcal{K} \models A(c)$, or $\mathcal{K} \models \neg A(c)$, or $\mathcal{K} \parallel A(c)$. For given $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, $\mathcal{N}$, and atom $A(c)$, these three relations give nine combinations, which are presented in Fig. 3.1. We refer to each such combination as *type of* $A(c)$ (w.r.t. $\mathcal{K}$ and $\mathcal{N}$) and consequently there are nine types: (T1)–(T9).

We recall that $\mathcal{J}_0[A(c)]$ is a minimal (w.r.t. set inclusion) submodel of $\mathcal{J}_0$ containing $A(c)$ and satisfying $\mathcal{T}$ (see Definition 3.9).

Proposition 3.14. *Let $\mathcal{T}$ be a DL-Lite^{pr} TBox, $\mathcal{I}$ and $\mathcal{J}$ models of $\mathcal{T}$, and $A(c)$ an atom. Then, the interpretation $(\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(\neg A(c))) \cup \mathcal{J}[A(c)]$ is a model of $\mathcal{T}$.*

We proceed to correctness of GSymbAlg.

Theorem 3.15. *Let $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ be a DL-Lite^{pr}-evolution setting. Then,*

$$\mathcal{K}' = \mathcal{T} \cup \mathsf{GSymbAlg}(\mathcal{E}) \tag{5}$$

is a DL-Lite^{pr} KB and a $\mathbf{G}^s_{\subseteq}$-evolution for $\mathcal{E}$. Moreover, GSymbAlg$(\mathcal{E})$ *is computable in time polynomial in $|\mathcal{E}|$.*

Proof. Let $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$. The fact that $\mathcal{K}'$ is a *DL-Lite^{pr}* KB follows from the fact that $\mathcal{T} \cup \mathsf{AtAlg}(\mathcal{E}) \cup \mathcal{N}$ is in *DL-Lite^{pr}* (see Theorem 3.10) and $\mathcal{K}' \subseteq \mathcal{T} \cup \mathsf{AtAlg}(\mathcal{E}) \cup \mathcal{N}$. Polynomiality of GSymbAlg follows from polynomiality of AtAlg, the fact that $|\mathsf{cl}_{\mathcal{T}}(\mathcal{N})|$ is worst case quadratic in $|\mathcal{N} \cup \mathcal{T}|$, and that the test $\neg\phi \in \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$ is polynomial in $|\mathcal{K} \cup \mathcal{N}|$. Let $\mathcal{M} = \mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{G}^s_{\subseteq}$. We now show that $\mathcal{M} = \mathsf{Mod}(\mathcal{K}')$ by showing the two inclusions separately.

$\mathcal{M} \subseteq \mathsf{Mod}(\mathcal{K}')$: Consider a model $\mathcal{J}_0 \in \mathcal{M}$. We show that $\mathcal{J}_0 \in \mathsf{Mod}(\mathcal{K}')$. By the definition of $\mathbf{G}^s_{\subseteq}$, we have $\mathcal{J}_0 \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$ and there exists a model $\mathcal{I}_0 \in \mathsf{Mod}(\mathcal{K})$ such that for every pair of models $\mathcal{J}_1 \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$ and $\mathcal{I}_1 \in \mathsf{Mod}(\mathcal{K})$ the following inclusion *does not* hold.

$$\mathsf{dist}^s_{\subseteq}(\mathcal{I}_1, \mathcal{J}_1) \subsetneq \mathsf{dist}^s_{\subseteq}(\mathcal{I}_0, \mathcal{J}_0). \tag{6}$$

Assume that $\mathcal{J}_0 \notin \mathsf{Mod}(\mathcal{K}')$. We now exhibit a pair of appropriate $\mathcal{I}_1$ and $\mathcal{J}_1$ that satisfies Eq. (6), thus, obtaining a contradiction. Since $\mathcal{J}_0 \notin \mathsf{Mod}(\mathcal{K}')$ and $\mathcal{J}_0 \models \mathcal{T} \cup \mathcal{N}$, by Line 6 of GSymbAlg (see Algorithm 3.2), $\mathcal{J}_0 \not\models X$. Hence, there exists a literal $L(c) \in X$ such that $\mathcal{J}_0 \not\models L(c)$. Proposition 3.2 and the fact that $\neg\phi \notin \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$, where ϕ is of the form $R(a,b)$ or $\exists R(a)$, for a *DL-Lite^{pr}* KB imply that $L(c)$ is of the form $A(c)$ or $\neg A(c)$. Moreover, from $\mathcal{I}_0 \models \mathcal{K}$ we conclude that $\mathcal{I}_0 \models X$ and consequently $\mathcal{I}_0 \models L(c)$. Therefore, $A^{\mathcal{I}_0} \neq A^{\mathcal{J}_0}$.

Now we construct $\mathcal{I}_1$ and $\mathcal{J}_1$ from $\mathcal{I}_0$ and $\mathcal{J}_0$, respectively, in a way that they agree on the interpretation of A. The construction of these models depends on the type of $A(d) \in (\mathcal{I}_0 \ominus \mathcal{J}_0) \setminus (\mathcal{I} \ominus \mathcal{J})$ for $d \in \Delta$ w.r.t. $\mathcal{K}$, $\mathcal{N}$ (Fig. 3.1). Observe that $A(d)$ *cannot* be of type (T1)–(T4). Indeed, if $A(d)$ is of type (T1) or (T2), then $A(d) \notin \mathcal{I}_0 \ominus \mathcal{J}_0$. If $A(d)$ is of type (T3) or (T4), then $A(d) \in \mathcal{I} \ominus \mathcal{J}$. Both cases contradict $A(d) \in (\mathcal{I}_0 \ominus \mathcal{J}_0) \setminus (\mathcal{I} \ominus \mathcal{J})$.

We construct $\mathcal{I}_1$ from $\mathcal{I}_0$ using atoms $A(d)$ of type (T5)–(T7), and then $\mathcal{J}_1$ from $\mathcal{J}_0$ using atoms $A(d)$ of type (T7)–(T9). The interpretation $\mathcal{I}_1$ is defined as follows:

$$\mathcal{I}_1 := \bigcup_{\substack{A(d) \in \mathcal{I}_0 \ominus \mathcal{J}_0,\, \mathcal{J}_0 \models A(d) \\ A(d) \text{ of type (T5) or (T7)}}} \big((\mathcal{I}_0 \setminus \mathsf{root}_{\mathcal{T}}(\neg A(d))) \cup \mathcal{J}_0[A(d)]\big) \setminus \bigcup_{\substack{A(d) \in \mathcal{I}_0 \ominus \mathcal{J}_0,\, \mathcal{J}_0 \models \neg A(d) \\ A(d) \text{ of type (T6)}}} \mathsf{root}_{\mathcal{T}}(A(d)). \tag{7}$$

Observe that $\mathcal{I}_1 \models \mathcal{K}$. Indeed, due to Proposition 3.6 and Proposition 3.14 we have that $\mathcal{I}_1 \models \mathcal{T}$. To see that $\mathcal{I}_1 \models \mathcal{A}$, recall that $A(d)$ is of type (T5)–(T7) and therefore $\mathcal{K} \parallel A(d)$. Moreover, due to the fact that $\mathcal{K}$ is in *DL-Lite^{pr}* and $\mathcal{J}_0 \models \neg A(d)$, each subtracted set $\mathsf{root}_{\mathcal{T}}(A(d))$ contains only unary atoms of the form $A'(d)$. Combining these two observations we conclude that $\mathcal{K} \parallel A'(d)$ and therefore $A'(d) \notin \mathcal{A}$. Thus, $\mathcal{I}_1 \models \mathcal{A}$.

The interpretation $\mathcal{J}_1$ is defined as follows:

$$\mathcal{J}_1 := \bigcup_{\substack{A(d) \in \mathcal{I}_0 \ominus \mathcal{J}_0,\, \mathcal{I}_0 \models A(d) \\ A(d) \text{ of type (T7) or (T8)}}} \big((\mathcal{J}_0 \setminus \mathsf{root}_{\mathcal{T}}(\neg A(d))) \cup \mathcal{I}_0[A(d)]\big) \setminus \bigcup_{\substack{A(d) \in \mathcal{I}_0 \ominus \mathcal{J}_0,\, \mathcal{I}_0 \models \neg A(d) \\ A(d) \text{ of type (T9)}}} \mathsf{root}_{\mathcal{T}}(A(d)). \tag{8}$$

One can show that $\mathcal{J}_1 \models \mathcal{T} \cup \mathcal{N}$ analogously to the proof of $\mathcal{I}_1 \models \mathcal{T} \cup \mathcal{A}$ above. Observe that by construction of $\mathcal{I}_1$ and $\mathcal{J}_1$, we have $A^{\mathcal{I}_1} = A^{\mathcal{J}_1}$ and $\mathsf{dist}^s_{\subseteq}(\mathcal{I}_1, \mathcal{J}_1) \subseteq \mathsf{dist}^s_{\subseteq}(\mathcal{I}_0, \mathcal{J}_0)$. Finally, the former equality gives that $A \notin \mathsf{dist}^s_{\subseteq}(\mathcal{I}_1, \mathcal{J}_1)$, which together with $A \in \mathsf{dist}^s_{\subseteq}(\mathcal{I}_0, \mathcal{J}_0)$ implies Eq. (6) and concludes the proof.

$\mathsf{Mod}(\mathcal{K}') \subseteq \mathcal{M}$: Let $\mathcal{J}_0 \in \mathsf{Mod}(\mathcal{K}') = \mathsf{Mod}(\mathcal{T} \cup \mathcal{A}')$ where $\mathcal{A}' = \mathsf{GSymbAlg}(\mathcal{E})$, and assume $\mathcal{J}_0 \notin \mathcal{M}$, that is: (i) $\mathcal{J}_0 \notin \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$, or (ii) for every $\mathcal{I} \models \mathcal{K}$ there is a pair of models $\mathcal{I}' \models \mathcal{K}$ and $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ s.t. $\mathsf{dist}^s_{\subseteq}(\mathcal{I}', \mathcal{J}') \subsetneq \mathsf{dist}^s_{\subseteq}(\mathcal{I}, \mathcal{J}_0)$. Case (i) is impossible since $\mathcal{N} \subseteq \mathcal{A}'$. If Case (ii) holds, then consider a model $\mathcal{I}_0$ as in Eq. (2). By Lemma 3.8, we have $\mathcal{I}_0 \models \mathcal{K}$. By our assumption, $\mathsf{dist}^s_{\subseteq}(\mathcal{I}', \mathcal{J}') \subsetneq \mathsf{dist}^s_{\subseteq}(\mathcal{I}_0, \mathcal{J}_0)$ holds for some $\mathcal{I}'$ and $\mathcal{J}'$. Due to Proposition 3.2, $\mathcal{I}_0$ and $\mathcal{J}_0$ coincide on how they interpret roles. Thus, there is a concept A such that $A^{\mathcal{I}'} = A^{\mathcal{J}'}$ while $A^{\mathcal{I}_0} \neq A^{\mathcal{J}_0}$, and consequently there is an atom $A(c) \in \mathcal{I}_0 \ominus \mathcal{J}_0$. Note that, by the construction of $\mathcal{I}_0$, it holds that $A(c) \in \mathsf{ucl}_{\mathcal{T}}(\mathcal{A})$ or $A(c) \in \mathsf{conf}_{\mathcal{T}}(\mathcal{J}_0, \mathcal{A})$. We have two cases: $A(c) \in \mathcal{I}_0 \setminus \mathcal{J}_0$ and $A(c) \in \mathcal{J}_0 \setminus \mathcal{I}_0$. In either case the fact $A^{\mathcal{I}'} \neq A^{\mathcal{J}'}$ is implied, which yields a contradiction with $A^{\mathcal{I}'} = A^{\mathcal{J}'}$ (see Appendix B for the details). Thus, $\mathcal{J}_0 \in \mathcal{M}$, which concludes the proof. $\square$

3.3. Approximation of $\mathbf{L}^s_{\subseteq}$-evolution

We start with the observation that $\mathbf{L}^s_{\subseteq}$ is not expressible in *DL-Lite^{pr}*, because capturing $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^s_{\subseteq}$ requires disjunction, which is not available in *DL-Lite*. Formally:

Algorithm 3.3: LSymbAlg($\mathcal{E}$).

INPUT : *DL-Lite*^{pr}-evolution setting $\mathcal{E} = (\mathcal{T} \cup \mathcal{A}, \mathcal{N})$
OUTPUT: A set $\mathcal{A}'' \subseteq \mathsf{cl}_{\mathcal{T}}(\mathcal{A}) \cup \mathsf{cl}_{\mathcal{T}}(\mathcal{N})$ of ABox assertions

1 $\mathcal{A}'' := \emptyset$; $X := \mathsf{AtAlg}(\mathcal{T} \cup \mathcal{A}, \mathcal{N})$; $Y := \mathsf{cl}_{\mathcal{T}}(\mathcal{N})$;
2 **repeat**
3 | **choose some** $\phi \in Y$; $Y := Y \setminus \{\phi\}$;
4 | **if** $\phi \notin X$ **then** $X := X \setminus \{\phi' \in \mathsf{cl}_{\mathcal{T}}(\mathcal{A}) \mid \phi$ and ϕ' have the same concept name$\}$
5 **until** $Y = \emptyset$;
6 $\mathcal{A}'' := X \cup \mathcal{N}$;
7 **return** $\mathcal{A}''$.

Theorem 3.16. *DL-Lite*^{pr} *is not closed under* $\mathbf{L}^s_{\subseteq}$ *semantics.*

Proof (Sketch). Let $S = \mathbf{L}^s_{\subseteq}$. Consider the KB $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, where $\mathcal{T} = \{A \sqsubseteq B\}$ and $\mathcal{A} = \{B(c)\}$, and $\mathcal{N} = \{B(d)\}$. It can be shown that (i) every $\mathcal{J} \models \mathcal{K} \diamond_S \mathcal{N}$ satisfies $A(d) \to B(c)$, and (ii) there are models $\mathcal{J}_0, \mathcal{J}_1 \in \mathcal{K} \diamond_S \mathcal{N}$ such that $\mathcal{J}_0 \not\models \neg A(c)$ and $\mathcal{J}_1 \not\models B(c)$. Due to Lemma 1 in [7], if these two conditions hold, then $\mathcal{K} \diamond_S \mathcal{N}$ is inexpressible in *DL-Lite*, and hence in *DL-Lite*^{pr}. □

The theorem above suggests to look for *DL-Lite*^{pr}-approximations of $\mathbf{L}^s_{\subseteq}$-evolution. We now show that the algorithm LSymbAlg in Algorithm 3.3 can be used for this purpose. Note that LSymbAlg differs from GSymbAlg in Line 4 only, i.e., LSymbAlg in Line 4 performs a test different from the one of GSymbAlg. Intuitively, for an assertion ϕ of the form $A(c)$, LSymbAlg checks whether $A(c)$ is in $\mathsf{cl}_{\mathcal{T}}(\mathcal{N})$ but not in X, and, if it is the case, then LSymbAlg deletes all the assertions over the concept A from $\mathsf{cl}_{\mathcal{T}}(\mathcal{A})$. Note that the test of LSymbAlg is weaker than the one of GSymbAlg since it is easier to get changes in the interpretation of A by choosing a model of $\mathcal{K}$ that does not include $A(c)$. We illustrate LSymbAlg on the following example.

Example 3.17. Consider a *DL-Lite*^{pr}-evolution setting $\mathcal{E} = (\mathcal{T} \cup \mathcal{A}, \mathcal{N})$ with

$$\mathcal{T} = \emptyset, \qquad \mathcal{A} = \{Wife(mary), Wife(chloe)\}, \quad \text{and} \quad \mathcal{N} = \{Wife(tanya)\}.$$

Observe that $\mathsf{LSymbAlg}(\mathcal{E}) = \{Wife(tanya)\}$. Indeed, $Y = \mathsf{cl}_{\mathcal{T}}(\mathcal{N}) = \mathcal{N}$, $\mathsf{cl}_{\mathcal{T}}(\mathcal{A}) = \mathcal{A}$, and $X = \mathsf{AtAlg}(\mathcal{E}) = \mathcal{A}$. Now, the assertion *Wife*(*tanya*) satisfies the condition of Line 4 of LSymbAlg, and therefore both atoms of X should be deleted, that is, LSymbAlg returns $\mathcal{A}' = \emptyset \cup \mathcal{N}$.

Observe that the KB $\mathcal{K}'' = \mathcal{T} \cup \mathsf{LSymbAlg}(\mathcal{E})$ that approximates $\mathbf{L}^s_{\subseteq}$-evolution for $\mathcal{E}$[6] is even less intuitive than $\mathbf{G}^s_{\subseteq}$-evolution $\mathcal{K}'$ for $\mathcal{E}$ from Example 3.12: $\mathbf{L}^s_{\subseteq}$-evolution erases all the old ABox information about a concept, say B (e.g., in our case such a B is *Wife*), as soon as we just add any new object in B that does not even conflict with anything in the old ABox (in our case we added *Wife*(*tanya*) and had to erase from the old knowledge the information about the other two wives, *Wife*(*mary*) and *Wife*(*chloe*)). □

Theorem 3.18. *Let $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ be a DL-Lite^{pr}-evolution setting. Then,*

$$\mathcal{K}'' = \mathcal{T} \cup \mathsf{LSymbAlg}(\mathcal{E}) \tag{9}$$

is a maximal sound DL-Lite^{pr}-approximation of $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^s_{\subseteq}$. Moreover, LSymbAlg($\mathcal{E}$) *is computable in time polynomial in* $|\mathcal{E}|$.

Proof. The fact that $\mathcal{K}''$ is a *DL-Lite*^{pr} KB follows from the fact that $\mathcal{T} \cup \mathsf{AtAlg}(\mathcal{E}) \cup \mathcal{N}$ is in *DL-Lite*^{pr} (see Theorem 3.10) and $\mathcal{K}'' \subseteq \mathcal{T} \cup \mathsf{AtAlg}(\mathcal{E}) \cup \mathcal{N}$. Polynomiality of LSymbAlg can be shown analogously to polynomiality of GSymbAlg (see Theorem 3.15). Let $S = \mathbf{L}^s_{\subseteq}$. The fact that $\mathcal{K}''$ is a sound approximation of $\mathcal{K} \diamond_S \mathcal{N}$, i.e., $\mathcal{K} \diamond_S \mathcal{N} \subseteq \mathsf{Mod}(\mathcal{K}'')$, can also be shown analogously to the soundness $\mathcal{M} \subseteq \mathsf{Mod}(\mathcal{K}')$ in Theorem 3.15.

Let $\mathcal{A}'' = \mathsf{LSymbAlg}(\mathcal{E})$. Suppose that $\mathcal{K}''$ is not a maximal sound approximation, which means we may add an assertion $A(c)$ to A'', where $A(c)$ is such that $\mathcal{K}'' \not\models A(c)$. That is, $\mathcal{K}''_1 = \mathcal{T} \cup \mathcal{A}'' \cup \{A(c)\}$ is another sound approximation. Consider a canonical model $\mathcal{J}''$ of $\mathcal{K}''$. Using a similar argument as in the proof of the completeness $\mathsf{Mod}(\mathcal{K}') \subseteq \mathcal{M}$ in Theorem 3.15, one can show that $\mathcal{J}'' \in \mathcal{K} \diamond_S \mathcal{N}$. Clearly, $A(c) \notin \mathcal{J}''$, thus $\mathcal{J}'' \not\models \mathcal{K}''_1$, which contradicts the fact that $\mathcal{K}''_1$ is a sound approximation. This concludes the proof. □

Summary of Section 3. $\mathbf{L}^a_{\subseteq}$ and $\mathbf{G}^s_{\subseteq}$-evolutions for *DL-Lite*^{pr}-evolution settings can be computed in polynomial time; $\mathbf{L}^s_{\subseteq}$-evolution for a *DL-Lite*^{pr}-evolution setting in general does not exist, but one can find a maximal sound *DL-Lite*^{pr}-approximation of it in polynomial time.

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6 Actually, one can check that in this particular case the algorithm returns not just an approximation, but an $\mathbf{L}^s_{\subseteq}$-evolution for $\mathcal{E}$.

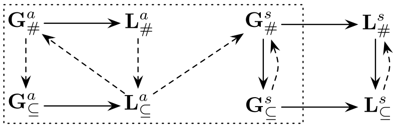

Fig. 4.1. Subsumptions for evolution semantics. The arrows stand for the subsumption $\preccurlyeq_{\mathsf{sem}}$: "$\rightarrow$": for any DL (Theorem 4.2). "$\dashrightarrow$": for *DL-Lite^{pr}* (Theorems 4.4, 4.6, 4.8). The dashed frame surrounds those semantics under which *DL-Lite^{pr}* is closed.

4. Relationships between model-based semantics

In this section we define a framework for comparing different model-based evolution semantics and apply it to the eight semantics that have been presented in Section 2.3. Then we show how to apply the results of Section 3 to all these eight semantics.

Definition 4.1 *(Subsumption for evolution semantics).* Let S_1 and S_2 be two evolution semantics and $\mathcal{D}$ a DL. Then, S_1 *is subsumed by* S_2 w.r.t. $\mathcal{D}$, denoted $(S_1 \preccurlyeq_{\mathsf{sem}} S_2)(\mathcal{D})$, or just $S_1 \preccurlyeq_{\mathsf{sem}} S_2$ when $\mathcal{D}$ is clear from the context, if $\mathcal{K} \diamond_{S_1} \mathcal{N} \subseteq \mathcal{K} \diamond_{S_2} \mathcal{N}$ for every $\mathcal{D}$-evolution setting $\mathcal{E} = (\mathcal{K}, \mathcal{N})$. Also, S_1 and S_2 are *equivalent* w.r.t. $\mathcal{D}$, denoted $(S_1 \equiv_{\mathsf{sem}} S_2)(\mathcal{D})$, if $(S_1 \preccurlyeq_{\mathsf{sem}} S_2)(\mathcal{D})$ and $(S_2 \preccurlyeq_{\mathsf{sem}} S_1)(\mathcal{D})$. $\square$

The following theorem shows the subsumption relations that hold between different semantics, independently of the chosen DL. We depict these relations in Fig. 4.1 using solid arrows. Note that Fig. 4.1 is complete in the following sense: there is a solid oriented path (a sequence of solid arrows) from a semantics S_1 to a semantics S_2 *if* $S_1 \preccurlyeq_{\mathsf{sem}} S_2(\mathcal{D})$ for every DL $\mathcal{D}$.

Theorem 4.2. *W.r.t. every DL it holds that*

$$\mathbf{G}^y_x \preccurlyeq_{\mathsf{sem}} \mathbf{L}^y_x, \quad \textit{where } x \in \{\subseteq, \#\} \textit{ and } y \in \{a, s\}, \qquad \mathbf{L}^s_\# \preccurlyeq_{\mathsf{sem}} \mathbf{L}^s_\subseteq, \quad \textit{and} \quad \mathbf{G}^s_\# \preccurlyeq_{\mathsf{sem}} \mathbf{G}^s_\subseteq.$$

Proof. Let $\mathcal{D}$ be a DL, $(\mathcal{K}, \mathcal{N})$ with $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ a $\mathcal{D}$-evolution setting,[7] and dist^y_x a distance function. We consider the three cases one by one.

$\mathbf{G}^y_x \preccurlyeq_{\mathsf{sem}} \mathbf{L}^y_x$: Let $\mathcal{M}_{\mathbf{G}} = \mathcal{K} \diamond_{S_1} \mathcal{N}$ with $S_1 = \mathbf{G}^y_x$ and $\mathcal{M}_{\mathbf{L}} = \mathcal{K} \diamond_{S_2} \mathcal{N}$ with $S_2 = \mathbf{L}^y_x$ be sets of models defined using respectively global (see Definition 2.8) and local (see Definition 2.6) semantics based on dist^y_x. Let $\mathcal{J}' \in \mathcal{M}_{\mathbf{G}}$, then there is $\mathcal{I}' \models \mathcal{K}$ such that for every $\mathcal{I}'' \models \mathcal{K}$ and $\mathcal{J}'' \models \mathcal{T} \cup \mathcal{N}$ it *does not* hold that

$$\mathsf{dist}^y_x\big(\mathcal{I}'', \mathcal{J}''\big) \lneq \mathsf{dist}^y_x\big(\mathcal{I}', \mathcal{J}'\big).$$

In particular, when $\mathcal{I}'' = \mathcal{I}'$, there is no $\mathcal{J}'' \models \mathcal{T} \cup \mathcal{N}$ such that $\mathsf{dist}^y_x(\mathcal{I}', \mathcal{J}'') \lneq \mathsf{dist}^y_x(\mathcal{I}', \mathcal{J}')$, which yields that $\mathcal{J}' \in \mathsf{loc\_min}^y_x(\mathcal{I}', \mathcal{T}, \mathcal{N})$, and hence $\mathcal{J}' \in \mathcal{M}_{\mathbf{L}}$, which concludes the proof.

$\mathbf{L}^s_\# \preccurlyeq_{\mathsf{sem}} \mathbf{L}^s_\subseteq$: Consider $\mathcal{M}_\# = \mathcal{K} \diamond_{S_1} \mathcal{N}$ with $S_1 = \mathbf{L}^s_\#$, which is based on the distance $\mathsf{dist}^s_\#$, and $\mathcal{M}_\subseteq = \mathcal{K} \diamond_{S_2} \mathcal{N}$ with $S_2 = \mathbf{L}^s_\subseteq$, which is based on $\mathsf{dist}^s_\subseteq$. We show that $\mathcal{M}_\# \subseteq \mathcal{M}_\subseteq$ holds. Towards a contradiction, assume that $\mathcal{J}' \in \mathcal{M}_\#$ and $\mathcal{J}' \notin \mathcal{M}_\subseteq$. Then, from the former assumption we conclude the existence of $\mathcal{I}' \models \mathcal{K}$ such that $\mathcal{J}' \in \mathsf{loc\_min}^s_\#(\mathcal{I}', \mathcal{T}, \mathcal{N})$. From the latter assumption, considering Definition 2.6, $\mathcal{J}' \notin \mathsf{loc\_min}^s_\subseteq(\mathcal{I}, \mathcal{T}, \mathcal{N})$ for every $\mathcal{I} \in \mathsf{Mod}(\mathcal{K})$, in particular for $\mathcal{I} = \mathcal{I}'$. Hence, $\mathcal{J}' \notin \mathsf{loc\_min}^s_\subseteq(\mathcal{I}', \mathcal{T}, \mathcal{N})$ and there exists an interpretation $\mathcal{J}'' \models \mathcal{T} \cup \mathcal{N}$ such that $\mathsf{dist}^s_\subseteq(\mathcal{I}', \mathcal{J}'') \subsetneq \mathsf{dist}^s_\subseteq(\mathcal{I}', \mathcal{J}')$. Since the signature of $\mathcal{K} \cup \mathcal{N}$ is finite (and a signature includes finitely many concept and role names), the distance $\mathsf{dist}^s_\subseteq$ between every two interpretations over this signature is also finite. Thus, we obtain that $\mathsf{dist}_\#(\mathcal{I}', \mathcal{J}'') \lneq \mathsf{dist}_\#(\mathcal{I}', \mathcal{J}')$, which contradicts $\mathcal{J}' \in \mathcal{M}_\#$ and concludes the proof.

$\mathbf{G}^s_\# \preccurlyeq_{\mathsf{sem}} \mathbf{G}^s_\subseteq$: analogous to $\mathbf{L}^s_\# \preccurlyeq_{\mathsf{sem}} \mathbf{L}^s_\subseteq$. $\square$

4.1. Relationships between atom-based semantics in DL-Lite^{pr}

The next theorem shows that all four atom-based semantics are equivalent w.r.t. *DL-Lite^{pr}*. Before proceeding to the formal statement and proof of this result, we present a technical proposition which is the analog of Proposition 3.2 for $\mathbf{L}^a_\#$-evolution semantics, i.e., Proposition 3.2 shows which MAs of the original KB are preserved by evolution under $\mathbf{L}^a_\subseteq$, while the following proposition shows the same under $\mathbf{L}^a_\#$.

Proposition 4.3. *Let $(\mathcal{K}, \mathcal{N})$ with $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ be a DL-Lite^{pr}-evolution setting and let $\mathcal{M} = \mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_\#$. Let g be an MA of the form $R(a, b)$ or $\exists R(a)$. If $\mathcal{A} \models_{\mathcal{T}} g$, then for every $\mathcal{J} \in \mathcal{M}$ it holds $\mathcal{J} \models g$.*

[7] Note that according to Definition 2.4, $\mathcal{N}$ should contain only positive MAs, while here we may weaken this requirement since it does not affect the proof.

Proof (Sketch). Similar to the case of $\mathbf{L}^a_\subseteq$, the semantics $\mathbf{L}^a_\#$ preserves role atoms since a KB $\mathcal{K}' \in$ *DL-Lite*^{pr} does not entail "negative" information involving role atoms. So, dropping role atoms from the models evolved under $\mathbf{L}^a_\#$ violates the principle of minimal change. $\square$

Theorem 4.4. $\mathbf{L}^a_\# \equiv_{\mathsf{sem}} \mathbf{L}^a_\subseteq \equiv_{\mathsf{sem}} \mathbf{G}^a_\# \equiv_{\mathsf{sem}} \mathbf{G}^a_\subseteq$ *w.r.t. DL-Lite*^{pr}.

Proof. Due to Theorem 4.2 and transitivity of the relation $\preccurlyeq_{\mathsf{sem}}$, to conclude the proof it suffices to show only three relations: $\mathbf{L}^a_\# \preccurlyeq_{\mathsf{sem}} \mathbf{L}^a_\subseteq$, $\mathbf{L}^a_\subseteq \preccurlyeq_{\mathsf{sem}} \mathbf{G}^a_\#$, and $\mathbf{G}^a_\# \preccurlyeq_{\mathsf{sem}} \mathbf{G}^a_\subseteq$. Consider a *DL-Lite*^{pr}-evolution setting $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ with $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$.

$\mathbf{L}^a_\# \preccurlyeq_{\mathsf{sem}} \mathbf{L}^a_\subseteq$: Consider a model $\mathcal{J} \in \mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_\#$. Then, there is a model $\mathcal{I} \models \mathcal{K}$ such that $\mathcal{J} \in \mathsf{loc\_min}^a_\#(\mathcal{I}, \mathcal{T}, \mathcal{N})$. Now consider a model $\mathcal{I}'$ built as in Eq. (2); then, we have that $\mathsf{dist}^a_\#(\mathcal{I}, \mathcal{J}) \leqslant \mathsf{dist}^a_\#(\mathcal{I}', \mathcal{J})$. The latter distance is finite by Lemma 3.8, so is the former distance, and therefore $\mathbf{L}^a_\# \preccurlyeq_{\mathsf{sem}} \mathbf{L}^a_\subseteq$ (this can be shown in a similar way as $\mathbf{L}^s_\# \preccurlyeq_{\mathsf{sem}} \mathbf{L}^s_\subseteq$ has been shown in the proof of Theorem 4.2). It only remains to be shown that $\mathcal{I}' \models \mathcal{K}$. This can be shown similarly to the proof of Lemma 3.8, using Proposition 4.3 instead of Proposition 3.2.

$\mathbf{L}^a_\subseteq \preccurlyeq_{\mathsf{sem}} \mathbf{G}^a_\#$: Let $\mathcal{M}_{\mathbf{L}} = \mathcal{K} \diamond_{S_1} \mathcal{N}$ with $S_1 = \mathbf{L}^a_\subseteq$ and $\mathcal{M}_{\mathbf{G}} = \mathcal{K} \diamond_{S_2} \mathcal{N}$ with $S_2 = \mathbf{G}^a_\#$. Consider a model $\mathcal{J} \in \mathcal{M}_{\mathbf{L}}$. We show that $\mathcal{J} \in \mathcal{M}_{\mathbf{G}}$, that is, that there is a model $\mathcal{I} \models \mathcal{K}$ such that for every $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ and $\mathcal{I}' \models \mathcal{K}$ it *does not* hold that $|\mathcal{I}' \ominus \mathcal{J}'| \lneq |\mathcal{I} \ominus \mathcal{J}|$. Consider $\mathcal{I}$ as in Eq. (2). Due to Lemma 3.8, $\mathcal{I} \models \mathcal{K}$. Assume there are $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ and $\mathcal{I}' \models \mathcal{T} \cup \mathcal{A}$ such that $|\mathcal{I}' \ominus \mathcal{J}'| \lneq |\mathcal{I} \ominus \mathcal{J}|$. Since the set $\mathcal{I} \ominus \mathcal{J}$ is at most countable, $\mathcal{I}' \ominus \mathcal{J}'$ is finite, so there exists an atom $A(c) \in (\mathcal{I} \ominus \mathcal{J}) \setminus (\mathcal{I}' \ominus \mathcal{J}')$. We have two cases: $A(c) \in \mathcal{I} \setminus \mathcal{J}$ and $A(c) \in \mathcal{J} \setminus \mathcal{I}$. In either case a contradiction can be obtained. Indeed:

(i) $A(c) \in \mathcal{I} \setminus \mathcal{J}$: By the definition of $\mathcal{I}$, this condition implies that $A(c) \in \mathsf{ucl}_\mathcal{T}(\mathcal{A})$. Observe that $\mathsf{ucl}_\mathcal{T}(\mathcal{A}) \subseteq \mathsf{cl}_\mathcal{T}(\mathcal{A})$, and consequently $A(c) \in \mathsf{cl}_\mathcal{T}(\mathcal{A})$. Due to Theorem 3.10, the inclusion $\mathcal{J} \in \mathcal{M}_L$ implies that $\mathcal{J}$ is a model of $\mathcal{K}' = \mathcal{T} \cup \mathcal{A}'$, where $\mathcal{A}' = \mathsf{AtAlg}(\mathcal{E}) \cup \mathcal{N}$. Due to $A(c) \notin \mathcal{J}$, we conclude that $A(c) \notin \mathsf{AtAlg}(\mathcal{E})$. From the last condition and $A(c) \in \mathsf{cl}_\mathcal{T}(\mathcal{A})$, we obtain $\{A(c)\} \cup \mathcal{N} \models_\mathcal{T} \bot$ (this follows from the definition of AtAlg). This entailment together with $\mathcal{J}' \models \mathcal{N}$ implies that $A(c) \notin \mathcal{J}'$. From $A(c) \notin \mathcal{J}'$ and $A(c) \notin \mathcal{I}' \ominus \mathcal{J}'$ we get $A(c) \notin \mathcal{I}'$. Finally, since $A(c) \in \mathsf{cl}_\mathcal{T}(\mathcal{A})$ and $\mathcal{I}' \models \mathsf{cl}_\mathcal{T}(\mathcal{A})$, we have that $A(c) \in \mathcal{I}'$, which yields a contradiction.

(ii) $A(c) \in \mathcal{J} \setminus \mathcal{I}$: By the definition of $\mathcal{I}$, this condition implies $A(c) \in \mathsf{conf}_\mathcal{T}(\mathcal{J}, \mathcal{A})$ and also $\{A(c)\} \cup \mathsf{cl}_\mathcal{T}(\mathcal{A}) \models_\mathcal{T} \bot$, which means that $\neg A(c) \in \mathsf{cl}_\mathcal{T}(\mathcal{A})$. Thus, $A(c) \notin \mathcal{I}'$ (otherwise $\mathcal{I}'$ would not be a model of $\mathcal{K}$). Recall that $A(c) \notin \mathcal{I}' \ominus \mathcal{J}'$, so $A(c) \notin \mathcal{J}'$. We obtain that $\mathcal{J}' \not\models A(c)$. Since $A(c) \in \mathcal{J}$, we have $\neg A(c) \notin \mathcal{A}'$, that is, $\{\neg A(c)\} \cup \mathcal{N} \models_\mathcal{T} \bot$. Now, combining the last entailment with $\mathcal{J}' \models \mathcal{N}$, we conclude $\mathcal{J}' \not\models \neg A(c)$, which contradicts $\mathcal{J}' \not\models A(c)$.

Thus, $\mathcal{J} \in \mathcal{M}_{\mathbf{G}}$ and consequently $\mathbf{L}^a_\subseteq \preccurlyeq_{\mathsf{sem}} \mathbf{G}^a_\#$.

$\mathbf{G}^a_\# \preccurlyeq_{\mathsf{sem}} \mathbf{G}^a_\subseteq$: Let $\mathcal{J} \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$. Assume $\mathcal{J} \in \mathcal{K} \diamond_{S_1} \mathcal{N}$ with $S_1 = \mathbf{G}^a_\#$, but $\mathcal{J} \notin \mathcal{K} \diamond_{S_2} \mathcal{N}$ with $S_2 = \mathbf{G}^a_\subseteq$. The former assumption implies that there is a model $\mathcal{I} \in \mathsf{Mod}(\mathcal{K})$ such that it is $\mathbf{G}^a_\#$-minimally distant from $\mathcal{J}$. The latter assumption implies the existence of models $\mathcal{I}' \in \mathsf{Mod}(\mathcal{K})$ and $\mathcal{J}' \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$ such that

$$\mathcal{I}' \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J}. \tag{10}$$

Due to the finite model property of *DL-Lite*^{pr}, it holds that $|\mathcal{I} \ominus \mathcal{J}|$ is finite. (See a detailed proof of this statement in Proposition D.1 in Appendix D.) This inequality together with Eq. (10) yields that $|\mathcal{I}' \ominus \mathcal{J}'| \lneq |\mathcal{I} \ominus \mathcal{J}|$, which contradicts the fact that $\mathcal{I}$ is $\mathbf{G}^a_\#$-minimally distant from $\mathcal{J}$. This concludes the proof. $\square$

From Theorems 3.10 and 4.4 we conclude the following:

Corollary 4.5. *Let $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ with $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ be a DL-Lite^{pr}-evolution setting. Then, $\mathcal{T} \cup \mathsf{AtAlg}(\mathcal{E}) \cup \mathcal{N}$ (cf. Eq. (3)) is an S-evolution, where $S \in \{\mathbf{L}^a_\#, \mathbf{L}^a_\subseteq, \mathbf{G}^a_\#, \mathbf{G}^a_\subseteq\}$.*

4.2. Relationships between symbol-based semantics in DL-Lite^{pr}

For symbol-based semantics, the local semantics based on cardinality and on set inclusion coincide, as well as the global ones, while local semantics are not subsumed by the global ones.

Theorem 4.6. $\mathbf{L}^s_\subseteq \equiv_{\mathsf{sem}} \mathbf{L}^s_\#$ *and* $\mathbf{G}^s_\subseteq \equiv_{\mathsf{sem}} \mathbf{G}^s_\#$, *while* $\mathbf{L}^s_\subseteq \not\preccurlyeq_{\mathsf{sem}} \mathbf{G}^s_\#$, *w.r.t. DL-Lite*^{pr}.

Proof. Let $(\mathcal{K}, \mathcal{N})$, where $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, be a *DL-Lite*^{pr}-evolution setting. We consider the three cases one by one.

$\mathbf{G}^s_\# \equiv_{\mathsf{sem}} \mathbf{G}^s_\subseteq$: Due to Theorem 4.2, it suffices to show $\mathbf{G}^s_\subseteq \preccurlyeq_{\mathsf{sem}} \mathbf{G}^s_\#$. Let $\mathcal{M}_\subseteq = \mathcal{K} \diamond_{S_1} \mathcal{N}$ with $S_1 = \mathbf{G}^s_\subseteq$ and $\mathcal{M}_\# = \mathcal{K} \diamond_{S_2} \mathcal{N}$ with $S_2 = \mathbf{G}^s_\#$. Consider a model $\mathcal{J}_0 \in \mathcal{M}_\subseteq$; we show that $\mathcal{J}_0 \in \mathcal{M}_\#$. By the definition of $\mathbf{G}^s_\subseteq$ semantics, there is a model $\mathcal{I}_0 \in \mathsf{Mod}(\mathcal{K})$ such that for every pair of models $\mathcal{I} \in \mathsf{Mod}(\mathcal{K})$ and $\mathcal{J} \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$ it *does not* hold that $\mathsf{dist}^s_\subseteq(\mathcal{I}, \mathcal{J}) \subsetneq \mathsf{dist}^s_\subseteq(\mathcal{I}_0, \mathcal{J}_0)$. Suppose that $\mathcal{J}_0 \notin \mathcal{M}_\#$, that is, for each model $\mathcal{I}' \in \mathsf{Mod}(\mathcal{K})$ there are models $\mathcal{I} \in \mathsf{Mod}(\mathcal{K})$ and $\mathcal{J} \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$ such that $|\mathsf{dist}^s_\subseteq(\mathcal{I}, \mathcal{J})| \lneq |\mathsf{dist}^s_\subseteq(\mathcal{I}', \mathcal{J}_0)|$. In particular, it holds when $\mathcal{I}' = \mathcal{I}_0$. This implies that there is an element in the signature of $\mathcal{K} \cup \mathcal{N}$ with the same interpretation in $\mathcal{I}$ and $\mathcal{J}$, and different interpretations in $\mathcal{I}_0$ and $\mathcal{J}_0$. If this element is a concept A, then $A^{\mathcal{I}} = A^{\mathcal{J}}$ and $A^{\mathcal{I}_0} \neq A^{\mathcal{J}_0}$ (the case when this element is a role symbol is analogous). Thus, there is an atom $A(c) \in (\mathcal{I}_0 \ominus \mathcal{J}_0) \setminus (\mathcal{I} \ominus \mathcal{J})$ for some $c \in \Delta$. We now exhibit models $\mathcal{I}_1 \models \mathcal{K}$ and $\mathcal{J}_1 \models \mathcal{T} \cup \mathcal{N}$ s.t.

$$\mathsf{dist}^s_{\subseteq}(\mathcal{I}_1, \mathcal{J}_1) \subsetneq \mathsf{dist}^s_{\subseteq}(\mathcal{I}_0, \mathcal{J}_0), \tag{11}$$

which contradicts the assumption $\mathcal{J}_0 \in \mathcal{M}_{\subseteq}$. Now we construct $\mathcal{I}_1$ and $\mathcal{J}_1$ as in Eqs. (7) and (8), respectively. The proof that Eq. (11) holds for these $\mathcal{I}_1$ and $\mathcal{J}_1$ is similar to the proof that Eq. (6) holds for $\mathcal{I}_1$ and $\mathcal{J}_1$ in the proof of Theorem 3.15. Thus, $\mathcal{J}_0 \in \mathcal{M}_{\#}$.

$\mathbf{L}^s_{\#} \equiv_{\mathsf{sem}} \mathbf{L}^s_{\subseteq}$: Due to Theorem 4.2, it suffices to show $\mathbf{L}^s_{\subseteq} \preccurlyeq_{\mathsf{sem}} \mathbf{L}^s_{\#}$. This can be done similarly to the case of $\mathbf{G}^s_{\subseteq} \preccurlyeq_{\mathsf{sem}} \mathbf{G}^s_{\#}$, by proving $\mathsf{dist}^s_{\subseteq}(\mathcal{I}_0, \mathcal{J}_1) \subsetneq \mathsf{dist}^s_{\subseteq}(\mathcal{I}_0, \mathcal{J}_0)$ with $\mathcal{J}_1$ for $A(c)$ of types (T7)–(T9).

$\mathbf{L}^s_{\subseteq} \not\preccurlyeq_{\mathsf{sem}} \mathbf{G}^s_{\#}$: Consider the evolution setting $(\mathcal{K}, \mathcal{N})$, where $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, $\mathcal{T} = \{B \sqsubseteq \neg C\}$, $\mathcal{A} = \{A(c), B(a), B(d)\}$, and $\mathcal{N} = \{A(e), C(d)\}$. Consider the following model of $\mathcal{T} \cup \mathcal{N}$: $\mathcal{J} = \{A(e), C(d)\}$. To conclude the proof, observe that $\mathcal{J} \in \mathcal{K} \diamond_{S_1} \mathcal{N}$ with $S_1 = \mathbf{L}^s_{\subseteq}$ and $\mathcal{J} \notin \mathcal{K} \diamond_{S_2} \mathcal{N}$ with $S_2 = \mathbf{G}^s_{\#}$. □

From Theorems 3.15, 3.18, and 4.6 we conclude the following:

Corollary 4.7. *DL-Lite^{pr} is not closed under $\mathbf{L}^s_{\#}$-evolution, and for a DL-Lite^{pr}-evolution setting $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ with $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$*

(i) *the KB $\mathcal{T} \cup \mathsf{GSymbAlg}(\mathcal{E})$ (cf. Eq. (5)) is an S-evolution for $\mathcal{E}$, where $S \in \{\mathbf{G}^s_{\#}, \mathbf{G}^s_{\subseteq}\}$;*
(ii) *the KB $\mathcal{T} \cup \mathsf{LSymbAlg}(\mathcal{E})$ (cf. Eq. (9)) is a maximal sound DL-Lite^{pr}-approximation of $\mathcal{K} \diamond_S \mathcal{N}$ with $S \in \{\mathbf{L}^s_{\#}, \mathbf{L}^s_{\subseteq}\}$.*

4.3. Symbol vs. atom-based semantics in DL-Lite^{pr}

Finally, we show that all atom-based semantics are subsumed by the global symbol-based semantics, while the contrary does not hold. Due to Theorems 4.2, 4.4, and 4.6, this statement follows from $\mathbf{L}^a_{\subseteq} \preccurlyeq_{\mathsf{sem}} \mathbf{G}^s_{\#}$ and $\mathbf{G}^s_{\#} \not\preccurlyeq_{\mathsf{sem}} \mathbf{L}^a_{\subseteq}$. Thus, for *DL-Lite^{pr}* we essentially have three different evolution semantics: atom-based, local symbol-based, and global symbol-based.

Theorem 4.8. *$\mathbf{L}^a_{\subseteq} \preccurlyeq_{\mathsf{sem}} \mathbf{G}^s_{\#}$, while $\mathbf{G}^s_{\#} \not\preccurlyeq_{\mathsf{sem}} \mathbf{L}^a_{\subseteq}$, w.r.t. DL-Lite^{pr}.*

Proof. We will consider the two cases separately.

$\mathbf{L}^a_{\subseteq} \preccurlyeq_{\mathsf{sem}} \mathbf{G}^s_{\#}$: Let $(\mathcal{K}, \mathcal{N})$ be a *DL-Lite^{pr}*-evolution setting. Consider a model $\mathcal{J} \in \mathcal{K} \diamond_{S_1} \mathcal{N}$ with $S_1 = \mathbf{L}^a_{\subseteq}$. We show that $\mathcal{J} \in \mathcal{K} \diamond_{S_2} \mathcal{N}$ with $S_2 \in \mathbf{G}^s_{\#}$. Due to Theorem 3.10, $\mathcal{J}$ is a model of $\mathcal{K}' = \mathcal{T} \cup \mathcal{A}'$ as in Eq. (3). Let $\mathcal{I}$ be an interpretation built as in Eq. (2). Due to Lemma 3.8, we obtain $\mathcal{I} \models \mathcal{K}$. Suppose that $\mathcal{J} \notin \mathcal{K} \diamond_{S_2} \mathcal{N}$ with $S_2 = \mathbf{G}^s_{\#}$, that is, there exists a model $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ such that $|\mathsf{dist}^s_{\subseteq}(\mathsf{Mod}(\mathcal{K}), \mathcal{J}')| \lneq |\mathsf{dist}^s_{\subseteq}(\mathsf{Mod}(\mathcal{K}), \mathcal{J})|$. Note that by the definition of $|\mathsf{dist}^s_{\subseteq}(\mathsf{Mod}(\mathcal{K}), \mathcal{J})|$, it holds that $|\mathsf{dist}^s_{\subseteq}(\mathsf{Mod}(\mathcal{K}), \mathcal{J})| \leqslant |\mathsf{dist}^s_{\subseteq}(\mathcal{I}, \mathcal{J})|$. By the definition of the distance between a set of interpretations and an interpretation, there exists a model $\mathcal{I}' \models \mathcal{K}$ such that $|\mathsf{dist}^s_{\subseteq}(\mathcal{I}', \mathcal{J}')| \lneq |\mathsf{dist}^s_{\subseteq}(\mathcal{I}, \mathcal{J})|$. This implies that there is an element in the signature of $\mathcal{K} \cup \mathcal{N}$ with the same interpretation in $\mathcal{I}'$ and $\mathcal{J}'$, but with different interpretation in $\mathcal{I}$ and $\mathcal{J}$. Note that there is no role $P \in \Sigma(\mathcal{K} \cup \mathcal{N})$ such that $P^{\mathcal{I}} \neq P^{\mathcal{J}}$ due to the construction of $\mathcal{I}$, hence it suffices to consider the case when this element is a concept A, i.e., $A^{\mathcal{I}} \neq A^{\mathcal{J}}$ and $A^{\mathcal{I}'} = A^{\mathcal{J}'}$. From $A^{\mathcal{I}} \neq A^{\mathcal{J}}$ and Eq. (2) we imply that there is an atom $A(c)$ that is either in $\mathsf{conf}_{\mathcal{T}}(\mathcal{J}, \mathcal{A})$ or in $\mathsf{ucl}_{\mathcal{T}}(A)$.

- If $A(c) \in \mathsf{conf}_{\mathcal{T}}(\mathcal{J}, \mathcal{A})$, then $\neg A(c) \in \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$, and, since $\mathcal{J} \not\models \neg A(c)$, the literal $\neg A(c)$ was deleted from $\mathsf{cl}_{\mathcal{T}}(\mathcal{A})$ while building $\mathcal{A}'$ (see Algorithm 3.1), i.e., $\{\neg A(c)\} \cup \mathcal{N} \models_{\mathcal{T}} \bot$. From this entailment and $\mathcal{J}' \models \mathcal{N}$ we conclude that $\mathcal{J}' \not\models \neg A(c)$ and consequently $\mathcal{I}' \not\models \neg A(c)$ (since $A^{\mathcal{I}_1} = A^{\mathcal{J}_1}$). We obtain a contradiction with $\mathcal{I}_1 \models \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$.
- If $A(c) \in \mathsf{ucl}_{\mathcal{T}}(A)$, then $A(c) \in \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$ and $\mathcal{I}' \models A(c)$. Due to $A^{\mathcal{I}'} = A^{\mathcal{J}'}$, we have that $\mathcal{J}' \models A(c)$. On the other hand, since $\mathcal{I} \models \mathsf{ucl}_{\mathcal{T}}(\mathcal{A})$ and $A^{\mathcal{I}} \neq A^{\mathcal{J}}$, we conclude that $\mathcal{J} \not\models A(c)$. Thus, $A(c)$ was deleted from $\mathsf{cl}(\mathcal{A})$ while building $\mathcal{A}'$ (see Algorithm 3.1) and therefore $\{A(c)\} \cup \mathcal{N} \models_{\mathcal{T}} \bot$. Recall that $\mathcal{J}' \models \mathcal{N}$, thus, $\mathcal{J}' \not\models A(c)$ and we obtain a contradiction.

$\mathbf{G}^s_{\#} \not\preccurlyeq_{\mathsf{sem}} \mathbf{L}^a_{\subseteq}$: Consider the evolution setting $(\mathcal{K}, \mathcal{N})$ as in the proof of the case $\mathbf{L}^s_{\subseteq} \not\preccurlyeq_{\mathsf{sem}} \mathbf{G}^s_{\#}$ in Theorem 4.6 and take the following model of $\mathcal{T} \cup \mathcal{N}$: $\mathcal{J} = \{A(c), A(e), C(d)\}$. To conclude the proof observe that $\mathcal{J} \in \mathcal{K} \diamond_{S_1} \mathcal{N}$ with $S_1 = \mathbf{G}^s_{\#}$, while $\mathcal{J} \notin \mathcal{K} \diamond_{S_2} \mathcal{N}$ with $S_2 = \mathbf{L}^a_{\subseteq}$. □

Summary of Section 4 and on DL-Lite^{pr}. Atom-based approaches (which all coincide) can be captured using a polynomial-time algorithm AtAlg. Moreover, the evolution results produced under these MBAs are intuitive and expected. Symbol-based approaches on the contrary produce quite unexpected and counter-intuitive results since the corresponding semantics delete too much data. Two global symbol-based approaches coincide and can be captured using the polynomial-time algorithm GSymbAlg. Two local symbol-based approaches also coincide, cannot be captured in *DL-Lite^{pr}*, but can be approximated using the polynomial-time algorithm LSymbAlg. Based on these results we conclude that using atom-based approaches for applications seems to be more practical. In Fig. 4.1, using dashed arrows, we illustrate all the subsumptions between semantics that have been established in this section. Note that Fig. 4.1 is complete for *DL-Lite^{pr}* in the following sense: there is an oriented path with solid or dashed arrows (a sequence of such arrows) between any two semantics S_1 and S_2 *if and only if* ($S_1 \preccurlyeq_{\mathsf{sem}} S_2$) (*DL-Lite*^{pr}). Moreover, in Fig. 4.1 we framed with a dashed rectangle the six out of eight MBAs under which *DL-Lite*^{pr} is closed.

5. Understanding $\mathbf{L}^a_\subseteq$-evolution of *DL-Lite*$_{core}$ KBs

In the previous sections, we showed that atom-based MBAs behave better than symbol-based ones for *DL-Lite*^{pr}-evolution. This suggests to investigate atom-based MBAs for the entire *DL-Lite*$_{core}$. Here we focus on one of four atom-based MBAs, namely $\mathbf{L}^a_\subseteq$. The remaining three semantics are subjects of future work.

As a further motivation for the study of $\mathbf{L}^a_\subseteq$, note that $\mathbf{L}^a_\subseteq$ is essentially the same as so-called *Winslett's semantics* (WS) [20], which was widely studied in the literature [8,10]. Liu, Lutz, Milicic, and Wolter studied WS for expressive DLs [8]. Most of the DLs they considered are not closed under WS. Poggi, Lembo, De Giacomo, Lenzerini, and Rosati studied WS in a similar setting as the one adopted in this paper. They called it instance-level update for *DL-Lite* [10] and proposed an algorithm to compute the result of updates. However, the algorithm turned out to have technical issues, and it was later shown that it is neither sound nor complete [7]. Note that the extension of this algorithm that approximates ABox updates in fragments of *DL-Lite* [10] inherits these technical issues. Actually, such an ABox update algorithm cannot exist since it was shown that *DL-Lite* is not closed under $\mathbf{L}^a_\subseteq$-evolution [17].

The remaining part of the section is organized as follows. In Section 5.1, we explain *why DL-Lite*$_{core}$ is not closed under $\mathbf{L}^a_\subseteq$-evolution and show which combination of *DL-Lite*$_{core}$ formulas is responsible for inexpressibility. In Section 5.2, we introduce so-called *prototypes*, which give a characterization of $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_\subseteq$ and are further used to approximate $\mathbf{L}^a_\subseteq$-evolution. In Section 5.3, we present the procedure BP, which constructs prototypes for *DL-Lite*$_{core}$-evolution settings, and in Section 5.4 we show correctness of BP.

5.1. Understanding inexpressibility of $\mathbf{L}^a_\subseteq$-evolution in *DL-Lite*$_{core}$

Using the following example, we illustrate why $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_\subseteq$ is not expressible in *DL-Lite*$_{core}$.

Example 5.1. Consider a *DL-Lite*$_{core}$ KB $\mathcal{K}_2 = \mathcal{T}_2 \cup \mathcal{A}_2$, new information $\mathcal{N}_2$, and $\mathcal{I} \models \mathcal{K}_2$:

$$\mathcal{T}_2 = \{\mathit{Wife} \sqsubseteq \exists \mathit{HasHusb}, \exists \mathit{HasHusb}^- \sqsubseteq \neg \mathit{Priest}\};$$
$$\mathcal{A}_2 = \{\mathit{Priest}(\mathit{pedro}), \mathit{Priest}(\mathit{ivan}), \exists \mathit{HasHusb}^-(\mathit{john})\};$$
$$\mathcal{N}_2 = \{\mathit{Priest}(\mathit{john})\};$$
$$\mathcal{I}:\quad \mathit{Wife}^{\mathcal{I}} = \{\mathit{girl}\},\qquad \mathit{Priest}^{\mathcal{I}} = \{\mathit{pedro}, \mathit{ivan}\},\qquad \mathit{HasHusb}^{\mathcal{I}} = \{(\mathit{girl}, \mathit{john})\},$$

where $\mathit{girl} \in \Delta \setminus \mathsf{adom}(\mathcal{K}_2)$ is an element of the domain. The following models belong to $\mathsf{loc\_min}^a_\subseteq(\mathcal{I}, \mathcal{T}_2, \mathcal{N}_2)$ and consequently to $\mathcal{K}_2 \diamond_S \mathcal{N}_2$ with $S = \mathbf{L}^a_\subseteq$:

$$\mathcal{J}_0:\quad \mathit{Wife}^{\mathcal{J}_0} = \emptyset,\qquad \mathit{Priest}^{\mathcal{J}_0} = \{\mathit{john}, \mathit{pedro}, \mathit{ivan}\},\qquad \mathit{HasHusb}^{\mathcal{J}_0} = \emptyset,$$
$$\mathcal{J}_1:\quad \mathit{Wife}^{\mathcal{J}_1} = \{\mathit{girl}\},\qquad \mathit{Priest}^{\mathcal{J}_1} = \{\mathit{john}, \mathit{ivan}\},\qquad \mathit{HasHusb}^{\mathcal{J}_1} = \{(\mathit{girl}, \mathit{pedro})\},$$
$$\mathcal{J}_2:\quad \mathit{Wife}^{\mathcal{J}_2} = \{\mathit{girl}\},\qquad \mathit{Priest}^{\mathcal{J}_2} = \{\mathit{john}, \mathit{pedro}\},\qquad \mathit{HasHusb}^{\mathcal{J}_2} = \{(\mathit{girl}, \mathit{ivan})\},$$
$$\mathcal{J}_3:\quad \mathit{Wife}^{\mathcal{J}_3} = \{\mathit{girl}\},\qquad \mathit{Priest}^{\mathcal{J}_3} = \{\mathit{john}, \mathit{pedro}, \mathit{ivan}\},\qquad \mathit{HasHusb}^{\mathcal{J}_3} = \{(\mathit{girl}, \mathit{guy})\},$$

where $\mathit{guy} \in \Delta \setminus \mathsf{adom}(\mathcal{K}_2) \setminus \{\mathit{girl}\}$ is an element of the domain.

Indeed, all $\mathcal{J}_i$'s satisfy $\mathcal{N}_2$ and $\mathcal{T}_2$. To see that they are in $\mathsf{loc\_min}^a_\subseteq(\mathcal{I}, \mathcal{T}_2, \mathcal{N}_2)$, observe that every model $\mathcal{J} \in \mathsf{loc\_min}^a_\subseteq(\mathcal{I}, \mathcal{T}_2, \mathcal{N}_2)$ can be obtained from $\mathcal{I}$ by making modifications that guarantee that $\mathcal{J} \models \mathcal{N}_2 \cup \mathcal{K}_2$ and that the distance between $\mathcal{I}$ and $\mathcal{J}$ is minimal. What are these modifications? Clearly, $\mathit{Priest}(\mathit{john})$ should hold in $\mathcal{J}$. Moreover, no priest can be in the *HasHusb* relation since $\mathit{Priest} \sqsubseteq \neg \exists \mathit{HasHusb}^- \in \mathcal{T}_2$. Hence, *john* cannot be in the *HasHusb* relation with *girl* after evolution of $\mathcal{K}_2$, and the first necessary modification in $\mathcal{I}$ is to drop the atom $\mathit{HasHusb}(\mathit{girl}, \mathit{john})$ and to add the atom $\mathit{Priest}(\mathit{john})$:

$$\mathcal{J}' = (\mathcal{I} \setminus \{\mathit{HasHusb}(\mathit{girl}, \mathit{john})\}) \cup \{\mathit{Priest}(\mathit{john})\}.$$

Observe that this modification is not enough, i.e., $\mathcal{J}' \notin \mathsf{loc\_min}^a_\subseteq(\mathcal{I}, \mathcal{T}_2, \mathcal{N}_2)$, since $\mathcal{J}'$ does not satisfy the TBox, namely, the assertion $\mathit{Wife} \sqsubseteq \exists \mathit{HasHusb}$. Indeed, *girl* is still a wife in $\mathcal{J}'$, while there is no husband for her, that is, no atom of the form $\mathit{HasHusb}(\mathit{girl}, x)$ for any x is in $\mathcal{J}'$. This problem can be solved by either dropping $\mathit{Wife}(\mathit{girl})$ from $\mathcal{J}'$ or by assigning to *girl* a husband, that is, adding $\mathit{HasHusb}(\mathit{girl}, x)$ to $\mathcal{J}'$ for some x. The model $\mathcal{J}_0$ corresponds to the former option, that is

$$\mathcal{J}_0 = \mathcal{J}' \setminus \{\mathit{Wife}(\mathit{girl})\}, \tag{12}$$

and the other three $\mathcal{J}_i$'s correspond to the latter one.

Regarding the other option, who should be the husband x of *girl* in $\mathcal{J}$? There are two possibilities in general: the husband is either one of the two priests (i.e., $x = pedro$ or $x = ivan$), or some other person (e.g., $x = guy$). Clearly, if a priest, say *pedro*, is a husband of *girl* in $\mathcal{J}$, then he should quit the priesthood due to the TBox assertion $Priest \sqsubseteq \neg HasHusb^-$, i.e., $Priest(pedro)$ should not be in $\mathcal{J}$. Thus, further modifications corresponding to these possibilities give exactly the models $\mathcal{J}_1$, $\mathcal{J}_2$, and $\mathcal{J}_3$ defined above, i.e.:

$$\mathcal{J}_1 = \left(\mathcal{J}_0 \setminus \left\{Priest(pedro)\right\}\right) \cup \left(\left\{HasHusb(girl, pedro)\right\} \cup \left\{Wife(girl)\right\}\right), \tag{13}$$

$$\mathcal{J}_2 = \left(\mathcal{J}_0 \setminus \left\{Priest(ivan)\right\}\right) \cup \left(\left\{HasHusb(girl, ivan)\right\} \cup \left\{Wife(girl)\right\}\right), \tag{14}$$

$$\mathcal{J}_3 = (\mathcal{J}_0 \setminus \emptyset) \cup \left(\left\{HasHusb(girl, guy)\right\} \cup \left\{Wife(girl)\right\}\right).$$

Note that we wrote the three formulas above in a specific way: first we subtract atoms about *Priest* from $\mathcal{J}_0$ (whenever it is needed), and then we add *HasHusb* and *Wife*-atoms that are required to comply with the TBox $\mathcal{T}_2$. This is done in order to be coherent with the BP procedure, which we present later in this section (see Section 5.3). □

Lack of canonical models. Recall that for every *DL-Lite*$_{core}$ KB $\mathcal{K}$, the set $\mathsf{Mod}(\mathcal{K})$ has a canonical model. At the same time, continuing with Example 5.1, one can verify that any model $\mathcal{J}_{can}$ that can be homomorphically embedded into the four $\mathcal{J}_i$s is such that $Wife^{\mathcal{J}_{can}} = HasHusb^{\mathcal{J}_{can}} = \emptyset$, and $pedro \notin Priest^{\mathcal{J}_{can}}$ and $ivan \notin Priest^{\mathcal{J}_{can}}$. It is easy to check that any such $\mathcal{J}_{can}$ is *not* in $\mathcal{K}_1 \diamond_S \mathcal{N}_1$ with $S = \mathbf{L}^a_\subseteq$. Thus, there is no canonical model in $\mathcal{K}_2 \diamond_S \mathcal{N}_2$ with $S = \mathbf{L}^a_\subseteq$ and, hence, this set is not expressible in *DL-Lite*$_{core}$. This gives us the first reason why *DL-Lite*$_{core}$ is not closed under $\mathbf{L}^a_\subseteq$-evolution.

Local functionality. Another problem with models $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_\subseteq$ is that they satisfy a special kind of functionality constraints on roles.

Definition 5.2 *(Local functionality).* Let R be a role and c a constant; then we call *local functionality of R on c* the formula

$$\mathsf{func}(R, c) = \forall x \forall y.\left(R(x, c) \wedge R(x, y) \to y = c\right). \qquad \square$$

Example 5.3. Continuing with Example 5.1, one can see that $\mathcal{K}_2 \diamond \mathcal{N}_2$ satisfies local functionality of *HasHusb* on both priests *pedro* and *ivan*, for example:

$$\mathsf{func}(HasHusb, pedro) = \forall x \forall y.\left(HasHusb(x, pedro) \wedge HasHusb(x, y) \to (y = pedro)\right).$$

That is, if in $\mathcal{J} \in \mathcal{K}_2 \diamond \mathcal{N}_2$ either *pedro* or *ivan* is a husband of *girl*, then she cannot be married to anyone else. For example, the following model $\mathcal{J}'$, which violates the local functionality, is *not* in $\mathcal{K}_2 \diamond \mathcal{N}_2$ since it is not minimally $\mathbf{L}^a_\subseteq$-distant from $\mathcal{I}$ (or any other model of $\mathcal{K}_2$):

$$Wife^{\mathcal{J}'} = \{girl\}, \qquad Priest^{\mathcal{J}'} = \{john, ivan\}, \qquad HasHusb^{\mathcal{J}'} = \left\{(girl, pedro), (girl, guy)\right\}.$$

To see this, one can check that it holds that $\mathcal{I} \ominus \mathcal{J}_1 \subset \mathcal{I} \ominus \mathcal{J}'$ for every model $\mathcal{I} \in \mathsf{Mod}(\mathcal{K})$.

At the same time, if *girl* has a husband in $\mathcal{J}$ who is neither *pedro* nor *ivan*, then she can have several husbands with the same property. For example, the following model $\mathcal{J}''$ is in $\mathcal{K}_2 \diamond \mathcal{N}_2$:

$$Wife^{\mathcal{J}''} = \{girl\}, \qquad Priest^{\mathcal{J}''} = \{john, pedro, ivan\}, \qquad HasHusb^{\mathcal{J}''} = \left\{(girl, guy_1), (girl, guy_2)\right\}. \qquad \square$$

The following proposition shows that local functionality is not expressible in *DL-Lite*$_{core}$.

Proposition 5.4. *Let $\mathcal{K}$ be a satisfiable DL-Lite$_{core}$ KB, R a role, and c a constant. Then, $\mathcal{K} \models \mathsf{func}(R, c)$ iff $\mathcal{K} \models \neg\exists R^-(c)$.*

As a corollary of the proposition above, the set $\mathcal{K}_2 \diamond_S \mathcal{N}_2$ from Example 5.1 is not axiomatizable in *DL-Lite*$_{core}$. Indeed, $\mathcal{K}_2 \diamond_S \mathcal{N}_2$ satisfies local functionality $\mathsf{func}(HasHusb, pedro)$, but, due to $\mathcal{J}_1 \not\models \neg\exists HasHusb^-(pedro)$ and $\mathcal{J}_1 \in \mathcal{K}_2 \diamond_S \mathcal{N}_2$, it holds that $\mathcal{K}_2 \diamond_S \mathcal{N}_2 \not\models \neg\exists HasHusb^-(pedro)$. This gives us the second argument why *DL-Lite*$_{core}$ is not closed under $\mathbf{L}^a_\subseteq$-evolution.

Dually-affected roles. Both lack of canonical models and local functionality for $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_\subseteq$ observed above are due to *dual-affection* and *triggering* defined as follows.

Definition 5.5 *(Dual-affection and triggering).* Let $\mathcal{E} = (\mathcal{K}, \mathcal{N})$, where $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, be a *DL-Lite*$_{core}$-evolution setting. Then a role R is *dually-affected* in $\mathcal{T}$ if there are atomic concepts A_1 and A_2 such that $\mathcal{T} \models A_1 \sqsubseteq \exists R$ and $\mathcal{T} \models \exists R^- \sqsubseteq \neg A_2$. A dually-affected role R is *triggered* in $\mathcal{E}$ if $\mathcal{A} \not\models_{\mathcal{T}} \neg\exists R^-(b)$ and $\mathcal{N} \models_{\mathcal{T}} \neg\exists R^-(b)$, for some constant $b \in \mathsf{adom}(\mathcal{E})$. □

Example 5.6. In Example 5.1 the role *HasHusb* is dually-affected in $\mathcal{T}_2$. Indeed, $\mathcal{T}_2 \models Wife \sqsubseteq HasHusb$ and $\mathcal{T}_2 \models \exists HasHusb^- \sqsubseteq \neg Priest$. This role is also triggered in $(\mathcal{K}_2, \mathcal{N}_2)$ since $\mathcal{A}_2 \not\models_{\mathcal{T}_2} \exists HasHusb^-(john)$, and $\mathcal{N}_2 \models_{\mathcal{T}_2} \neg\exists HasHusb^-(john)$. □

The following theorem shows that if there is a dually-affected role, we can always find $\mathcal{A}$ and $\mathcal{N}$ to trigger it and thus, to guarantee that $(\mathcal{T} \cup \mathcal{A}) \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_{\subseteq}$ is inexpressible in *DL-Lite*$_{core}$.

Theorem 5.7. *Let $\mathcal{T}$ be a DL-Lite$_{core}$ TBox and R a role dually-affected in $\mathcal{T}$. Then there are ABoxes $\mathcal{A}$ and $\mathcal{N}$ such that $(\mathcal{T} \cup \mathcal{A}, \mathcal{N})$ is a DL-Lite$_{core}$-evolution setting and $(\mathcal{T} \cup \mathcal{A}) \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_{\subseteq}$ is inexpressible in DL-Lite$_{core}$.*

Proof. By definition, there are concepts A and C such that $\mathcal{T} \models A \sqsubseteq \exists R$ and $\mathcal{T} \models \exists R^- \sqsubseteq \neg C$. Now it is enough to take $\mathcal{A}$ and $\mathcal{N}$ analogous to $\mathcal{A}_2$ and $\mathcal{N}_2$ from Example 5.1, respectively. Then $(\mathcal{T} \cup \mathcal{A}) \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_{\subseteq}$ is non-axiomatizable in *DL-Lite*$_{core}$ since it has no canonical model or since it violates Proposition 5.4. $\square$

5.2. Prototypes

As we discussed above, the set of models $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_{\subseteq}$ may not have a canonical model. A closer look at this $\mathcal{K} \diamond_S \mathcal{N}$ gives a surprising result: this set can be divided (but in general not partitioned) into a finite number of subsets $X_0, \ldots, X_n$, that is, $X_i \subseteq \mathcal{K} \diamond_S \mathcal{N}$ for $i \in \{0, \ldots, n\}$ and $\bigcup_{i=0}^{n} X_i = \mathcal{K} \diamond_S \mathcal{N}$, where each X_i includes a canonical model for X_i itself.

Definition 5.8 *(Prototypal set).* Let $\mathcal{M}$ be a set of models. A *prototypal set for* $\mathcal{M}$ is a finite subset $\mathfrak{J} = \{\mathcal{J}_0, \ldots, \mathcal{J}_n\}$ of $\mathcal{M}$ satisfying the following property: for every model $\mathcal{J} \in \mathcal{M}$ there exists $\mathcal{J}_i \in \mathfrak{J}$ such that $\mathcal{J}_i \hookrightarrow \mathcal{J}$. We call each $\mathcal{J}_i$ in $\mathfrak{J}$ a *prototype for* $\mathcal{M}$. $\square$

The notion of prototypes generalizes the notion of canonical model: for example, if $\mathcal{K}$ is a *DL-Lite*$_{core}$ KB, then $\{\mathcal{I}_{\mathsf{can}}\}$ is a prototypal set for $\mathsf{Mod}(\mathcal{K})$. Clearly, since a prototypal set is required to be finite, not every set of models has a prototypal set.

In [29] the notion of *F-universal model set* and in [30] the notion of *universal basis* (a special case of F-universal model set), which are similar to the notion of prototypal set, have been considered. Intuitively, an F-universal model set is defined as follows: given a set $\mathcal{X}$ of models and a set F of mappings between models, a finite subset $\mathcal{U} \subseteq \mathcal{X}$ of finite models is an F-universal model set if for every $X \in \mathcal{X}$ there is $U \in \mathcal{U}$ and a mapping $f \in F$ s.t. f maps U to X in the "right" way; moreover, $\mathcal{U}$ should be minimal (w.r.t. set inclusion and F-embedding) among all finite subsets of finite models of $\mathcal{X}$ that satisfy this condition. If one considers F to be the class of all homomorphisms, then prototypal sets look similar to F-universal sets of models. Crucial differences between the two are that (i) each element of $\mathcal{U}$ is finite, but elements of $\mathfrak{J}$ are in general infinite, (ii) $\mathcal{U}$ is required to be finite and minimal (w.r.t. set inclusion and F-embedding), but $\mathfrak{J}$ is only required to be finite, and (iii) in [29] F-universal model sets are applied to $\mathcal{X}$, which consists of models defined by constraints, and in that settings $\mathcal{U}$ does not always exist; we will apply prototypal sets to $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_{\subseteq}$, where $(\mathcal{K}, \mathcal{N})$ is a *DL-Lite*$_{core}$-evolution setting, and in our setting prototypal sets always exist. Furthermore, the BP procedure for constructing prototypal sets, which we present later in this section, has different properties from the *extended core chase* algorithm for computing F-universal model sets: (i) BP is an abstract procedure and not an algorithm (it manipulates infinite objects), (ii) BP and extended core chase are completely different in the approach to computation, and (iii) BP is sound and complete (see Theorem 5.21), while the extended core chase is only complete. Due to these differences, the applicability of results from [29] to our setting and vise-versa is unclear and requires further investigation.

Definition 5.9 *(Prototypal set for evolution settings).* Let S be a model-based semantics, $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ an evolution setting, and $\mathfrak{J}$ a prototypal set for $\mathcal{K} \diamond_S \mathcal{N}$. Then $\mathfrak{J}$ is called an *S-prototypal set for $\mathcal{E}$*. $\square$

Since we will study only $\mathbf{L}^a_{\subseteq}$-prototypal sets, in the following we will refer to them as *prototypal sets for* $(\mathcal{K}, \mathcal{N})$ and omit the $\mathbf{L}^a_{\subseteq}$ prefix.

Example 5.10. Continuing with Example 5.1, one can check that the sets $X = \{\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_4\}$ and $Y = \{\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_4\}$ are prototypal for $(\mathcal{K}_2, \mathcal{N}_2)$, where $\mathcal{J}_0, \ldots, \mathcal{J}_3$ are as in Example 5.1, and $\mathcal{J}_4$ is:

$$\mathcal{J}_4: \quad \mathit{Wife}^{\mathcal{J}_4} = \{\mathit{girl}_1, \mathit{girl}_2\}, \qquad \mathit{Priest}^{\mathcal{J}_4} = \{\mathit{john}\}, \qquad \mathit{HasHusb}^{\mathcal{J}_4} = \big\{(\mathit{girl}_1, \mathit{pedro}), (\mathit{girl}_2, \mathit{ivan})\big\}.$$

Note that $X = Y \setminus \{\mathcal{J}_3\}$, i.e., $\mathcal{J}_3$ is not needed in the prototypal set X. This holds due to the fact that $\mathcal{J}_0 \subsetneq \mathcal{J}_3$ and $\mathcal{J}_0$ is homomorphically embeddable in $\mathcal{J}_3$. At the same time, if we drop any model from X, then the resulting set of models is not a prototypal for $(\mathcal{K}_2, \mathcal{N}_2)$ anymore.

Observe that the prototypes $\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2$ were obtained by manipulations of the model $\mathcal{I}$ from Example 5.1, while $\mathcal{J}_4$ can be obtained from the following interpretation $\mathcal{I}' \models \mathcal{K}_2$:

$$\mathcal{I}': \quad \mathit{Wife}^{\mathcal{I}'} = \{\mathit{girl}_1, \mathit{girl}_2\}, \qquad \mathit{Priest}^{\mathcal{I}'} = \{\mathit{pedro}, \mathit{ivan}\}, \qquad \mathit{HasHusb}^{\mathcal{I}'} = \big\{(\mathit{girl}_1, \mathit{john}), (\mathit{girl}_2, \mathit{john})\big\}. \qquad \square$$

We will show later in this section that for *every DL-Lite$_{core}$*-evolution setting $\mathcal{E}$ there is a prototypal set of size exponential in $|\mathcal{E}|$. To this effect we will present a procedure BP (where BP stands for *Build Prototypes*) that, given $\mathcal{E}$, constructs such a prototypal set. For the ease of exposition of BP, we consider a restricted form of evolution settings.

Definition 5.11 *(Simple evolution setting).* A *DL-Lite$_{core}$*-evolution setting $(\mathcal{K}, \mathcal{N})$, where $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, is *simple*, if

(i) for every role R there is *no* atomic concept A such that $\mathcal{T} \models \exists R \sqsubseteq A$,
(ii) for every two different roles R and R', neither $\mathcal{T} \models \exists R \sqsubseteq \exists R'$ nor $\mathcal{T} \models \exists R \sqsubseteq \neg\exists R'$ holds,
(iii) if $\mathcal{N} \models_{\mathcal{T}} \exists R(a)$, then $\mathcal{N} \models_{\mathcal{T}} R(a, b)$ for some constant b, and
(iv) if for an atomic concept A there are B and R such that $B \sqsubseteq \exists R$ and $\exists R^- \sqsubseteq \neg A$ are in $\mathsf{cl}(\mathcal{T})$, then for every role R' it holds that $A \sqsubseteq \exists R' \notin \mathsf{cl}(\mathcal{T})$. $\square$

These four restrictions guarantee that evolution under $\mathbf{L}^a_{\subseteq}$ semantics affects roles independently one from another. Later on we explain how the following techniques can be extended to the case of general *DL-Lite$_{core}$*-evolution settings.

5.3. *Procedure* BP *to build prototypal sets for evolution settings*

We now introduce the procedure $\mathsf{BP}(\mathcal{E})$ that takes a *DL-Lite$_{core}$*-evolution setting $\mathcal{E}$ as input and returns the prototypal set for $(\mathcal{K}, \mathcal{N})$.

The components of the BP *procedure.* Before introducing BP, we will introduce several notions and notations that the procedure is based upon. We start with the notion of alignment of a model, in which the facts that contradict a given ABox are removed from the model.

Definition 5.12 *($\mathcal{T}$-alignment of a model with an ABox).* Let $\mathcal{T}$ be a *DL-Lite$_{core}$* TBox, $\mathcal{N}$ an ABox with only positive assertions and satisfiable with $\mathcal{T}$, and $\mathcal{I}$ a model of $\mathcal{T}$. Then the *$\mathcal{T}$-alignment of $\mathcal{I}$ with $\mathcal{N}$*, denoted $\mathsf{Align}_{\mathcal{T}}(\mathcal{I}, \mathcal{N})$, is defined as follows:

$$\mathsf{Align}_{\mathcal{T}}(\mathcal{I}, \mathcal{N}) = \mathcal{I} \setminus \bigcup_{g \in \mathcal{I} \text{ s.t. } \{g\} \cup \mathcal{N} \models_{\mathcal{T}} \bot} \mathsf{root}_{\mathcal{T}}(g). \qquad \square$$

In Example 5.1, the only atom g of $\mathcal{I}$ such that $\{g\} \cup \mathcal{N}_2 \models_{\mathcal{T}} \bot$ is $g = \mathit{HasHusb}(\mathit{girl}, \mathit{john})$; then, $\mathsf{root}_{\mathcal{T}_2}(g) = \{\mathit{HasHusb}(\mathit{girl}, \mathit{john}), \mathit{Wife}(\mathit{girl})\}$; thus, $\mathsf{Align}_{\mathcal{T}_2}(\mathcal{I}, \mathcal{N}_2) = \{\mathit{Priest}(\mathit{pedro}), \mathit{Priest}(\mathit{ivan})\}$.

The next definition introduces the set of triggered roles, building on the notion of dually-affected roles (cf. Definition 5.5).

Definition 5.13 *(Set of triggered roles).* Let $(\mathcal{K}, \mathcal{N})$ be a simple *DL-Lite$_{core}$*-evolution setting. Then, $\mathsf{TR}[\mathcal{K}, \mathcal{N}]$ (where TR stands for *triggered roles*), or simply TR when $\mathcal{K}$ and $\mathcal{N}$ are clear, is the set of all roles dually-affected in $\mathcal{K}$ and triggered in $(\mathcal{K}, \mathcal{N})$. $\square$

In Example 5.1, $\mathsf{TR}[\mathcal{K}_2, \mathcal{N}_2] = \{\mathit{HasHusb}\}$. The next definition introduces the notion of disjoint atoms.

Definition 5.14 *(Disjoint atoms).* Let R be dually-affected and triggered in a simple *DL-Lite$_{core}$*-evolution setting $(\mathcal{K}, \mathcal{N})$. Then, the set of unary atoms $\mathsf{DjnAts}[\mathcal{K}, \mathcal{N}](R) \subseteq \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$ (where DjnAts stands for *Disjoint Atoms*) contains $D(c)$ if $\mathcal{T}$ entails that the range of R is disjoint with D, while $\mathcal{N}$ "says" nothing about $D(c)$. Formally:

$$\begin{aligned}\mathsf{DjnAts}[\mathcal{K}, \mathcal{N}](R) = \bigl\{ D(c) \in \mathsf{cl}_{\mathcal{T}}(\mathcal{A}) \bigm| & R \in \mathsf{TR}[\mathcal{K}, \mathcal{N}],\ \bigl\{\exists R^-(c), D(c)\bigr\} \models_{\mathcal{T}} \bot, \\ & \mathcal{N} \not\models_{\mathcal{T}} D(c), \text{ and } \mathcal{N} \not\models_{\mathcal{T}} \neg D(c) \bigr\}. \qquad \square\end{aligned}$$

In Example 5.1, $\mathsf{DjnAts}[\mathcal{K}_2, \mathcal{N}_2](\mathit{HasHusb}) = \{\mathit{Priest}(\mathit{pedro}), \mathit{Priest}(\mathit{ivan})\}$. The set of disjoint atoms for the entire KB $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ and $\mathcal{N}$, denoted $\mathsf{DjnAts}(\mathcal{K}, \mathcal{N})$, or DjnAts when the parameters are clear, is $\bigcup_{R \in \mathsf{TR}} \mathsf{DjnAts}[\mathcal{K}, \mathcal{N}](R)$. The next definition introduces immediate subconcepts.

Definition 5.15 *(Immediate sub-concept).* For a role R, the set $\mathtt{ISubCon}[\mathcal{T}](\exists R)$ (where $\mathtt{ISubCon}$ stands for *Immediate Sub-Concepts*) is the set of atomic concepts that are subsumed by $\exists R$ and are "immediately" under $\exists R$ in the concept hierarchy generated by $\mathcal{T}$. Formally:

$$\mathtt{ISubCon}[\mathcal{T}](\exists R) = \bigl\{ A \bigm| \mathcal{T} \models A \sqsubseteq \exists R \text{ and there is no } A' \text{ s.t. } \mathcal{T} \models A \sqsubseteq A',\ \mathcal{T} \not\models A' \sqsubseteq A, \text{ and } \mathcal{T} \models A' \sqsubseteq \exists R \bigr\}. \qquad \square$$

In Example 5.1, $\mathtt{ISubCon}[\mathcal{T}_2](\exists \mathit{HasHusb}) = \{\mathit{Wife}\}$.

BZP($\mathcal{E}$)

1. $\mathcal{J}_0 := \mathsf{Align}_{\mathcal{T}}(\mathcal{I}_{\mathsf{can}}, \mathcal{N})$, where $\mathcal{I}_{\mathsf{can}}$ is the canonical model of $\mathcal{K}$;
2. For each $R \in \mathsf{TR}[\mathcal{K}, \mathcal{N}]$ do
   for each $A \in \mathtt{ISubCon}[\mathcal{K}](\exists R)$ do
   if $A(x) \in \mathcal{J}_0$ for some $x \in \Delta$, and for every $b \in \mathsf{adom}(\mathcal{K} \cup \mathcal{N})$: $\mathcal{J}_0 \not\models_{\mathcal{T}} R(x, b)$, $\mathcal{N} \not\models_{\mathcal{T}} R(x, b)$
   then $\mathcal{J}_0 := \mathcal{J}_0 \setminus (\mathsf{root}_{\mathcal{T}}(A(x)) \cup \bigcup_{y \in \Delta \setminus \mathsf{adom}(\mathcal{K})} \{R(x, y)\})$;
3. $\mathcal{J}_0 := \mathsf{chase}_{\mathcal{T}}(\mathcal{J}_0 \cup \mathcal{N})$;
4. Return $\mathcal{J}_0$.

Fig. 5.1. BZP($\mathcal{E}$) procedure for building the zero prototype $\mathcal{J}_0$ for a simple *DL-Lite$_{core}$*-evolution setting $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ where $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$.

BP($\mathcal{E}$)

1. $\mathcal{J}_0 := \mathsf{BZP}(\mathcal{E})$;
2. For each $1 \leqslant k \leqslant |\mathsf{DjnAts}[\mathcal{K}, \mathcal{N}]|$ and for each set $\mathcal{D} = \{D_1(c_1), \ldots, D_k(c_k)\} \subseteq \mathsf{DjnAts}[\mathcal{K}, \mathcal{N}]$ do
   for each vector $\mathcal{R} = \langle R_1, \ldots, R_k \rangle$, s.t. $R_j \in \mathsf{TR}$ and $D_j(c_j) \in \mathsf{DjnAts}(R_j)$ for $j \in \{1, \ldots, k\}$ do
   for each vector $\mathcal{B} = \langle A_1, \ldots, A_k \rangle$ s.t. $A_j \in \mathtt{ISubCon}[\mathcal{T}](\exists R_j)$ for $j \in \{1, \ldots, k\}$ do
   $\mathcal{J}[\mathcal{D}, \mathcal{R}, \mathcal{B}] := (\mathcal{J}_0 \setminus \bigcup_{i=1}^{k} \mathsf{root}_{\mathcal{T}}(D_i(c_i))) \cup \bigcup_{i=1}^{k} \mathsf{chase}_{\mathcal{T}}(\{R_i(x_i, c_i), A_i(x_i)\})$,
   where $\{x_1, \ldots, x_k\}$ are pairwise distinct constants from $\Delta \setminus \mathsf{adom}(\mathcal{K})$ and fresh for $\mathfrak{J}$;
   $\mathfrak{J} := \mathfrak{J} \cup \{\mathcal{J}[\mathcal{D}, \mathcal{R}, \mathcal{B}]\}$.
3. Return $\mathfrak{J}$;

Fig. 5.2. BP($\mathcal{E}$) procedure of building the prototypal set $\mathfrak{J}$ for a simple *DL-Lite$_{core}$*-evolution setting $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ with $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$.

We are ready to proceed to the description of the BP procedure. It works similar to the way we described in Eqs. (12)–(14) of Example 5.1, i.e., by first constructing one prototype $\mathcal{J}_0$ by "aligning" $\mathcal{I}_{\mathsf{can}}$ of $\mathcal{K}$ with $\mathcal{N}$ and post-processing the result (in $\mathcal{J}_0$ of Example 5.1, *girl* is not a *Wife* anymore and all the priests of $\mathcal{I}$ remain priests), and then manipulating $\mathcal{J}_0$ in order to get all the other prototypes. We will further refer to such a model $\mathcal{J}_0$ as the *zero prototype*. We start with a procedure BZP for constructing the zero prototype.

Procedure BZP *for building zero prototype.* The procedure BZP($\mathcal{E}$) (where BZP stands for *Build Zero Prototype*) in Fig. 5.1 constructs the zero prototype $\mathcal{J}_0$ for a simple *DL-Lite$_{core}$*-evolution setting $\mathcal{E} = (\mathcal{K}, \mathcal{N})$. It works as follows. First, it deletes from the canonical model $\mathcal{I}_{\mathsf{can}}$ of $\mathcal{K}$ all the atoms that are not $\mathcal{T}$-satisfiable with $\mathcal{N}$ (Step 1). Then, in Step 2 the procedure does the following: from the interpretation $\mathcal{J}_0$ resulting from Step 1 it deletes all atoms of the form $A(a)$ (together with the atoms that $\mathcal{T}$-entail $A(a)$) for which there is no constant b from $\mathsf{adom}(\mathcal{K} \cup \mathcal{N})$ such that $\mathcal{J}_0 \models R(a, b)$ or $\mathcal{N} \models_{\mathcal{T}} R(a, b)$ for some role $R \in \mathsf{TR}[\mathcal{K}, \mathcal{N}]$ s.t. $A \in \mathtt{ISubCon}[\mathcal{K}](\exists R)$. Moreover, it further deletes from $\mathcal{J}_0$ all atoms of the form $R(a, x)$ where $x \in \Delta \setminus \mathsf{adom}(\mathcal{K} \cup \mathcal{N})$. Intuitively, Step 2 works as follows: if neither $\mathcal{J}_0$ nor $\mathcal{N}$ entails an atom of the form $R(a, b)$ (e.g., in Example 5.1, there is no active-domain husband b of a *girl* provided by $\mathcal{J}_0$ or $\mathcal{N}_2$), then the zero prototype should not contain $A(a)$ (e.g., in Example 5.1, *girl* stops to be a wife) and also all atoms $R(a, x)$ for some non-active x. Step 3 combines $\mathcal{J}_0$ resulting from Step 2 with $\mathcal{N}$ and chases them in order to obtain a model of $\mathcal{T} \cup \mathcal{N}$. Finally, Step 4 returns $\mathcal{J}_0$.

We illustrate BZP on the following example.

Example 5.16. In Example 5.1, the zero prototype obtained by $\mathsf{BZP}(\mathcal{K}_2, \mathcal{N}_2)$ is $\mathcal{J}_0$. Indeed, the canonical model of $\mathcal{K}_2$ is $\mathcal{I}_{\mathsf{can}} = \{\mathit{Priest}(\mathit{pedro}), \mathit{Priest}(\mathit{ivan}), \mathit{HasHusb}(x, \mathit{john})\}$. Step 1 of $\mathsf{BZP}(\mathcal{K}_1, \mathcal{N}_1)$ returns the interpretation $\mathcal{I}_{\mathsf{can}} \setminus \{\mathit{HasHusb}(x, \mathit{john})\}$ and Step 2 does nothing. Finally, Step 3 returns $\mathsf{chase}(\{\mathit{Priest}(\mathit{pedro}), \mathit{Priest}(\mathit{ivan})\} \cup \{\mathit{Priest}(\mathit{john})\})$, which coincides with $\mathcal{J}_0$ of Example 5.1.

Consider another example: $\mathcal{A} = \{C(a)\}$, $\mathcal{T} = \{C \sqsubseteq A, A \sqsubseteq \exists R, \exists R^- \sqsubseteq \neg B\}$, and $\mathcal{N} = \{B(b)\}$. Then, $\mathcal{I}_{\mathsf{can}} = \{C(a), A(a), R(a, x)\}$. Step 1 of $\mathsf{BZP}(\mathcal{K}, \mathcal{N})$ returns the model $\mathcal{I}_{\mathsf{can}}$; Step 2 deletes from $\mathcal{I}_{\mathsf{can}}$ the atom $R(a, x)$ and $\mathsf{root}_{\mathcal{T}}(A(a)) = \{C(a), A(a)\}$, that is, it returns $\emptyset$; finally, Step 3 returns $\mathcal{J}_0 = \mathsf{chase}_{\mathcal{T}}\{\emptyset \cup \{B(b)\}\} = \{B(b)\}$. □

Procedure BP *for building prototypes.* The procedure BP($\mathcal{E}$) for constructing $\mathfrak{J}$ (see Fig. 5.2) takes a simple *DL-Lite$_{core}$*-evolution setting $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ with $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ as input, constructs the zero prototype $\mathcal{J}_0$ by calling BZP (at Step 1), and based on $\mathcal{J}_0$ builds the other prototypes of $\mathfrak{J}$ (Step 2). Each element in $\mathfrak{J}$ corresponds to a distinct triple consisting of a set $\mathcal{D}$, a tuple $\mathcal{R}$ (which depends on $\mathcal{D}$), and a tuple $\mathcal{B}$ (which depends on $\mathcal{R}$) that are constructed from $\mathcal{K}$ and $\mathcal{N}$. More precisely, BP first chooses a triple $\mathcal{D}$, $\mathcal{R}$, $\mathcal{B}$ that is composed of

(i) a set $\mathcal{D}$ of disjoint atoms from $\mathsf{DjnAts}[\mathcal{K}, \mathcal{N}]$ (in Example 5.1, $\mathcal{D}$ is any subset of the priests from $\mathcal{A}$, that is, of $\{\mathit{Priest}(\mathit{pedro}), \mathit{Priest}(\mathit{ivan})\}$);
(ii) a tuple $\mathcal{R}$ of roles R, one for each $D(c) \in \mathcal{D}$, such that $D(c)$ is a disjoint atom for R, that is, $D(c) \in \mathsf{DjnAts}(R)$ (in Example 5.1, $\mathcal{R} = \langle \mathit{HasHusb} \rangle$ for every possible $\mathcal{D}$ since $D(c)$ can be either $\mathit{Priest}(\mathit{pedro})$ or $\mathit{Priest}(\mathit{ivan})$ and it holds that $\mathit{Priest}(\mathit{pedro}) \in \mathsf{DjnAts}(\mathit{HasHusb})$ and $\mathit{Priest}(\mathit{ivan}) \in \mathsf{DjnAts}(\mathit{HasHusb})$);
(iii) a tuple $\mathcal{B}$ of immediate subconcepts A of $\exists R$ for each $R \in \mathcal{R}$ (in Example 5.1, $\mathcal{B} = \langle \mathit{Wife} \rangle$ since *Wife* is the only immediate subconcept of *HasHusb*).

Then, BP

(a) deletes from $\mathcal{J}_0$ all the atoms $D(c)$ of $\mathcal{D}$ (that is, it deletes c from $D^{\mathcal{J}_0}$) together with all the atoms that $\mathcal{T}$-entail them. In Eqs. (13) and (14) of Example 5.1, this corresponds to

$$\mathcal{J}_0 \setminus \{Priest(pedro)\} \quad \text{and} \quad \mathcal{J}_0 \setminus \{Priest(ivan)\};$$

(b) adds to what remains from $\mathcal{J}_0$ the chase of pairs of atoms of the form $R(x, c)$, $A(x)$, that is, it connects with R some elements x of $\Delta \setminus \mathsf{adom}(\mathcal{K})$ to the constants c. In Eqs. (13) and (14) of Example 5.1, this respectively corresponds to adding

$$\{HasHusb(girl, ivan)\} \cup \{Wife(girl)\} \quad \text{and} \quad \{HasHusb(girl, guy)\} \cup \{Wife(girl)\}.$$

Note that $\mathsf{BZP}(\mathcal{E}) \subseteq \mathsf{BP}(\mathcal{E})$ and $\mathcal{J}_0$ corresponds to $\mathcal{J}[\emptyset, \varepsilon, \varepsilon]$, where ε is the empty tuple.

5.4. Correctness of the BP *procedure*

Before we proceed to the main result of this section, i.e., to the proof that for a simple *DL-Lite*$_{core}$-evolution settings $\mathcal{E}$, the set $\mathsf{BP}(\mathcal{E})$ is prototypal for $\mathcal{E}$, we present a number of technical lemmas, propositions and observations that will help us to prove this result.

Let $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ be a simple *DL-Lite*$_{core}$-evolution setting where $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ and $\mathcal{S} = \mathbf{L}^a_{\subseteq}$. Let $\mathcal{J}_0$ be the zero prototype for $\mathcal{E}$ and $\mathcal{J}[\mathcal{D}, \mathcal{R}, \mathcal{B}]$ a prototype for $\mathcal{E}$ as defined in the BP procedure. We now exhibit models $\mathcal{I}_0 \models \mathcal{K}$ and $\mathcal{I}[\mathcal{D}, \mathcal{R}, \mathcal{B}] \models \mathcal{K}$ which can be considered as “preimages” of $\mathcal{J}_0$ and $\mathcal{J}[\mathcal{D}, \mathcal{R}, \mathcal{B}]$ under $\mathbf{L}^a_{\subseteq}$, respectively, in the sense that $\mathcal{J}_0 \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}_0, \mathcal{T}, \mathcal{N})$ and $\mathcal{J}[\mathcal{D}, \mathcal{R}, \mathcal{B}] \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}[\mathcal{D}, \mathcal{R}, \mathcal{B}], \mathcal{T}, \mathcal{N})$.

Lemma 5.17. *Let $S = \mathbf{L}^a_{\subseteq}$ and let $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ be a simple DL-Lite$_{core}$-evolution setting with $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$. For the model $\mathcal{J}_0$ returned by $\mathsf{BZP}(\mathcal{E})$ and every other model $\mathcal{J}[\mathcal{D}, \mathcal{R}, \mathcal{B}]$ returned by $\mathsf{BP}(\mathcal{E})$, it holds that $\mathcal{J}_0 \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}_0, \mathcal{T}, \mathcal{N})$ and $\mathcal{J}[\mathcal{D}, \mathcal{R}, \mathcal{B}] \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}[\mathcal{D}, \mathcal{R}, \mathcal{B}], \mathcal{T}, \mathcal{N})$, where $\mathcal{I}_0$ and $\mathcal{I}[\mathcal{D}, \mathcal{R}, \mathcal{B}]$ are models of $\mathcal{K}$ defined as follows:*

$$\mathcal{I}[\mathcal{D}, \mathcal{R}, \mathcal{B}] = \mathcal{I}_{\mathsf{can}} \cup \bigcup_{1 \leqslant i \leqslant |\mathcal{D}|} \mathsf{chase}_{\mathcal{T}}\big(\big\{R_i(x_i, d), A_i(x_i) \bigm| R_i \in \mathcal{R},\ A_i \in \mathcal{B},\ \mathcal{N} \models_{\mathcal{T}} \neg\exists R_i^-(d),\ \mathcal{A} \not\models_{\mathcal{T}} \neg\exists R_i^-(d), \big\}\big) \tag{15}$$

$$\mathcal{I}_0 = \mathsf{chase}_{\mathcal{T}}\Bigg(\mathcal{A} \cup \bigcup_{A(a) \in \mathcal{A}_1} \big\{R_a(a, b_a) \bigm| \textit{for corresponding } R_a \textit{ and } b_a\big\}\Bigg), \tag{16}$$

where the auxiliary ABox $\mathcal{A}_1$ is as follows:

$$\mathcal{A}_1 = \big\{A(a) \in \mathsf{cl}_{\mathcal{T}}(\mathcal{A}) \bigm| \textit{there is } R_a \in \mathsf{TR}[\mathcal{K}, \mathcal{N}],\ \textit{s.t. } A \in \mathtt{ISubCon}[\mathcal{T}](\exists R_a) \textit{ and } \forall x \in \Delta\text{:}$$
$$\mathcal{N} \not\models_{\mathcal{T}} R_a(a, x),\ \mathcal{A} \not\models_{\mathcal{T}} R_a(a, x) \textit{ and there is } b_a \in \Delta\text{: } \mathcal{N} \models_{\mathcal{T}} \neg\exists R^-(b_a)\big\}.$$

Our next observation is that all $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$ share the alignment of $\mathcal{I}$ without disjoint atoms and immediate sub-concepts. In terms of Example 5.1, these $\mathcal{J}$'s share $\mathcal{I}$ without *Priest*(*pedro*), *Priest*(*ivan*), and *Wife*(*girl*).

Lemma 5.18. *Let $S = \mathbf{L}^a_{\subseteq}$ and let $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ be a simple DL-Lite$_{core}$-evolution setting.*

(i) *For every $\mathcal{I} \in \mathsf{Mod}(\mathcal{K})$ and $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$ it holds that*:

$$\mathsf{Align}_{\mathcal{T}}(\mathcal{I} \setminus \mathcal{B}_{\mathcal{I}}, \mathcal{N}) \subseteq \mathcal{J}, \quad \textit{where } \mathcal{B}_{\mathcal{I}} = \bigcup_{D(c) \in \mathcal{S}} \mathsf{root}^{\mathcal{I}}_{\mathcal{T}}\big(D(c)\big) \quad \textit{and}$$
$$\mathcal{S} = \big\{D(c) \in \mathcal{I} \bigm| \mathcal{N} \not\models_{\mathcal{T}} D(c),\ \mathcal{N} \not\models_{\mathcal{T}} \neg D(c),\ \textit{and there is } R \textit{ dually-affected in } \mathcal{K} \textit{ s.t.}$$
$$\big\{\exists R^-(c), D(c)\big\} \models_{\mathcal{T}} \bot,\ \textit{and there are } x, d \in \Delta \textit{ s.t. } \mathcal{I} \models R(x, d),\ \mathcal{N} \models \neg\exists R^-(d)\big\}.$$

(ii) *In particular, if $\mathcal{I}$ is $\mathcal{I}_{\mathsf{can}}$, i.e., a canonical model of $\mathcal{K}$, then for every $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}_{\mathsf{can}}, \mathcal{T}, \mathcal{N})$ it holds that*

$$\mathsf{Align}_{\mathcal{T}}\big(\mathcal{I}_{\mathsf{can}} \setminus \mathsf{DjnAts}[\mathcal{K}, \mathcal{N}], \mathcal{N}\big) \subseteq \mathcal{J}.$$

Note that $\mathcal{B}_{\mathcal{I}}$ in the lemma above can be seen as an extension of the set of disjoint atoms $\mathsf{DjnAts}[\mathcal{K}, \mathcal{N}]$ from KBs to models of this KBs, in the sense that $\mathcal{B}_{\mathcal{I}} \cap \mathsf{cl}_{\mathcal{T}}(\mathcal{A}) = \mathsf{DjnAts}[\mathcal{K}, \mathcal{N}]$. As a consequence of Lemma 5.18, consider the following definition.

Definition 5.19 *(Constant and variable parts of models).* For a given $\mathcal{I} \in \mathsf{Mod}(\mathcal{K})$, every model $\mathcal{J} \in \mathsf{loc\_min}^a_\subseteq(\mathcal{I}, \mathcal{T}, \mathcal{N})$ can be partitioned in two parts:

(i) a *constant* part $\mathcal{J}_c = \mathsf{Align}_\mathcal{T}(\mathcal{I} \setminus \mathcal{B}_\mathcal{I}, \mathcal{N})$, which is the same across all elements of $\mathsf{loc\_min}^a_\subseteq(\mathcal{I}, \mathcal{T}, \mathcal{N})$, and
(ii) a *variable* part $\mathcal{J}_v = \mathcal{J} \setminus \mathcal{J}_c$, which varies from one element of $\mathsf{loc\_min}^a_\subseteq(\mathcal{I}, \mathcal{T}, \mathcal{N})$ to another. □

Note that $\mathcal{J}_c$ is the constant part of the entire set $\mathsf{loc\_min}^a_\subseteq(\mathcal{I}, \mathcal{T}, \mathcal{N})$ in the sense that $\mathcal{J}_c \subseteq \bigcap_{\mathcal{J}' \in \mathsf{loc\_min}^a_\subseteq(\mathcal{I}, \mathcal{T}, \mathcal{N})} \mathcal{J}'$.

Finally, consider the following property of models $\mathcal{I}$ and $\mathcal{J}$ w.r.t. disjoint atoms, where $\mathcal{I}$ and $\mathcal{J}$ are related as follows: $\mathcal{J} \in \mathsf{loc\_min}^a_\subseteq(\mathcal{I}, \mathcal{T}, \mathcal{N})$.

Lemma 5.20. *Let $(\mathcal{K}, \mathcal{N})$, where $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, be a simple DL-Lite$_{core}$-evolution setting, $\mathcal{I} \models \mathcal{K}$, and $\mathcal{J} \in \mathsf{loc\_min}^a_\subseteq(\mathcal{I}, \mathcal{T}, \mathcal{N})$. If $D(c) \in \mathsf{DjnAts}[\mathcal{K}, \mathcal{N}]$, then the following holds*:

(i) *If $D(c) \notin \mathcal{J}$, then there exist $R \in \mathsf{TR}$ and $A \in \mathtt{ISubCon}[\mathcal{T}](\exists R)$ s.t. $D(c) \in \mathsf{DjnAts}[\mathcal{K}, \mathcal{N}](R)$ and*

$$\{R(x, c), A(x)\} \subseteq \mathcal{J} \quad \textit{for some } x \in \Delta. \tag{17}$$

(ii) *If $D(c) \in \mathcal{J}$, then for every unary MA, an atom $A(c)$ satisfying $\mathcal{K} \models A(c)$, where $\mathcal{T} \models A \sqsubseteq D$ and $A(c) \in \mathsf{Align}_\mathcal{T}(\mathcal{I}, \mathcal{N})$, the inclusion $A(c) \in \mathcal{J}$ holds.*[8]

The following result establishes the fundamental property of the set of models computed by BP: this set is prototypal for the input evolution setting and exponential in the size of the setting.

Theorem 5.21. *Let $\mathcal{E}$ be a simple DL-Lite$_{core}$-evolution setting. Then, the set $\mathsf{BP}(\mathcal{E})$ is a prototypal set for $\mathcal{E}$. Moreover, $|\mathsf{BP}(\mathcal{E})|$ is exponential in $|\mathcal{E}|$.*

Proof. Let $S = \mathbf{L}^a_\subseteq$. To see the bound on $|\mathsf{BP}(\mathcal{E})|$ observe that the number of prototypes is polynomial in the number of triples $\mathcal{D}$, $\mathcal{R}[\mathcal{D}]$ and $\mathcal{B}[\mathcal{R}]$, where the number of different components in each triple is exponential in $|\mathcal{E}|$.

By the definition of prototypal sets, in order to prove that $\mathsf{BP}(\mathcal{E}) = \mathfrak{J}$ is a prototypal set, we need to show the following two conditions:

1. $\mathsf{BP}(\mathcal{E}) \subseteq \mathcal{K} \diamond_S \mathcal{N}$ and
2. for every model $\mathcal{J} \in \mathcal{K} \diamond_S \mathcal{N}$ there exists a model $\mathcal{J}' \in \mathsf{BP}(\mathcal{E})$ such that

$$\mathcal{J}' \hookrightarrow \mathcal{J}. \tag{18}$$

Condition 1 follows from Lemma 5.17. Indeed, for every element $\mathcal{J}$ of $\mathsf{BP}(\mathcal{E})$ the lemma presents a model $\mathcal{I}$ of $\mathcal{K}$ that "evolves" in $\mathcal{J}$, i.e., for $\mathcal{J}_0$ and $\mathcal{J}[\mathcal{D}, \mathcal{R}, \mathcal{B}]$ it presents $\mathcal{I}_0$ and $\mathcal{I}[\mathcal{D}, \mathcal{R}, \mathcal{B}]$ such that $\mathcal{J}_0 \in \mathsf{loc\_min}^a_\subseteq(\mathcal{I}_0, \mathcal{T}, \mathcal{N})$ and $\mathcal{J}[\mathcal{D}, \mathcal{R}, \mathcal{B}] \subseteq \mathsf{loc\_min}^a_\subseteq(\mathcal{I}, \mathcal{T}, \mathcal{N})$. Therefore $\mathsf{BP}(\mathcal{E}) \subseteq \mathcal{K} \diamond_S \mathcal{N}$.

Now we prove Condition 2. Let $\mathcal{J} \in \mathcal{K} \diamond_S \mathcal{N}$, we will exhibit $\mathcal{J}' \in \mathsf{BP}(\mathcal{E})$ such that $\mathcal{J}' \hookrightarrow \mathcal{J}$. To this effect, consider $\mathcal{D}_\mathcal{J}$, the set of all (redundant) atoms from $\mathsf{DjnAts}[\mathcal{K}, \mathcal{N}]$ that are *not* in $\mathcal{J}$, that is, $\mathcal{D}_\mathcal{J} \subseteq \mathsf{DjnAts}[\mathcal{K}, \mathcal{N}]$ and

- for every $D(c) \in \mathcal{D}_\mathcal{J}$ we have $D(c) \notin \mathcal{J}$, while
- for every $D(c) \in \mathsf{DjnAts}[\mathcal{K}, \mathcal{N}] \setminus \mathcal{D}_\mathcal{J}$ we have $D(c) \in \mathcal{J}$.

Assume that $\mathcal{D}_\mathcal{J} = \{D_1(c_1), \ldots, D_n(c_n)\}$ for some $n \in \mathbb{N}$ and for every $D_i(c_i)$ from $\mathcal{D}_\mathcal{J}$ let $R_i^{D_i(c_i)}$ be a role name such that $D_i(c_i) \in \mathsf{DjnAts}[\mathcal{K}, \mathcal{N}](R_i^{D_i(c_i)})$. Note that for two different $D_i(c_i)$ and $D_j(c_j)$, the corresponding roles $R_i^{D_i(c_i)}$ and $R_j^{D_j(c_j)}$ may coincide. Moreover, for every $R_i^{D_i(c_i)}$ let $A_i^{R_i}$ be a concept name such that $A_i^{R_i} \in \mathtt{ISubCon}[\mathcal{K}](R_i^{D_i(c_i)})$. Now take

$$\mathcal{R}_\mathcal{J} = \langle R_1^{D_1(c_1)}, \ldots, R_n^{D_n(c_n)} \rangle, \qquad \mathcal{B}_\mathcal{J} = \langle A_1^{R_1}, \ldots, A_n^{R_n} \rangle,$$

and define the model $\mathcal{J}'$ using the BP procedure in Fig. 5.2 for constructing $\mathcal{J}[\mathcal{D}_\mathcal{J}, \mathcal{R}_\mathcal{J}, \mathcal{B}_\mathcal{J}]$ as follows:

$$\mathcal{J}' := \mathcal{J}[\mathcal{D}_\mathcal{J}, \mathcal{R}_\mathcal{J}, \mathcal{B}_\mathcal{J}].$$

We now show that this $\mathcal{J}'$ together with $\mathcal{J}$ from which it is constructed satisfies Eq. (18).

[8] Recall that the evolution setting $(\mathcal{K}, \mathcal{N})$ is simple and therefore there is no role R such that $\exists R \sqsubseteq D$.

Note that

(a) $\mathcal{J} \in \mathcal{K} \diamond_S \mathcal{N}$ implies that there exists $\mathcal{I} \in \mathsf{Mod}(\mathcal{K})$ such that $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$, and
(b) since, by its definition, $\mathcal{J}'$ belongs to $\mathsf{BP}(\mathcal{E})$, Condition 1 implies that $\mathcal{J}' \in \mathcal{K} \diamond_S \mathcal{N}$ and therefore there exists $\mathcal{I}' \in \mathsf{Mod}(\mathcal{K})$, which may differ from $\mathcal{I}$, such that $\mathcal{J}' \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}', \mathcal{T}, \mathcal{N})$.

Recall that due to Lemma 5.18 we can partition $\mathcal{J}$ (resp., $\mathcal{J}'$) in two parts: a constant part $\mathcal{J}_c$ (resp., $\mathcal{J}'_c$) and a variable part $\mathcal{J}_v = \mathcal{J} \setminus \mathcal{J}_c$ (resp., $\mathcal{J}'_v$).

Now, to conclude that $\mathcal{J}' \hookrightarrow \mathcal{J}$ holds, i.e., $(\mathcal{J}'_c \cup \mathcal{J}'_v) \hookrightarrow (\mathcal{J}_c \cup \mathcal{J}_v)$ holds, we show that there are homomorphisms h_c and h_v such that:

$$\text{(a)}\quad h_c : \mathcal{J}'_c \to \mathcal{J}_c \quad \text{and} \quad \text{(b)}\quad h_v : \mathcal{J}'_v \to \mathcal{J}_v,$$

and the combination of h_c and h_v will be a homomorphism from $\mathcal{J}'$ to $\mathcal{J}$.

- We prove Condition (a) using Lemmas 5.17 and 5.18. Indeed, combining Eq. (15) of Lemma 5.17 if $\mathcal{D}_\mathcal{J} \neq \emptyset$ and Eq. (16) of Lemma 5.17 otherwise, with Lemma 5.18, Case (ii), we obtain that the constant part $\mathcal{J}'_c$ of $\mathcal{J}'$ is:

$$\mathcal{J}'_c = \mathsf{Align}_\mathcal{T}\big(\mathcal{I}_{\mathsf{can}} \setminus \mathsf{DjnAts}[\mathcal{K}, \mathcal{N}], \mathcal{N}\big).$$

  Due to Lemma 5.18, Case (i), we have that the constant part of $\mathcal{J}$ is $\mathcal{J}_c = \mathsf{Align}_\mathcal{T}(\mathcal{I} \setminus \mathsf{DjnAts}[\mathcal{K}, \mathcal{N}], \mathcal{N})$. Now recall that by the definition of canonical models, $\mathcal{I}_{\mathsf{can}} \hookrightarrow \mathcal{I}$ holds and this implies that $\mathsf{Align}_\mathcal{T}(\mathcal{I}_{\mathsf{can}} \setminus \mathsf{DjnAts}[\mathcal{K}, \mathcal{N}], \mathcal{N}) \hookrightarrow \mathsf{Align}_\mathcal{T}(\mathcal{I} \setminus \mathsf{DjnAts}[\mathcal{K}, \mathcal{N}], \mathcal{N})$ holds as well. I.e., there is a homomorphism h_c from $\mathcal{J}'_c$ to $\mathcal{J}_c$.
- We prove Condition (b) using Lemma 5.20. Observe that by the definition of the BP procedure the variable part $\mathcal{J}'_v = \mathcal{J}[\mathcal{D}_\mathcal{J}, \mathcal{R}_\mathcal{J}, \mathcal{B}_\mathcal{J}]_v$ of $\mathcal{J}'$ is the following:

$$\mathcal{J}'_v = \bigcup_{D(c) \in \mathcal{D}_\mathcal{J}} \big\{R^{D(c)}(x', c), A^{D(c)}(x') \bigm| x' \text{ is fresh}\big\} \tag{19}$$

$$\cup \bigcup_{D(c) \in \mathsf{DjnAts}[\mathcal{K}, \mathcal{N}] \setminus \mathcal{D}_\mathcal{J}} \big(\big\{D(c)\big\} \cup \big\{A(c) \bigm| \mathcal{K} \models A(c), A(c) \notin \mathcal{D}_\mathcal{J},\ A(c) \in \mathsf{AtAlg}(\mathcal{E}),\ \mathcal{T} \models A \sqsubseteq D\big\}\big), \tag{20}$$

  where "x' is fresh" means that it is in $\Delta \setminus \mathsf{adom}(\mathcal{K})$, distinct for each set $\{R^{D(c)}(x', c), A^{D(c)}(x')\}$ defined by $D(c)$, and does not appear in $\mathcal{J}'_c$. Also observe that if $\mathcal{D} = \emptyset$, then $\mathcal{J}[\mathcal{D}_\mathcal{J}, \mathcal{R}_\mathcal{J}, \mathcal{B}_\mathcal{J}]$ is the zero prototype $\mathcal{J}_0$.
  It remains to show that there is a homomorphism from $\mathcal{J}'_v$ to $\mathcal{J}_v$. Consider a mapping h_v such that it is the identity mapping on every element of $\mathcal{J}'_v$, but x' (see Eq. (19)), and $h_v(x') = x$, where x is from Eq. (17) of Lemma 5.20, for every such x'. To see that h_v is a homomorphism, observe how it works on every atom that occurs in Eqs. (19) and (20):
  (i) on atoms from Eq. (19):

$$R^{D(c)}\big(h_v(x'), h_v(c)\big) = R^{D(c)}(x, c) \quad \text{and} \quad A^{D(c)}\big(h_v(x')\big) = A^{D(c)}(x),$$

  where $R^{D(c)}(x, c) \in \mathcal{J}_v$ and $A^{D(c)}(x) \in \mathcal{J}_v$ hold due to Lemma 5.20, Case (i);
  (ii) on atoms from Eq. (20):

$$D\big(h_v(c)\big) = D(c) \quad \text{and} \quad A\big(h_v(c)\big) = A(c),$$

  where $A(c) \in \mathcal{J}_v$ holds since $A(c)$ satisfies the conditions of Lemma 5.20, Case (ii), and $D(c) \in \mathcal{J}_v$ holds since $D(c) \in \mathsf{DjnAts} \setminus \mathcal{D}_\mathcal{J}$ and consequently $D(c) \in \mathcal{J}$.

Now we consider the following mapping from $\mathcal{J}'$ to $\mathcal{J}$, which we prove to be a homomorphism:

$$h(x) = \begin{cases} h_c(x) & \text{if } x \text{ appear in } \mathcal{J}'_c, \\ h_v(x) & \text{otherwise.} \end{cases}$$

The correctness of this definition of h follows from the following observation. Note that, by the construction of $\mathcal{J}'$ by the BP procedure, the elements appearing in $\mathcal{J}'$ are either constants (i.e., from $\mathsf{adom}(\mathcal{E})$) or "fresh" elements. The fact that $h_c(a) = h_v(a)$ for a being a constant guarantees the correctness of homomorphic embedding of atoms containing constants, no matter whether an atom with this constant is in $\mathcal{J}'_c$ or $\mathcal{J}'_v$; the fact that "fresh" elements appear only in $\mathcal{J}'_v$ again guarantees that this part will be homomorphically embedded into $\mathcal{J}$ by h correctly, since h_v does so. The elements of Δ which are neither constants nor "fresh" will be homomorphically embedded into $\mathcal{J}$ by h as well, since h_c does so. □

As a corollary from the theorem, note that a prototypal set always exists for every simple *DL-Lite*$_{core}$-evolution setting.

Extension of BP *to DL-Lite$_{core}$ KBs without restrictions.* The results of Theorem 5.21 can be extended to the general case when the *DL-Lite$_{core}$*-evolution setting is not simple. Observe that in the general case the BP procedure does return prototypes but not all of them. Weakening the restrictions in Cases (i) and (iii) in the definition of simple evolution settings (i.e., allowing entailments from $\mathcal{K}$ of the form $\exists R \sqsubseteq A$ and $\mathcal{T}$-entailments from $\mathcal{N}$ of the form $\exists R(a)$) results in more than one zero prototype. Weakening the restrictions in Case (ii) (i.e., allowing entailment from $\mathcal{K}$ of direct role interactions of the form $\exists R \sqsubseteq \exists R'$ and $\exists R \sqsubseteq \neg\exists R'$) leads to the need to iterate BP over constructed prototypes. More precisely, to gain the missing prototypes in this case one should run BP several times (finitely many times) iterating over (already constructed) prototypes until no new prototypes can be constructed. Intuitively, the reason is that BP deletes disjoint atoms (atoms of DjnAts) and adds new atoms of the form $R(a,b)$ for some triggered dually-affected role R, which may in turn trigger another dually-affected role, say P, and such triggering may require further modifications, already for P. These further modifications require a new run of BP. For example, if we have $\exists R^- \sqsubseteq \neg\exists P^-$ in the TBox and we set $R(a,b)$ in a prototype, say $\mathcal{J}_k$, this modification triggers role P and we should run BP again with the prototype $\mathcal{J}_k$ as if it were the zero prototype. We will not discuss the general procedures in more detail due to space limitations.

Summary of Section 5. We discussed why $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_\subseteq$ in general cannot be axiomatized in *DL-Lite$_{core}$*, introduced prototypal sets, and a procedure that constructs these sets with exponentially many prototypes for simple *DL-Lite$_{core}$*-evolution settings.

6. Approximating $\mathbf{L}^a_\subseteq$-evolution

Capturing $\mathbf{L}^a_\subseteq$-evolution in richer logics. We start with a discussion on *how* to capture $\mathbf{L}^a_\subseteq$-evolution of *DL-Lite$_{core}$* KBs in logics richer than *DL-Lite$_{core}$*. As we saw in the previous section, for every *DL-Lite$_{core}$*-evolution setting $(\mathcal{K},\mathcal{N})$, the evolution result $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_\subseteq$ is a finite union of sets of models, $\mathcal{K} \diamond_S \mathcal{N} = \bigcup_i \mathcal{M}_i$, where each $\mathcal{M}_i$ contains a prototype $\mathcal{J}_i$. Thus, for $S = \mathbf{L}^a_\subseteq$ axiomatization of $\mathcal{K} \diamond_S \mathcal{N}$ boils down to axiomatization of each $\mathcal{M}_i$ with some theory Th_i, that is, $\mathcal{M}_i = \mathsf{Mod}(\mathsf{Th}_i)$, and taking the disjunction across these theories. As shown in [18], each Th_i can be computed based on a prototype of $\mathcal{M}_i$ using a *DL-Lite$_{core}$* KB $\mathcal{K}_i[\mathcal{J}_i]$ (whose canonical model is precisely $\mathcal{J}_i$) and a compensation formula Ψ, which is not expressible in *DL-Lite$_{core}$*, as $\mathsf{Th}_i \equiv \mathcal{K}_i[\mathcal{J}_i] \wedge \Psi$. The compensation formula Ψ cuts off the models which are not in $\mathcal{K} \diamond_S \mathcal{N}$, but cannot be filtered out by a *DL-Lite$_{core}$* KB, e.g., models not satisfying local functionality. It turned out [18] that Ψ is the same for each Th_i, hence

$$\mathcal{K} \diamond_S \mathcal{N} = \mathsf{Mod}\left(\Psi \wedge \bigvee_{i=1}^{n} \mathcal{K}[\mathcal{J}_i]\right).$$

Moreover, $\mathsf{Th}_{\mathcal{K}\diamond_S\mathcal{N}} = \Psi \wedge \bigvee_{i=1}^{n} \mathcal{K}[\mathcal{J}_i]$ can be expressed in FO[2], the fragment of first-order logic that restricts formulas to have at most two variables, and even in $\mathcal{SHOIQ}$ [2], the DL underlying the Web Ontology Language OWL 2.

To the best of our knowledge, it is unknown how to do $\mathbf{L}^a_\subseteq$-evolution of $\mathcal{SHOIQ}$ KBs: if for $S = \mathbf{L}^a_\subseteq$ one wants to apply evolution to the evolution setting $(\mathsf{Th}_{\mathcal{K}\diamond_S\mathcal{N}}, \mathcal{N}_1)$, where $\mathcal{N}_1$ is some new knowledge, then it is still unclear which logic is needed to capture the evolution result and how to compute it. Moreover, we would like to stay within *DL-Lite$_{core}$*, i.e., to return a *DL-Lite$_{core}$* KB as the evolution result. Therefore, we study now how for $S = \mathbf{L}^a_\subseteq$ we can *approximate* $\mathsf{Th}_{\mathcal{K}\diamond_S\mathcal{N}}$ in *DL-Lite$_{core}$*.

Approximating evolution results. Since for $S = \mathbf{L}^a_\subseteq$ neither the disjunction of $\mathcal{K}_i[\mathcal{J}_i]$ nor Ψ is expressible in *DL-Lite$_{core}$*, one way to approximate $\mathsf{Th}_{\mathcal{K}\diamond_S\mathcal{N}}$ is to take one of $\mathcal{K}_i[\mathcal{J}_i]$. Unfortunately, such an approximation is not sound, that is, for each i there are models of $\mathcal{K}_i[\mathcal{J}_i]$ that are not in $\mathcal{K} \diamond_S \mathcal{N}$. What we propose next is a *DL-Lite$_{core}$*-approximation that is sound and keeps the certain knowledge of $\mathcal{K} \diamond_S \mathcal{N}$, that is, ABox assertions shared by all $\mathcal{K}_i[\mathcal{J}_i]$.

We now formalize the notion of certain knowledge, which is the key component in our *DL-Lite$_{core}$*-approximation of $\mathbf{L}^a_\subseteq$-evolution.

Definition 6.1 *(S-certain MA).* Let S be an MBA, $(\mathcal{K},\mathcal{N})$ an evolution setting, and $\mathcal{K}'$ an S-evolution for $(\mathcal{K},\mathcal{N})$. Then, a membership assertion g (positive or negative) is *S-certain for* $(\mathcal{K},\mathcal{N})$ if $\mathcal{K}' \models g$. □

Consider the algorithm ApproxAlg that computes all $\mathbf{L}^a_\subseteq$-certain membership assertions for a given *DL-Lite$_{core}$*-evolution setting $\mathcal{E} = (\mathcal{T} \cup \mathcal{A}, \mathcal{N})$. As we show later in this section, the KB $\mathcal{T} \cup \mathsf{ApproxAlg}(\mathcal{E})$ is a maximal sound *DL-Lite$_{core}$*-approximation of $(\mathcal{T} \cup \mathcal{A}) \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_\subseteq$. ApproxAlg uses the algorithm Weeding (which was introduced in [7]) as a subroutine. Intuitively, Weeding works as follows: it takes as input a *DL-Lite$_{core}$* KB $\mathcal{T} \cup \mathcal{A}$ and a set $\mathcal{D}$ of ABox assertions to be deleted, and returns as output an ABox $\mathcal{A}^w$ such that $\mathcal{A}^w \not\models_{\mathcal{T}} \mathcal{D}$. It starts with $\mathcal{A}^w = \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$ (Line 1), and then for each MA $B_1(c)$ in $\mathcal{A}$, the algorithm deletes $B_1(c)$ and all the assertions of $\mathsf{cl}_{\mathcal{T}}(\mathcal{A})$ that $\mathcal{T}$-entail $B_1(c)$ (Lines 2–10). Coming back to ApproxAlg, intuitively it works as follows: it takes as input a simple *DL-Lite$_{core}$*-evolution setting $\mathcal{E} = (\mathcal{T} \cup \mathcal{A}, \mathcal{N})$ and returns a *DL-Lite$_{core}$* ABox $\mathcal{A}^{\mathsf{app}}$ such that $\mathcal{T} \cup \mathcal{A}^{\mathsf{app}}$ is a minimal sound approximation of $\mathcal{K} \diamond_S \mathcal{N}$. First, it computes $\mathsf{AtAlg}(\mathcal{E})$ (Line 1) in order to get rid of the assertions that $\mathcal{T}$-contradict $\mathcal{N}$. Second, it computes the positive MAs of $\mathsf{AtAlg}(\mathcal{E})$ (Lines 2–7) and negative MAs of $\mathsf{AtAlg}(\mathcal{E})$ (Lines 8–10) that are not certain. Finally, it deletes the uncertain MAs from

Algorithm 6.1: Weeding($\mathcal{T}, \mathcal{A}, \mathcal{D}$).

INPUT : TBox $\mathcal{T}$, and ABoxes $\mathcal{A}$, $\mathcal{D}$, each satisfiable with $\mathcal{T}$
OUTPUT: finite set $\mathcal{A}^w$ of membership assertions

1 $\mathcal{A}^w := \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$;
2 **for each** $B_1(c) \in \mathcal{D}$ **do**
3 $\quad \mathcal{A}^w := \mathcal{A}^w \setminus \{B_1(c)\}$ **and**;
4 $\quad$ **for each** $B_2 \sqsubseteq B_1 \in \mathsf{cl}(\mathcal{T})$ **do**
5 $\quad\quad \mathcal{A}^w := \mathcal{A}^w \setminus \{B_2(c)\}$;
6 $\quad\quad$ **if** $B_2(c) = \exists R(c)$ **then**
7 $\quad\quad\quad$ **for each** $R(c,d) \in \mathcal{A}^w$ **do** $\mathcal{D} := \mathcal{D} \cup \{R(c,d)\}$
8 $\quad\quad$ **end**
9 $\quad$ **end**
10 **end**
11 **return** $\mathcal{A}^w$.

Algorithm 6.2: ApproxAlg($\mathcal{E}$).

INPUT : *DL-Lite$_{core}$*-evolution setting $\mathcal{E} = (\mathcal{T} \cup \mathcal{A}, \mathcal{N})$
OUTPUT: ABox $\mathcal{A}^{\mathsf{app}}$

1 $\mathcal{A}^{\mathsf{app}} := \mathsf{AtAlg}(\mathcal{E})$; $X := \emptyset$;
2 **for each** $R \in \mathsf{TR}[\mathcal{T}, \mathcal{N}]$ **and for each** $a \in \mathsf{adom}(\mathcal{T} \cup \mathcal{A} \cup \mathcal{N})$ **do**
3 $\quad$ **if there is no** $R(a,b) \in \mathcal{A}^{\mathsf{app}} \cup \mathcal{N}$ **for some** $b \in \mathsf{adom}(\mathcal{T} \cup \mathcal{A} \cup \mathcal{N})$ **then**
4 $\quad\quad X := X \cup \{\exists R(a)\}$;
5 $\quad$ **end**
6 **end**
7 **for each** $A(c) \in \mathsf{DjnAts}[\mathcal{T} \cup \mathcal{A}, \mathcal{N}]$ **do** $X := X \cup \{A(c)\}$;
8 **for each** $R \in \mathsf{TR}[\mathcal{T}, \mathcal{N}]$ **do**
9 $\quad$ **for each** $A(c) \in \mathsf{DjnAts}[\mathcal{T} \cup \mathcal{A}, \mathcal{N}](R)$ **do** $\mathcal{A}^{\mathsf{app}} := \mathcal{A}^{\mathsf{app}} \setminus \{\neg\exists R^-(c)\}$;
10 **end**
11 $\mathcal{A}^{\mathsf{app}} := \mathcal{N} \cup \mathsf{Weeding}(\mathcal{T}, \mathcal{A}^{\mathsf{app}}, X)$;
12 **return** $\mathcal{A}^{\mathsf{app}}$.

$\mathsf{AtAlg}(\mathcal{E})$ by means of the Weeding algorithm and adds $\mathcal{N}$ to the result (Line 11). The following example illustrates how the algorithm works.

Example 6.2. Continuing with Example 5.1, we compute the approximation of $\mathcal{K}_2 \diamond_S \mathcal{N}_2$ with $S = \mathbf{L}^a_{\subseteq}$. First, we compute the necessary components $\mathsf{cl}_{\mathcal{T}}(\mathcal{A}_2)$, $\mathsf{AtAlg}(\mathcal{K}_2, \mathcal{N}_2)$, and X: we start with $X = \emptyset$, and

$$\mathsf{cl}_{\mathcal{T}}(\mathcal{A}_2) = \mathcal{A}_2 \cup \bigl\{\neg \mathit{Priest}(\mathit{john}), \neg\exists \mathit{HasHusb}^-(\mathit{pedro}), \neg\exists \mathit{HasHusb}^-(\mathit{ivan})\bigr\},$$
$$\mathsf{AtAlg}(\mathcal{K}_2, \mathcal{N}_2) = \bigl\{\mathit{Priest}(\mathit{pedro}), \mathit{Priest}(\mathit{ivan}), \neg\exists \mathit{HasHusb}^-(\mathit{pedro}), \neg\exists \mathit{HasHusb}^-(\mathit{ivan})\bigr\}.$$

Second, we re-compute X as in Lines 2–6: $X := X \cup \emptyset$. Third, we re-compute X as in Line 7: $X := X \cup \{\mathit{Priest}(\mathit{pedro}), \mathit{Priest}(\mathit{ivan})\}$. Then, we delete from $\mathcal{A}^{\mathsf{app}}$ the MAs $\neg\exists \mathit{HasHusb}(\mathit{pedro})$ and $\neg\exists \mathit{HasHusb}(\mathit{ivan})$ as in Lines 8–10; one can check that these MAs are not certain; indeed, it holds that $\mathcal{J}_1 \not\models \neg\exists \mathit{HasHusb}(\mathit{pedro})$ and $\mathcal{J}_2 \not\models \neg\exists \mathit{HasHusb}(\mathit{ivan})$. Finally, $\mathsf{Weeding}(\mathcal{T}, \mathcal{A}^{\mathsf{app}}, X)$ returns $\emptyset$. Therefore, $\mathcal{A}^{\mathsf{app}} = \mathcal{N} = \{\mathit{Priest}(\mathit{john})\}$.

To sum up, as soon as husband *john*, who is married to some unknown individual, decides to become a priest, the algorithm that computes maximal sound approximation forces us to delete all the priests and all the wives from the old knowledge. The reason is that we do not know who of the wives from the old knowledge were married to *john* and who are their new husbands: either some of the former priest, or even no one. To account for this uncertainty, the atoms about wives and priests should be erased from the old KB. Thus, the minimal sound approximation $\mathcal{K}^{\mathsf{app}}$ may erase a lot of old knowledge and the approximation result may be quite unexpected from the user's point of view. □

Before proceeding to the formal proof of the algorithm's correctness, consider the following two lemmas. They characterize positive and negative $\mathbf{L}^a_{\subseteq}$-certain membership assertions. The first lemma shows that positive $\mathbf{L}^a_{\subseteq}$-certain MAs are characterized by prototypes of $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_{\subseteq}$.

Lemma 6.3. *Let $(\mathcal{K}, \mathcal{N})$ be a simple DL-Lite$_{core}$-evolution setting, g a positive membership assertion, and $\mathfrak{J}$ a prototypal set for $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_{\subseteq}$. Then, g is $\mathbf{L}^a_{\subseteq}$-certain for $(\mathcal{K}, \mathcal{N})$ if and only if $\mathcal{J} \models g$ for every $\mathcal{J} \in \mathfrak{J}$.*

Proof. Let $S = \mathbf{L}^a_{\subseteq}$, let $\mathcal{K}'$ be an $\mathcal{S}$-evolution for $(\mathcal{K}, \mathcal{N})$, and $\mathfrak{J}$ a prototypal set for $\mathcal{K} \diamond_S \mathcal{N}$. The "only-if" direction is trivial. Indeed, if a positive MA g is $\mathbf{L}^a_{\subseteq}$-certain for $(\mathcal{K}, \mathcal{N})$, then $\mathcal{K}' \models g$ and consequently, by the definition of S-evolution, $\mathcal{K} \diamond_S \mathcal{N} \models g$. To conclude the proof, it suffices to observe that $\mathfrak{J} \subseteq \mathcal{K} \diamond_S \mathcal{N}$.

We now show the "if" direction. Suppose that g is a positive MA such that $\mathcal{J} \models g$ for every $\mathcal{J} \in \mathfrak{J}$. Let $\mathcal{J}_0$ be a model in $\mathcal{K} \diamond_S \mathcal{N}$. Consider a prototype $\mathcal{J}_0'$ in $\mathfrak{J}$ for which there exists a homomorphism h such that $h : \mathcal{J}_0' \hookrightarrow \mathcal{J}_0$. Since $\mathcal{J}_0' \models g$, we have three possibilities:

(i) If g is of the form $A(b)$ and $b \in \mathsf{adom}(\mathcal{K} \cup \mathcal{N})$, then $A(b) \in \mathcal{J}_0'$ and hence $A(h(b)) = A(b) \in \mathcal{J}_0$.
(ii) If g is of the form $R(b,c)$ and $b, c \in \mathsf{adom}(\mathcal{K} \cup \mathcal{N})$, then, analogously to the previous case, $R(b,c) \in \mathcal{J}_0$.
(iii) If g is of the form $\exists R(b)$ and $b \in \mathsf{adom}(\mathcal{K} \cup \mathcal{N})$, then there exists an element $\alpha \in \Delta$ such that $R(b,\alpha) \in \mathcal{J}_0'$, and therefore $R(h(b), h(\alpha)) = R(b, h(\alpha)) \in \mathcal{J}_0$.

In all the cases, we have that $\mathcal{J}_0 \models g$, which concludes the proof. □

The next lemma (which proof can be found in Appendix H) shows that negative $\mathbf{L}^a_{\subseteq}$-certain MAs for $(\mathcal{K}, \mathcal{N})$ are characterized by the elements of $\mathsf{cl}_{\mathcal{T}}(\mathcal{N})$ or $\mathsf{AtAlg}(\mathcal{K}, \mathcal{N})$.

Lemma 6.4. *Let $(\mathcal{K}, \mathcal{N})$, where $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, be a simple DL-Lite$_{core}$-evolution setting, and g a negative membership assertion. Then g is $\mathbf{L}^a_{\subseteq}$-certain for $(\mathcal{N}, \mathcal{K})$ if and only if*

$$g \in \mathsf{cl}_{\mathcal{T}}(\mathcal{N}) \cup \left(\mathsf{AtAlg}(\mathcal{K}, \mathcal{N}) \setminus \bigcup_{R \in \mathsf{TR}} \bigcup_{A(c) \in \mathsf{DjnAts}[\mathcal{K},\mathcal{N}](R)} \{\neg \exists R^-(c)\} \right).$$

Based on the preceding lemmas, we conclude with the main result of this section:

Theorem 6.5. *Let $\mathcal{E} = (\mathcal{K}, \mathcal{N})$, where $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, be a simple DL-Lite$_{core}$-evolution setting and let $\mathcal{A}^{\mathsf{app}} = \mathsf{ApproxAlg}(\mathcal{E})$. Then,*

(i) *g is $\mathbf{L}^a_{\subseteq}$-certain for $\mathcal{E}$ if and only if $g \in \mathcal{A}^{\mathsf{app}}$, and*
(ii) *$\mathcal{K}^{\mathsf{app}} = \mathcal{T} \cup \mathcal{A}^{\mathsf{app}}$ is a maximal sound DL-Lite$_{core}$-approximation of $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_{\subseteq}$.*

Moreover, $\mathsf{ApproxAlg}(\mathcal{E})$ can be computed in time polynomial in $|\mathcal{E}|$.

Proof (Sketch). The proof follows from Lemma 6.4, and Lemma 6.3 coupled with the correctness of the BP procedure. More precisely, polynomiality follows from polynomiality of the AtAlg and Weeding algorithms and polynomiality of computing $\mathsf{TR}[\mathcal{T}, \mathcal{N}]$ and $\mathsf{DjnAts}[\mathcal{K}, \mathcal{N}]$.

Case (i): Suppose that an MA g is certain. Consider the following two cases.

- Assume that g is positive. Then, by Lemma 6.3, for every $\mathcal{J} \in \mathsf{BP}(\mathcal{E})$ it holds that $\mathcal{J} \models g$. By the definition of the BP and BZP procedures, the prototypes from $\mathsf{BZP}(\mathcal{E})$ differ only on: (a) atoms over roles from $\mathsf{TR}[\mathcal{T}, \mathcal{N}]$; (b) MAs from $\mathsf{DjnAts}[\mathcal{K}, \mathcal{N}]$ and; (c) atoms which are in $\mathsf{root}_{\mathcal{T}}$ of the ones in Items (a) and (b). Atoms described in Items (a)–(c) are uncertain by their definition. Note, that the algorithm ApproxAlg adds the atoms of Items (a)–(c) to X and then deletes them at Line 11 by means of the Weeding algorithm.
- Assume that g is negative. Then, by Lemma 6.4, it belongs to $\mathsf{cl}_{\mathcal{T}}(\mathcal{N})$ or $\mathsf{AtAlg}(\mathcal{K}, \mathcal{N}) \setminus Z$, where $Z = \bigcup_{R \in \mathsf{TR}} \bigcup_{A(c) \in \mathsf{DjnAts}[\mathcal{K},\mathcal{N}](R)} \{\neg \exists R^-(c)\}$. In the former case, it holds that $\mathcal{A}^{\mathsf{app}} \models_{\mathcal{T}} g$ since $\mathcal{N} \subseteq \mathcal{A}^{\mathsf{app}}$. In the latter case, it again holds that $\mathcal{A}^{\mathsf{app}} \models_{\mathcal{T}} g$ since the algorithm deletes negative MAs from $\mathsf{AtAlg}(\mathcal{K}, \mathcal{N})$ only in Line 9, and those MAs are from Z.

Case (ii): Suppose that $\mathcal{T} \cup \mathcal{A}^{\mathsf{app}}$ is not a maximal sound approximation, that is, there is an MA g' such that $\mathcal{T} \cup \mathcal{A}^{\mathsf{app}} \cup \{g'\}$ is a sound approximation. Note that g' is not certain since $\mathcal{T} \cup \mathcal{A}^{\mathsf{app}}$ entails all the certain MAs. Thus, by Definition 6.1, there is a model $\mathcal{J}' \in \mathcal{K} \diamond_S \mathcal{N}$ such that $\mathcal{J}' \not\models g'$. Clearly, $\mathcal{J}' \notin \mathsf{Mod}(\mathcal{T} \cup \mathcal{A}^{\mathsf{app}} \cup \{g'\})$, that is, $\mathcal{T} \cup \mathcal{A}^{\mathsf{app}} \cup \{g'\}$ is not a sound approximation. The obtained contradiction concludes the proof. □

As a corollary of the theorem above, consider the case of a simple evolution setting $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ such that $\mathcal{K}$ is a *DL-Lite^{pr}* KB. Then, observe that in this case both $\mathsf{TR}[\mathcal{T}, \mathcal{N}]$ and $\mathsf{DjnAts}[\mathcal{K}, \mathcal{N}]$ are empty sets, and therefore the algorithm $\mathsf{ApproxAlg}(\mathcal{E})$ runs only Lines 1 and 12. That is, in this case $\mathsf{ApproxAlg}(\mathcal{E}) = \mathsf{AtAlg}(\mathcal{E})$. This is quite expected since, as shown in Section 3.1, $\mathbf{L}^a_{\subseteq}$-semantics is expressible in *DL-Lite^{pr}* and a maximal sound approximation should be logically equivalent to the exact evolution result.

Summary of Section 6. For *DL-Lite$_{core}$*-evolution settings, $\mathbf{L}^a_{\subseteq}$-evolution can be efficiently *DL-Lite$_{core}$*-approximated and we presented an algorithm ApproxAlg which can be used to compute these approximations.

7. Practical considerations and conclusion

We summarize here how one can use the results of this paper to do ABox evolution of *DL-Lite*$_{core}$ KBs in practice. Given a *DL-Lite*$_{core}$-evolution setting $(\mathcal{K}, \mathcal{N})$, one can first check (in polynomial time) whether $\mathcal{K}$ is in *DL-Lite*^{pr}. If this is the case, then one can compute in polynomial time S-evolution for $\mathcal{E}$, where S is any of atom-based semantics $\mathbf{G}^a_\subseteq$, $\mathbf{L}^a_\subseteq$, $\mathbf{G}^a_\#$, and $\mathbf{L}^a_\#$, or global symbol-based semantics $\mathbf{G}^s_\subseteq$ and $\mathbf{G}^s_\#$, using the techniques of Theorem 3.10 or 3.15, respectively. One can also compute a maximal sound *DL-Lite*^{pr}-approximation of S-evolution under the remaining two local symbol-based semantics $\mathbf{L}^s_\subseteq$ and $\mathbf{L}^s_\#$ using the techniques of Theorem 3.18. The choice of evolution semantics for *DL-Lite*^{pr} is up to the user, although we believe that atom-based semantics behave more intuitively. For the case when $\mathcal{K}$ is *not* in *DL-Lite*^{pr}, the set of evolved models $\mathcal{K} \diamond_S \mathcal{N}$ is in general not axiomatizable in *DL-Lite*$_{core}$ for S being any of the eight MBAs [7,17]. At the same time, if $S = \mathbf{L}^a_\subseteq$, then one can compute in polynomial time a maximal sound *DL-Lite*$_{core}$-approximation of $\mathcal{K} \diamond_S \mathcal{N}$ using the techniques of Theorem 6.5.

We studied model-based approaches to ABox evolution (update and revision) over *DL-Lite*$_{core}$ and its fragment *DL-Lite*^{pr}, which extends (the first-order fragment of) RDFS. *DL-Lite*^{pr} is closed under most of the MBAs, while *DL-Lite*$_{core}$ is *not* closed under any of them. We showed that if the TBox of $\mathcal{K}$ entails a pair of assertions of the form $A_1 \sqsubseteq \exists R$ and $\exists R^- \sqsubseteq \neg \mathcal{A}_2$, then an interplay of $\mathcal{N}$ and $\mathcal{A}$ may lead to inexpressibility of $\mathcal{K} \diamond_S \mathcal{N}$ with $\mathcal{S} = \mathbf{L}^a_\subseteq$. For *DL-Lite*^{pr} we provided the algorithms to compute evolution results for six model-based approaches and approximate evolution for the remaining two. For *DL-Lite*$_{core}$ (under some restrictions) we studied the properties of evolution under the local model-based approach $\mathbf{L}^a_\subseteq$. In particular, we introduced the notion of prototypal sets that extends the notion of canonical models. We proved that prototypal sets for $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_\subseteq$ exist, and that they are of exponential size in $|\mathcal{K} \cup \mathcal{N}|$, and showed an abstract procedure that constructs them. Based on the insights gained, we proposed a polynomial-time algorithm to compute a maximal sound *DL-Lite*$_{core}$-approximation of $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_\subseteq$. We also believe that prototypes are important since they can be used to study evolution for ontology languages other than *DL-Lite*$_{core}$. In general, we provided some understanding on why *DL-Lite* is not closed under MBAs to evolution, and what are the properties of sets of models $\mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{L}^a_\subseteq$. This understanding is a prerequisite to proceed with the study of evolution in more expressive DLs and to understand what to expect from MBAs in such logics.

Future directions for work include also to study complete approximations of $\mathbf{L}^a_\subseteq$-evolution for *DL-Lite*$_{core}$-evolution settings and to gain a good understudying of how results on F-universal model sets from [29] and universal bases from [30] are related to our results on prototypal sets.

Appendix A. Proofs for Section 3.1

In this and the following appendix we will need the following property of *DL-Lite*$_{core}$.

Proposition A.1. *Let $\mathcal{T} \cup \mathcal{A}$ be a satisfiable DL-Lite$_{core}$ KB and L be a membership assertion. If $\mathcal{A} \models_\mathcal{T} L$, then there exists a membership assertion $L_0 \in \mathcal{A}$ such that $L_0 \models_\mathcal{T} L$.*

Proof. Assume that L is a positive assertion, i.e., of the form $P(a,b)$, $A(c)$, or $\exists R(c)$. Since $\mathsf{chase}_\mathcal{T}(\mathcal{A})$ is a model of $\mathcal{T} \cup \mathcal{A}$ [9], the entailment $\mathcal{A} \models_\mathcal{T} L$ implies that $\mathsf{chase}_\mathcal{T}(\mathcal{A})$ models L. Suppose that $L \in \mathcal{A}$, then, by taking $L_0 = L$, the lemma trivially holds. Suppose that $L \notin \mathcal{A}$, then we have that either $L \in \mathsf{chase}_\mathcal{T}(\mathcal{A})$ with $a, b, c \in \mathsf{adom}(\mathcal{K})$, when $L \in \{P(a,b), A(c)\}$, or $R(c,x) \in \mathsf{chase}_\mathcal{T}(\mathcal{A})$ with $c \in \mathsf{adom}(\mathcal{K})$ and $x \notin \mathsf{adom}(\mathcal{A})$, when $L = \exists R(c)$. By the definition of chase, for every atom in $\mathsf{chase}_\mathcal{T}(\mathcal{A})$, there is a sequence of atoms $f_1, \ldots, f_n$, where (i) $f_n = L$ or $f_n = R(c,x)$, depending on the shape of L; (ii) $f_1 \in \mathcal{A}$, or $\exists R'(c') \in \mathcal{A}$ and $f_1 = R'(c', x')$, where $x' \notin \mathsf{adom}(\mathcal{A})$; (iii) each f_{i+1} is derivable from f_i by triggering a positive inclusion assertion of $\mathcal{T}$, that is, $f_i \models_\mathcal{T} f_{i+1}$. Due to transitivity of $\models_\mathcal{T}$, due to $f_1 \models_\mathcal{T} f_n$, and by taking $L_0 = f_1$ or $L_0 = \exists R'(c')$ depending on the shape of f_1, we obtain $L_0 \models_\mathcal{T} L$ and conclude the proof.

Assume that L is a negative inclusion assertion of the form $\neg A(c)$. If $L \in \mathcal{A}$, then, by taking $L_0 = L$, we conclude the proof. Assume that $L \notin \mathcal{A}$. Assume that

$$\text{for every assertion } L' \in \mathcal{A} \text{ it holds } L' \not\models_\mathcal{T} L. \tag{A.1}$$

Let $L_1, \ldots, L_n$ be all the PIs of $\mathcal{A}$. Consider the interpretation:

$$\mathcal{I} = \bigcup_{i=1}^{n} \mathsf{chase}_\mathcal{T}(L_i) \cup \mathsf{chase}_\mathcal{T}\big(A(c)\big).$$

Clearly, $\mathcal{I} \not\models L$. We now show that $\mathcal{I} \models \mathcal{A} \cup \mathcal{T}$, so we will obtain a contradiction with $\mathcal{A} \models_\mathcal{T} L$. Observe that $\mathcal{I}$ is a model of $\mathcal{A}$. Indeed, it models all the positive MAs of $\mathcal{A}$ by construction. Each $\mathsf{chase}_\mathcal{T}(L_i)$ (and consequently their union) satisfies all negative MAs of $\mathcal{A}$. Assume there is i for which $\mathsf{chase}_\mathcal{T}(L_i)$ does not satisfy a negative MA $\neg g$ of $\mathcal{A}$. Thus, $\{L_i, \neg g\} \models_\mathcal{T} \bot$ which contradicts satisfiability of $\mathcal{T} \cup \mathcal{A}$. Finally, $\mathsf{chase}_\mathcal{T}(A(c))$ satisfies all negative MAs of $\mathcal{A}$. Assume it is not the case, and there is a negative MA $\neg g \in \mathcal{A}$ such that $\mathsf{chase}_\mathcal{T}(A(c)) \models g$. Then, $A(c) \models_\mathcal{T} g$, and $\neg g \models_\mathcal{T} \neg A(c)$, thus we found an assertion in $\mathcal{A}$ that $\mathcal{T}$-entails L, which contradicts Eq. (A.1). Clearly, $\mathcal{I}$ models all the PIs of $\mathcal{T}$. It remains to show that

$\mathcal{I}$ models each NI of $\mathcal{T}$. Assume there is an NI α such that $\mathcal{I} \not\models \alpha$. Then, there are two atoms f and f' in $\mathcal{I}$ such that $f \to \neg f'$ is an instantiation of the first-order interpretation of α. This implies that $\{f, \alpha\} \models \neg f'$. Clearly, ABoxes $\{L_i\}$ for $1 \leqslant i \leqslant n$ and $\{A(c)\}$ satisfy α, so does $\mathsf{chase}_{\mathcal{T}}(L_i)$ for $1 \leqslant i \leqslant n$ and $\mathsf{chase}_{\mathcal{T}}(A(c))$ due to Lemma 12 of [9]. This implies that $\{f, f'\} \not\subseteq \mathsf{chase}_{\mathcal{T}}(L_i)$ for each $1 \leqslant i \leqslant n$ and $\{f, f'\} \not\subseteq \mathsf{chase}_{\mathcal{T}}(A(c))$. Thus, two cases are possible:

(i) $f \in \mathsf{chase}_{\mathcal{T}}(L_i)$ for some $i \in \{1, \ldots, n\}$ and $f' \in \mathsf{chase}_{\mathcal{T}}(A(c))$ (the case when $f' \in \mathsf{chase}_{\mathcal{T}}(L)$ and $f \in \mathsf{chase}_{\mathcal{T}}(A(c))$ is symmetric),
(ii) $f \in \mathsf{chase}_{\mathcal{T}}(L_i)$ and $f' \in \mathsf{chase}_{\mathcal{T}}(L_j)$ for some different $i, j \in \{1, \ldots, n\}$.

In Case (i), $f' \in \mathsf{chase}_{\mathcal{T}}(A(c))$ implies that $\neg f' \models_{\mathcal{T}} \neg A(c)$. Combining the latter entailment with $\{f, \alpha\} \models \neg f'$ we obtain $f \models_{\mathcal{T}} \neg A(c)$. Since $L_i \models_{\mathcal{T}} f$, we conclude that $L_i \models_{\mathcal{T}} \neg A(c)$ which contradicts the assumption in Eq. (A.1) and concludes the proof. In Case (ii), analogously to Case (i), we conclude that $L_i \models_{\mathcal{T}} \neg L_j$, thus A does not satisfy α which yields a contradiction with satisfiability of $\mathcal{T} \cup \mathcal{A}$.

The case when $L = \neg\exists R(c)$ is analogous to the previous one. □

Proof of Proposition 3.2. It follows from the definition of AtAlg and the facts that in *DL-Lite*^{pr} disjointnesses that involve roles or their projections are forbidden and $\mathcal{N}$ contains only positive membership assertions. Indeed, let $\mathcal{A} \models_{\mathcal{T}} R(a,b)$ and $\mathsf{AtAlg}(\mathcal{K}, \mathcal{N}) \cup \mathcal{N} \not\models_{\mathcal{T}} R(a,b)$. Then, $\{R(a,b)\} \cup \mathcal{N} \models_{\mathcal{T}} \bot$ (see Algorithm 3.1). Thus, there are membership assertions L_1 and L_2, and an NI $\alpha \in \mathsf{cl}(\mathcal{T})$ such that $\{R(a,b)\} \cup \mathcal{N} \models_{\mathcal{T}} \{L_1, L_2\}$ and $\alpha \models L_1 \to \neg L_2$. Note that $L_1 \to \neg L_2$ should be seen as a first-order formula with two subformulas L_1 and L_2, both without free variables. The semantics of this formula is defined straightforwardly: $\mathcal{I} \models L_1 \to \neg L_2$ if $\mathcal{I} \models L_1$ and $\mathcal{I} \not\models L_2$ for every interpretation $\mathcal{I}$. Due to Proposition A.1, one of the following two cases holds: $R(a,b) \models_{\mathcal{T}} L_1$ or $R(a,b) \models_{\mathcal{T}} L_2$. Consider the first case (the second one is symmetric). Combining $R(a,b) \models_{\mathcal{T}} L_1$ and $\alpha \models L_1 \to \neg L_2$ we obtain that $\alpha \models R(a,b) \to \neg L_2$. Thus, α is of the form $\exists R \sqsubseteq \neg B$ or $\exists R^- \sqsubseteq \neg B$ for some basic concept B. Either case contradicts the fact that $(\mathcal{T} \cup \mathcal{A})$ is a *DL-Lite*^{pr} KB. Similarly, the case when $\mathcal{A} \models_{\mathcal{T}} \exists R(a)$ can be proved. □

Proof of Proposition 3.3. $\Leftarrow$. Assume that for every $f_1 \in \mathcal{I}_1$ and $f_2 \in \mathcal{I}_2$ it holds that $\{f_1, f_2\} \not\models_{\mathcal{T}} \bot$, but $\mathcal{I}_1 \cup \mathcal{I}_2 \not\models \mathcal{T}$. Then, there is an assertion $\alpha \in \mathsf{cl}(\mathcal{T})$ such that $\mathcal{I}_1 \cup \mathcal{I}_2 \not\models \alpha$.

Suppose α is a PI, then there is a ground atom g_1 in $\mathcal{I}_1 \cup \mathcal{I}_2$ satisfying the property: for every ground atom g_2 such that $g_1 \to g_2$ is an instantiation of the first-order translation of α, it holds that $g_2 \notin \mathcal{I}_1 \cup \mathcal{I}_2$. The fact that $g_1 \in \mathcal{I}_1 \cup \mathcal{I}_2$ implies that one of the two cases holds: (i) $g_1 \in \mathcal{I}_1$ or (ii) $g_1 \in \mathcal{I}_2$. From Case (i) together with $g_2 \notin \mathcal{I}_1 \cup \mathcal{I}_2$, we conclude that $\mathcal{I}_1 \not\models \alpha$, and from Case (ii) together with $g_2 \notin \mathcal{I}_1 \cup \mathcal{I}_2$, we conclude that $\mathcal{I}_2 \not\models \alpha$. Either case contradicts the fact that $\mathcal{I}_1$ and $\mathcal{I}_2$ are models of $\mathcal{T}$.

Suppose α is an NI, then, due to Lemma 12 (more precisely, its straightforward extended to the case when $\mathcal{A}$ is a possibly infinite set of atoms) of [7], there are two atoms f_1 and f_2 in $\mathcal{I}_1 \cup \mathcal{I}_2$ such that $\{f_1, f_2\} \models_{\mathcal{T}} \bot$. Since $\mathcal{I}_1 \models \alpha$ and $\mathcal{I}_2 \models \alpha$, neither $\{f_1, f_2\} \subseteq \mathcal{I}_1$ nor $\{f_1, f_2\} \subseteq \mathcal{I}_2$ holds. Thus, either of the two cases holds: $f_1 \in \mathcal{I}_1$ and $f_2 \in \mathcal{I}_2$, or $f_2 \in \mathcal{I}_1$ and $f_1 \in \mathcal{I}_2$. Either case contradicts the assumption of the "if" direction.

$\Rightarrow$. Assume that there are $f_1 \in \mathcal{I}_1$ and $f_2 \in \mathcal{I}_2$ such that $\{f_1, f_2\} \models_{\mathcal{T}} \bot$. Then, $\mathcal{I}_1 \cup \mathcal{I}_2 \not\models \mathcal{T}$, which contradicts the assumption of the "only if" direction. □

Proof of Proposition 3.6. Assume that there is a general MA g such that $\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g) \not\models \mathcal{T}$. Then, there is an assertion $\alpha \in \mathsf{cl}(\mathcal{T})$ s.t.

$$\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g) \not\models \alpha. \tag{A.2}$$

Assume that α is an NI. Clearly, if a set of atoms satisfies a negative inclusion assertion, then any subset of this set of atoms does so. This implies that, since $\mathcal{I} \models \alpha$ and $\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g) \subseteq \mathcal{I}$, $\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g) \models \alpha$, which contradicts the assumption in Eq. (A.2).

Assume that g is a positive MA and α is a PI. Then, Eq. (A.2) implies that there is a ground atom f_1 in $\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g)$ satisfying the property: for every ground atom f_2 such that $f_1 \to f_2$ is an instantiation of the first-order translation of α, $f_2 \notin \mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g)$. Observe that $f_1 \in \mathcal{I}$ and $\mathcal{I} \models \alpha$, thus at least one such f_2, say $\hat{f}_2$, is in $\mathcal{I}$. Since $\hat{f}_2 \notin \mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g)$, we have that $\hat{f}_2 \in \mathsf{root}_{\mathcal{T}}(g)$. Therefore, by the definition of $\mathsf{root}_{\mathcal{T}}(g)$, $f_1 \in \mathsf{root}_{\mathcal{T}}(g)$ and $f_1 \notin \mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g)$, which contradicts the fact that $f_1 \in \mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g)$ and concludes the proof.

Assume that g is a negative MA and α is a PI. Let α be $B \sqsubseteq B'$. By exactly the same reason as the case of positive g, there are the atoms f_1 and $\hat{f}_2$ such that $f_1 \to \hat{f}_2$ instantiate $B \sqsubseteq B'$. Since $\hat{f}_2 \in \mathsf{root}_{\mathcal{T}}(g)$, there is an NI of the form $B' \sqsubseteq \neg B''$ such that $\mathcal{T} \models B' \sqsubseteq \neg B''$ and $\hat{f}_2 \to g$ is its instantiation. From $\mathcal{T} \models B' \sqsubseteq \neg B''$ and $\mathcal{T} \models B \sqsubseteq B'$, we conclude that $\mathcal{T} \models B \sqsubseteq \neg B''$ and therefore $f_1 \in \mathsf{root}_{\mathcal{T}}(g)$ which contradicts the fact that $f_1 \in \mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(g)$ and concludes the proof. □

Proof of Lemma 3.8. In the proof of the lemma in the paper, due to the space limitation, we shortened the proof of the fact that $\mathcal{J} \in \mathsf{loc\_min}^{a}_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$. The full proof of the fact can be found below.

$\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$: By the definition of $\mathbf{L}^a_{\subseteq}$-evolution, we need to show that there is *no* $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ such that $\mathcal{I} \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J}$. Assume there exists such $\mathcal{J}'$. Thus, there is an atom f such that $f \notin \mathcal{I} \ominus \mathcal{J}'$ while $f \in \mathcal{I} \ominus \mathcal{J}$. By the definition of $\mathcal{I}$, interpretations $\mathcal{I}$ and $\mathcal{J}$ differ only on atoms of the form $A(c)$; hence, f is of the form $A(c)$ (it cannot be of the form $R(a, b)$). We have two cases:

(i) $A(c) \in \mathcal{I}$, $A(c) \notin \mathcal{J}$, and $A(c) \in \mathcal{J}'$: By construction of $\mathcal{I}$, $A(c) \in \mathsf{ucl}_{\mathcal{T}}(\mathcal{A})$, while $A(c) \notin \mathcal{J}$ implies $\mathcal{A}(c) \notin \mathcal{A}'$. Thus, $\{A(c)\} \cup \mathcal{N} \models_{\mathcal{T}} \bot$. On the other hand, $A(c) \in \mathcal{J}'$ and $\mathcal{J}' \models \mathcal{N}$ imply that $\{A(c)\} \cup \mathcal{N} \not\models \bot$, which yields a contradiction.
(ii) $A(c) \notin \mathcal{I}$, $A(c) \in \mathcal{J}$, and $A(c) \notin \mathcal{J}'$: From $A(c) \notin \mathcal{J}'$ and $\mathcal{J}' \models \mathcal{N}$ we imply that $\mathcal{N} \not\models A(c)$. By the definition of $\mathcal{I}$, the assumptions $A(c) \notin \mathcal{I}$ and $A(c) \in \mathcal{J}$ imply that $\{A(c)\} \cup \mathcal{A} \models_{\mathcal{T}} \bot$, and therefore $\neg A(c) \in \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$. By the definition of $\mathcal{J}$, the assumption $A(c) \in \mathcal{J}$ implies $\neg A(c) \notin \mathsf{AtAlg}(\mathcal{K}, \mathcal{N})$. From $\neg A(c) \in \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$ and $\neg A(c) \notin \mathsf{AtAlg}(\mathcal{K}, \mathcal{N})$ we conclude that $\{\neg A(c)\} \cup \mathcal{N} \models_{\mathcal{T}} \bot$, and, therefore, $\mathcal{N} \models A(c)$ holds, which yields a contradiction.

Thus, $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$ and we conclude the proof. □

Appendix B. Proofs for Section 3.2

Proof of Proposition 3.14. Due to Proposition 3.3, it suffices to show that for every $f_1 \in (\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(\neg A(c)))$ and $f_2 \in \mathcal{J}[A(c)]$ we have $\{f_1, f_2\} \not\models_{\mathcal{T}} \bot$. Assume this is not the case, that is, there is $f_1 \in (\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(\neg A(c)))$ and $f_2 \in \mathcal{J}[A(c)]$ such that $\{f_1, f_2\} \models_{\mathcal{T}} \bot$. We now show that

$$\{f_1, A(c)\} \models_{\mathcal{T}} \bot. \tag{B.1}$$

From $\{f_1, f_2\} \models_{\mathcal{T}} \bot$ it clearly follows that there is an NI of the form $A_1 \sqsubseteq \neg A_2$ such that $\mathcal{T} \models A_1 \sqsubseteq \neg A_2$, and there are atoms $A_1(d) \in (\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(\neg A(c)))[f_1]$ and $A_2(d) \in \mathcal{J}[A(c)]$. From $A_2(d) \in \mathcal{J}[A(c)]$ we conclude that $d = c$. Indeed, from $A_2(d) \in \mathcal{J}[A(c)]$ we conclude that there is a sequence of atoms $f_1, \ldots, f_n$, where (i) $f_1 = A(c)$; (ii) $f_n = A_2(d)$, (iii) each f_{i+1} is derivable from f_i by triggering a positive inclusion assertion α_i of $\mathsf{cl}(\mathcal{T})$. If $c \neq d$, then there is a role symbol occurring in at least one α_i. Indeed, if each α_i has no role symbol, then due to transitivity of $\models_{\mathcal{T}}$, we have $f_1 \models_{\mathcal{T}} f_n$ and therefore $c = d$. Let R be a role symbol occurring in α_j with the highest index, that is, $j = n$, or $j < n$ and for each α_i where $j < i < n$ there is no role symbol occurring in α_i. Then, α_j is of the form $\exists R \sqsubseteq A'$. Thus, $\mathcal{T} \models \exists R \sqsubseteq A_2$. Combining this with $\mathcal{T} \models A_1 \sqsubseteq \neg A_2$, we obtain that $\mathcal{T} \models_{\mathcal{T}} \exists R \sqsubseteq \neg A_1$, which contradicts the fact that $\mathcal{T}$ is in *DL-Lite*^{pr}. Thus, there is no role symbol in each α_t, where $1 \leqslant t \leqslant n$. Therefore, $c = d$ and $A(c) \models_{\mathcal{T}} A_2(c)$.

Analogously, one can show that $A_1(c) \in (\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(\neg A(c)))[f_1]$ implies that $f_1 \models_{\mathcal{T}} A_1(c)$. To sum up, we proved that

$$f_1 \models_{\mathcal{T}} A_1(c), \qquad A(c) \models_{\mathcal{T}} A_2(c), \qquad A_1(c) \models_{\{A_1 \sqsubseteq \neg A_2\}} \neg A_2(c),$$

thus Eq. (B.1) holds.

Since for every $f_2 \in \mathcal{J}[A(c)]$ it holds that $A(c) \models_{\mathcal{T}} f_2$, we have $\{f_1, A(c)\} \models_{\mathcal{T}} \bot$. Thus, $\{f_1, A(c)\} \models_{\mathcal{T}} \neg A(c)$. There are only two literals in $\{f_1, A(c)\}$ and $A(c) \not\models_{\mathcal{T}} \neg A(c)$. Thus, due to Proposition A.1, we conclude that $f_1 \models_{\mathcal{T}} \neg A(c)$. This contradicts the assumption that $f_1 \in (\mathcal{I} \setminus \mathsf{root}_{\mathcal{T}}(\neg A(c)))$ and concludes the proof. □

Proof of Theorem 3.15. In the proof of the theorem in the body of the paper, due to the space limitations, we shortened the proof of the fact that $\mathsf{Mod}(\mathcal{K}') \subseteq \mathcal{M}$. The full proof of this fact is given below.

$\mathsf{Mod}(\mathcal{K}') \subseteq \mathcal{M}$: Let $\mathcal{J}_0 \in \mathsf{Mod}(\mathcal{K}') = \mathsf{Mod}(\mathcal{T} \cup \mathcal{A}')$ where $\mathcal{A}' = \mathsf{GSymbAlg}(\mathcal{E})$, and assume $\mathcal{J}_0 \notin \mathcal{M}$, that is: (i) $\mathcal{J}_0 \notin \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$, or (ii) for every $\mathcal{I} \models \mathcal{K}$ there is a pair of models $\mathcal{I}' \models \mathcal{K}$ and $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ s.t. $\mathsf{dist}^s_{\subseteq}(\mathcal{I}', \mathcal{J}') \subsetneq \mathsf{dist}^s_{\subseteq}(\mathcal{I}, \mathcal{J}_0)$. Case (i) is impossible since $\mathcal{N} \subseteq \mathcal{A}'$. If Case (ii) holds, then consider a model $\mathcal{I}_0$ as in Eq. (2). By Lemma 3.8 we have $\mathcal{I}_0 \models \mathcal{K}$. By our assumption, $\mathsf{dist}^s_{\subseteq}(\mathcal{I}', \mathcal{J}') \subsetneq \mathsf{dist}^s_{\subseteq}(\mathcal{I}_0, \mathcal{J}_0)$ holds for some $\mathcal{I}'$ and $\mathcal{J}'$. Due to Proposition 3.2, $\mathcal{I}_0$ and $\mathcal{J}_0$ coincide on how they interpret roles. Thus, there is a concept A such that $A^{\mathcal{I}'} = A^{\mathcal{J}'}$ while $A^{\mathcal{I}_0} \neq A^{\mathcal{J}_0}$, and consequently there is an atom $A(c) \in \mathcal{I}_0 \ominus \mathcal{J}_0$. Note that, by the construction of $\mathcal{I}_0$, it holds that $A(c) \in \mathsf{ucl}_{\mathcal{T}}(\mathcal{A})$ or $A(c) \in \mathsf{conf}_{\mathcal{T}}(\mathcal{J}_0, \mathcal{A})$. We have two cases:

(i) $A(c) \in \mathcal{I}_0 \setminus \mathcal{J}_0$: From $A(c) \in \mathcal{I}_0$ we conclude that $A(c) \in \mathsf{ucl}_{\mathcal{T}}(\mathcal{A}) \subseteq \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$. From $A(c) \notin \mathcal{J}_0$ and $\mathcal{J}_0 \models \mathcal{K}'$, we conclude that $A(c) \notin X$ (see Line 6 in Algorithm 3.2). From these two statements, $A(c) \in \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$ and $A(c) \notin X$, we imply that for some constant $b \in \mathsf{adom}(\mathcal{T} \cup \mathcal{A})$ (including c) one of the two cases holds: $\neg A(b) \in \mathsf{cl}_{\mathcal{T}}(\mathcal{N})$ and $A(b) \in \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$, or $A(b) \in \mathsf{cl}_{\mathcal{T}}(\mathcal{N})$ and $\neg A(b) \in \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$. Either case together with $\mathcal{J}' \models \mathsf{cl}_{\mathcal{T}}(\mathcal{N})$ and $\mathcal{I}' \models \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$ implies $A^{\mathcal{I}'} \neq A^{\mathcal{J}'}$ and yields a contradiction with $A^{\mathcal{I}'} = A^{\mathcal{J}'}$.
(ii) $A(c) \in \mathcal{J}_0 \setminus \mathcal{I}_0$: In this case $A(c) \in \mathsf{conf}_{\mathcal{T}}(\mathcal{J}_0, \mathcal{A})$ (see Eq. (2)) which means that $\{A(c)\} \cup \mathsf{cl}_{\mathcal{T}}(\mathcal{A}) \models_{\mathcal{T}} \bot$ and consequently $\neg A(c) \in \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$. From $\mathcal{J}_0 \models A(c)$ we conclude that $\neg A(c) \notin X$. Finally, from the two statements $\neg A(c) \in \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$ and $\neg A(c) \notin X$ we conclude that $A^{\mathcal{I}'} \neq A^{\mathcal{J}'}$ using the same argument as for $A(c) \in \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$ and $A(c) \notin X$ of Case (i) above.

Thus, $\mathcal{J}_0 \in \mathcal{M}$, which concludes the proof. □

Appendix C. Proofs for Section 3.3

Proof of Theorem 3.16. Let $S = \mathbf{L}^s_{\subseteq}$. Consider the KB $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, where: $\mathcal{T} = \{A \sqsubseteq B\}$, $\mathcal{A} = \{B(c)\}$, and $\mathcal{N} = \{B(d)\}$. It can be shown that (i) every $\mathcal{J} \models \mathcal{K} \diamond_S \mathcal{N}$ satisfies $A(d) \to B(c)$, and (ii) there are models $\mathcal{J}_0, \mathcal{J}_1 \in \mathcal{K} \diamond_S \mathcal{N}$ such that $\mathcal{J}_1 \not\models \neg A(c)$ and $\mathcal{J}_2 \not\models B(c)$. Due to Lemma 1 in [7], if these two conditions hold, then $\mathcal{K} \diamond_S \mathcal{N}$ is inexpressible in *DL-Lite*, and hence in *DL-Lite*^{pr}.

To see that Condition (i) holds, assume there is a model $\mathcal{J} \in \mathcal{K} \diamond \mathcal{N}$ such that $\mathcal{J} \not\models A(d) \to B(c)$, i.e., $A(d) \in \mathcal{J}$ but $B(c) \notin \mathcal{J}$. By the definition of $\mathbf{L}^s_{\subseteq}$, there is $\mathcal{I} \models \mathcal{K}$ such that: for every $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ it *does not* hold that $\mathsf{dist}^s_{\subseteq}(\mathcal{I}, \mathcal{J}') \subsetneq \mathsf{dist}^s_{\subseteq}(\mathcal{I}, \mathcal{J})$. Since $\mathcal{I} \models \mathcal{K}$ and $B(c) \in \mathcal{I}$ we have that $B \in \mathsf{dist}^s_{\subseteq}(\mathcal{I}, \mathcal{J})$. There are two cases:

- If $A(d) \in \mathcal{I}$, then also $B(d) \in \mathcal{I}$; thus, $\mathcal{I} \models \mathcal{T} \cup \mathcal{N}$ and, by taking $\mathcal{J}' = \mathcal{I}$, one obtains $\mathsf{dist}^s_{\subseteq}(\mathcal{I}, \mathcal{J}') \subsetneq \mathsf{dist}^s_{\subseteq}(\mathcal{I}, \mathcal{J})$ which yields a contradiction.
- If $A(d) \notin \mathcal{I}$, then $\{A, B\} \in \mathsf{dist}^s_{\subseteq}(\mathcal{I}, \mathcal{J})$. Now consider an interpretation $\mathcal{J}' = \mathcal{I} \cup \{B(d)\}$, which is clearly a model of $\mathcal{T} \cup \mathcal{N}$. If $B(d) \in \mathcal{I}$ then $\mathsf{dist}^s_{\subseteq}(\mathcal{I}, \mathcal{J}') = \emptyset$, otherwise $\mathsf{dist}^s_{\subseteq}(\mathcal{I}, \mathcal{J}') = \{B\}$. In either case $\mathsf{dist}^s_{\subseteq}(\mathcal{I}, \mathcal{J}') \subsetneq \mathsf{dist}^s_{\subseteq}(\mathcal{I}, \mathcal{J})$, which yields a contradiction.

Thus, every $\mathcal{J} \in \mathcal{K} \diamond \mathcal{N}$ satisfies $A(d) \to B(c)$.

To see that Condition (ii) holds, consider the following interpretations $\mathcal{J}_1$ and $\mathcal{J}_2$:

$$\mathcal{J}_1 = \{A(d), B(d), B(c)\}, \qquad \mathcal{J}_2 = \{B(d)\}.$$

It is easy to see that (a) $\mathcal{J}_i \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$ for $i = 1, 2$, (b) $\mathcal{J}_1 \not\models \neg A(d)$, and (c) $\mathcal{J}_2 \not\models B(c)$. It remains to show that $\mathcal{J}_i \in \mathcal{K} \diamond_S \mathcal{N}$, so the conditions of Lemma 1 in [7] will be satisfied, and inexpressibility of $\mathbf{L}^s_{\subseteq}$ in *DL-Lite*^{pr} will be proved. Let us show this.

- $\mathcal{J}_1 \in \mathcal{K} \diamond_S \mathcal{N}$: Note that $\mathcal{J}_1 \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{A})$, so we conclude that $\mathsf{dist}^s_{\subseteq}(\mathcal{J}_1, \mathcal{J}_1) = \emptyset$. Thus, $\mathcal{J}_1 \in \mathsf{loc\_min}^s_{\subseteq}(\mathcal{J}_1, \mathcal{T}, \mathcal{N})$ and therefore $\mathcal{J}_1 \in \mathcal{K} \diamond_S \mathcal{N}$.
- $\mathcal{J}_2 \in \mathcal{K} \diamond_S \mathcal{N}$: Consider an interpretation $\mathcal{I}_2 = \{B(c)\}$, which is clearly a model of $\mathcal{T} \cup \mathcal{A}$. Note that $\mathsf{dist}^s_{\subseteq}(\mathcal{I}_2, \mathcal{J}_2) = \{B\}$. Then, for every model $\mathcal{J} \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$, it holds that $\{B\} \subseteq \mathsf{dist}^s_{\subseteq}(\mathcal{I}_2, \mathcal{J})$ since $\mathcal{I}_2 \not\models B(d)$ and $\mathcal{J} \models B(d)$. Thus, we conclude that it *does not* hold $\mathsf{dist}^s_{\subseteq}(\mathcal{I}_2, \mathcal{J}) \subsetneq \mathsf{dist}^s_{\subseteq}(\mathcal{I}_2, \mathcal{J}_2)$, that is, $\mathcal{J}_2 \in \mathsf{loc\_min}^s_{\subseteq}(\mathcal{I}_2, \mathcal{T}, \mathcal{N})$ and therefore $\mathcal{J}_2 \in \mathcal{K} \diamond_S \mathcal{N}$.

Thus, Conditions (i) and (ii) hold and we conclude the proof. □

Appendix D. Proofs for Section 4.1

Proof of Theorem 4.2. It remains to show the case $\mathbf{G}^s_{\#} \preccurlyeq_{\mathsf{sem}} \mathbf{G}^s_{\subseteq}$. Consider $\mathcal{M}_{\#} = \mathcal{K} \diamond_{S_1} \mathcal{N}$ with $S_1 = \mathbf{G}^s_{\#}$, which is based on the distance $\mathsf{dist}^s_{\#}$, and $\mathcal{M}_{\subseteq} = \mathcal{K} \diamond_{S_2} \mathcal{N}$ with $S_2 = \mathbf{G}^s_{\subseteq}$, which is based on $\mathsf{dist}^s_{\subseteq}$, for an evolution setting $(\mathcal{K}, \mathcal{N})$. We now are interested in establishing whether $\mathcal{M}_{\#} \subseteq \mathcal{M}_{\subseteq}$ holds. Assume $\mathcal{J}' \in \mathcal{M}_{\#}$ and $\mathcal{J}' \notin \mathcal{M}_{\subseteq}$. From the former assumption, we conclude existence of a model $\mathcal{I}'$ such that for every pair of models $\mathcal{I} \in \mathsf{Mod}(\mathcal{K})$ and $\mathcal{J} \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$, it *does not* hold that $\mathsf{dist}^s_{\#}(\mathcal{I}, \mathcal{J}) \lneq \mathsf{dist}^s_{\#}(\mathcal{I}', \mathcal{J}')$. From the latter assumption, $\mathcal{J}' \notin \mathcal{M}_{\subseteq}$, we conclude existence of models $\mathcal{I}'' \in \mathsf{Mod}(\mathcal{K})$ and $\mathcal{J}'' \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$ such that $\mathsf{dist}^s_{\subseteq}(\mathcal{I}'', \mathcal{J}'') \subsetneq \mathsf{dist}^s_{\subseteq}(\mathcal{I}', \mathcal{J}')$. Since the signature of $\mathcal{K} \cup \mathcal{N}$ is finite, the distance $\mathsf{dist}^s_{\subseteq}$ between every two interpretations over this signature is also finite. Thus, we obtain that $\mathsf{dist}^s_{\#}(\mathcal{I}'', \mathcal{J}'') \lneq \mathsf{dist}^s_{\#}(\mathcal{I}', \mathcal{J}')$, which contradicts the fact that $\mathcal{J}' \in \mathcal{M}_{\#}$ and concludes the proof. □

Proof of Proposition 4.3. Let $S = \mathbf{L}^a_{\#}$. Let $g = R(a, b)$, then the case when $g = \exists R(a)$ is analogous. Assume $\mathcal{A} \models_{\mathcal{T}} R(a, b)$, while there is $\mathcal{J}_0 \in \mathcal{K} \diamond_S \mathcal{N}$ such that $\mathcal{J}_0 \not\models R(a, b)$. Let $\mathcal{I}_0$ be a model of $\mathcal{T} \cup \mathcal{A}$ such that $\mathcal{J}_0 \in \mathsf{loc\_min}^a_{\#}(\mathcal{I}_0, \mathcal{T}, \mathcal{N})$. We now exhibit $\mathcal{J}'_0 \models \mathcal{T} \cup \mathcal{N}$ such that $|\mathcal{I}_0 \ominus \mathcal{J}'_0| < |\mathcal{I}_0 \ominus \mathcal{J}_0|$. Consider $\mathcal{J}'_0 = \mathcal{J}_0 \cup \mathcal{I}_0[R(a, b)]$. Note that $\mathcal{A} \models_{\mathcal{T}} R(a, b)$ and therefore the set $\mathcal{I}_0[R(a, b)]$ is not empty.

Observe that $\mathcal{J}'_0 \models \mathcal{T} \cup \mathcal{N}$. Indeed, $\mathcal{J}'_0 \models \mathcal{N}$ since $\mathcal{N}$ contains positive MAs only and $\mathcal{J}_0 \models \mathcal{N}$. $\mathcal{J}'_0$ models all PIs from $\mathcal{T}$ since both $\mathcal{J}_0$ and $\mathcal{I}_0[R(a, b)]$ do so. Assume there is an NI $\alpha \in \mathsf{cl}(\mathcal{T})$ of the form $A_1 \sqsubseteq \neg A_2$, where A_1 and A_2 are atomic, such that $\mathcal{J}'_0 \not\models \alpha$.[9] Thus, there is a pair of atoms $\{A_1(c), A_2(c)\} \subseteq \mathcal{J}'_0$. Observe that $\{A_1(c), A_2(c)\} \nsubseteq \mathcal{I}_0[R(a, b)]$ and $\{A_1(c), A_2(c)\} \nsubseteq \mathcal{J}_0$. Indeed, the ABox $\{R(a, b)\}$ obviously satisfies α and, due to Lemma 12 of [9], so does the model $\mathcal{I}_0[R(a, b)]$, and therefore $\{A_1(c), A_2(c)\} \nsubseteq \mathcal{I}_0[R(a, b)]$. Since $\mathcal{J}_0 \models \mathcal{T}$, it holds that $\{A_1(c), A_2(c)\} \nsubseteq \mathcal{J}_0$. Therefore, one of the two cases holds: $A_1(c) \in \mathcal{J}_0$ and $A_2(c) \in \mathcal{I}_0[R(a, b)]$, or $A_2(c) \in \mathcal{J}_0$ and $A_1(c) \in \mathcal{I}_0[R(a, b)]$. Either case is possible since neither $\mathcal{J}_0$ nor $\mathcal{I}_0[R(a, b)]$ is empty. Consider the first case, the second case is symmetric. The membership $A_2(c) \in \mathcal{I}_0[R(a, b)]$ implies an existence of a sequence of atoms $f_1, \ldots, f_n$ in $\mathcal{I}_0[R(a, b)]$ such that $n \geqslant 2$, $f_1 = R(a, b)$, $f_n = A_2(c)$, and for each $1 \leqslant i \leqslant (n-1)$ there is a PI $\alpha_1 \in \mathsf{cl}(\mathcal{T})$ such that $f_i \to f_{i+1}$ is an instantiation of α_i. We now show by induction on n that $\mathsf{cl}(\mathcal{T})$ contains an NI of the form $\exists R' \sqsubseteq \neg A'$ for some role R' and atomic concept A', which will give

[9] Note that $\mathcal{T} \cup \mathcal{A}$ is a *DL-Lite*^{pr} KB and therefore all NIs in $\mathsf{cl}(\mathcal{T})$ has only atomic concepts on the left and the right of $\sqsubseteq$.

a contradiction with the fact that $\mathcal{T} \cup \mathcal{A}$ is a *DL-Lite*^{pr} KB. If $n = 2$, then $\alpha_1 = \exists R \sqsubseteq A_2$ or $\alpha_1 = \exists R^- \sqsubseteq A_2$. The former case combined with α gives that $\exists R \sqsubseteq \neg A_1 \in \mathsf{cl}(\mathcal{T})$, and the latter one: $\exists R^- \sqsubseteq \neg A_1 \in \mathsf{cl}(\mathcal{T})$. Thus, we obtain a contradiction. If $n > 2$, then consider α_{n-1}. The shape if α_{n-1} is either $A' \sqsubseteq A_2$ or $\exists R' \sqsubseteq A_2$. Combining the former case with α, we obtain that $A_1 \sqsubseteq \neg A'$ and we conclude the proof by the induction assumption. Combining the later case with α we obtain that $\exists R' \sqsubseteq \neg A_1$, which gives a contradiction. We conclude that $\mathcal{J}'_0 \models \mathcal{T} \cup \mathcal{N}$.

It remains to show that $|\mathcal{I}_0 \ominus \mathcal{J}'_0| < |\mathcal{I}_0 \ominus \mathcal{J}_0|$. By construction, $\mathcal{I}_0 \ominus \mathcal{J}'_0 \subsetneq \mathcal{I}_0 \ominus \mathcal{J}_0$. Since $\mathcal{T} \cup \mathcal{A}$ is a *DL-Lite*^{pr} KB, in particular, $\mathcal{T} \not\models \exists R \sqsubseteq \neg A$ for any role R, it holds that $|\mathcal{I}_0 \ominus \mathcal{J}_0|$ is finite. Thus, $|\mathcal{I}_0 \ominus \mathcal{J}'_0| < |\mathcal{I}_0 \ominus \mathcal{J}_0|$. $\square$

Proposition D.1. *Let $(\mathcal{K}, \mathcal{N})$ with $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ be a DL-Lite^{pr}-evolution setting. Let $\mathcal{J}_0 \in \mathcal{K} \diamond_S \mathcal{N}$ with $S = \mathbf{G}^a_\#$, $\mathcal{I}_0 \in \mathsf{Mod}(\mathcal{K})$, and $\mathcal{I}_0$ is $\mathbf{G}^a_\#$-minimally distant from $\mathcal{J}_0$. Then, $|\mathcal{I}_0 \ominus \mathcal{J}_0|$ is finite.*

Proof. Let $S = \mathbf{G}^a_\#$. Suppose that $|\mathcal{I}_0 \ominus \mathcal{J}_0|$ is infinite. If there exist models $\mathcal{I}' \in \mathsf{Mod}(\mathcal{K})$ and $\mathcal{J}' \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$ such that $|\mathcal{I}' \ominus \mathcal{J}'|$ is finite, this will contradict the fact that $\mathcal{J}_0 \in \mathcal{K} \diamond_S \mathcal{N}$ since $|\mathcal{I}' \ominus \mathcal{J}'| < |\mathcal{I}_0 \ominus \mathcal{J}_0|$. It is easy to see that these $\mathcal{I}'$ and $\mathcal{J}'$ always exist. Indeed, since *DL-Lite*^{pr} is a sub-language of *DL-Lite*$_{core}$ and *DL-Lite*$_{core}$ enjoys the final model property, one can choose finite models $\mathcal{I}'$ and $\mathcal{J}'$ (for them $\mathcal{I}' \ominus \mathcal{J}'$ is clearly a finite set). Then, $|\mathcal{I}' \ominus \mathcal{J}'| \leqslant |\mathcal{I}' \cup \mathcal{J}'|$, i.e., $|\mathcal{I}' \ominus \mathcal{J}'|$ is finite. $\square$

Appendix E. Proofs for Section 5.1

Proof of Proposition 5.4. Recall that $\mathsf{func}(R, c) = \forall x \forall y.(R(x, c) \wedge R(x, y) \to y = c)$. The "$\Leftarrow$" direction is trivial.

Now we show the "$\Rightarrow$" direction. Assume there is a *DL-Lite*$_{core}$ KB $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ such that $\mathcal{K} \models \mathsf{func}(R, c)$, but $\mathcal{K} \not\models \neg \exists R^-(c)$. Let a and b be constants *not* occurring in $\mathcal{K}$. Consider $\mathcal{A}' = \mathcal{A} \cup \{R(a, c), R(a, b)\}$. We now show that $\mathcal{A}'$ satisfies all the NIs of $\mathsf{cl}(\mathcal{T})$. If this is the case, then, due to Lemma 12 of [9], this observation gives that $\mathcal{T} \cup \mathcal{A}'$ is satisfiable. If in this case we consider a model $\mathcal{I}$ of $\mathcal{T} \cup \mathcal{A}'$, then it holds that $\mathcal{I} \models \mathcal{A}$ since $\mathcal{I} \models \mathcal{A}'$ and $\mathcal{A} \subseteq \mathcal{A}'$, and therefore $\mathcal{I} \models \mathcal{K}$ while $\mathcal{I} \models \{R(a, c), R(a, b)\}$, i.e., $\mathcal{I}$ does not satisfy $\mathsf{func}(R, c)$. This contradicts the fact that $\mathcal{K} \models \mathsf{func}(R, c)$.

So, it remains to show that $\mathcal{A}'$ satisfies all the NIs of $\mathsf{cl}(\mathcal{T})$. Assume there is an NI $\alpha \in \mathsf{cl}(\mathcal{T})$ such that $\mathcal{A}'$ does not satisfy α. Then, there are two MAs f and g in $\mathsf{cl}_{\mathcal{T}}(\mathcal{A}')$ such that $\{f, g\} \models_{\{\alpha\}} \perp$. Four cases are possible:

(i) $\{f, g\} \subseteq \{R(a, c), R(a, b)\}$. One possibility of $\{f, g\} \subseteq \{R(a, c), R(a, b)\}$ is when $f = R(a, c)$ and $g = R(a, b)$. Then, $\alpha = \exists R \sqsubseteq \neg \exists R$, which contradicts the coherency of $\mathcal{K}$. The other possibility of $\{f, g\} \subseteq \{R(a, c), R(a, b)\}$ is analogous.
(ii) $f \in \mathcal{A}$ and $g \in \{R(a, c), R(a, b)\}$. Let $g = R(a, b)$. Let $f' \to \neg g$ instantiate α, where $f' = f$ if $f \in \{A(x), P(x, y)\}$ or $f' = Q(x, z)$ if $f = \exists Q(x)$ for $x, y \in \mathsf{adom}(\mathcal{K})$. Then, f and g should share at least one constant. Since $f \in \mathcal{A}$ and neither a nor b occurs in $\mathcal{A}$, this constant is c. Hence, $g \neq R(a, b)$ and therefore $g = R(a, c)$. Thus, α is of the form $B \sqsubseteq \neg \exists R^-$, and $B(c) \in \mathcal{A}$. Thus, $\mathcal{K} \models \neg \exists R^-(c)$, which contradicts the assumptions of the proposition on $\mathcal{K}$.
(iii) $g \in \mathcal{A}$ and $f \in \{R(a, c), R(a, b)\}$. Analogous to Case (ii).
(iv) $\{f, g\} \in \mathcal{A}$. $\mathcal{A} \not\models \alpha$ and due to Lemma 12 of [9] $\mathcal{T} \cup \mathcal{A}$ is unsatisfiable, which contradicts the satisfiability of $\mathcal{K}$. $\square$

Appendix F. Proofs for Section 5.3

Proof of Lemma 5.17. First, we show that $\mathcal{I}_0 \models \mathcal{T} \cup \mathcal{N}$. Indeed, $\mathcal{I}_0 \models \mathcal{A}$ follows from the definition of chase and Eq. (16). To see that $\mathcal{I}_0 \models \mathcal{T}$, observe that, by the definition of $\mathcal{A}_1$, the set $\mathcal{A} \cup \{R_a(a, b_a)\}$ satisfies all the NIs in $\mathsf{cl}(\mathcal{T})$. Moreover, $\{R_a(a, b_a), R_{a'}(a', b_{a'})\}$, where a and a' are such that $A(a)$ and $A'(a')$ are in $\mathcal{A}_1$ for some concepts A and A', satisfies all the NIs in $\mathsf{cl}(\mathcal{T})$. Thus, the set of MAs that is chased in Eq. (16) satisfies all the NIs in $\mathsf{cl}(\mathcal{T})$. Hence, due to Lemma 12 of [9], $\mathcal{I}_0 \models \mathcal{T}$.

Now we show that $\mathcal{I}[\mathcal{D}, \mathcal{R}, \mathcal{B}] \models \mathcal{T} \cup \mathcal{N}$. The fact that $\mathcal{I}[\mathcal{D}, \mathcal{R}, \mathcal{B}] \models \mathcal{A}$ trivially follows from the fact that $\mathcal{I}_{\mathsf{can}} \models \mathcal{A}$. To see that $\mathcal{I}[\mathcal{D}, \mathcal{R}, \mathcal{B}] \models \mathcal{T}$, observe that for each i the set $\{R_i(x_i, d), A_i(x_i)\}$ satisfies all the NIs in $\mathsf{cl}(\mathcal{T})$, so does its chase with $\mathcal{T}$. Since x_is are fresh and $(\mathcal{K}, \mathcal{N})$ is a simple *DL-Lite*$_{core}$-evolution setting, for any $g_1 \in \mathsf{chase}_{\mathcal{T}}(\{R_i(x_i, d), A_i(x_i)\})$ and $g_2 \in \mathsf{chase}_{\mathcal{T}}(\{R_j(x_j, d), A_j(x_j)\})$, it holds that $\{g_1, g_2\} \not\models_{\mathcal{T}} \perp$. Therefore, we can apply Proposition 3.3 to the union of chases over $1 \leqslant i \leqslant |\mathcal{D}|$ and conclude that it satisfies $\mathcal{T}$. Clearly, for each $g_1 \in \mathcal{I}_{\mathsf{can}}$ and g_2 in the union of chases, $\{g_1, g_2\} \not\models_{\mathcal{T}} \perp$ and again, by applying Proposition 3.3, we conclude that $\mathcal{I}[\mathcal{D}, \mathcal{R}, \mathcal{B}] \models \mathcal{T}$.

The proof of $\mathcal{J}_0 \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}_0, \mathcal{T}, \mathcal{N})$ and $\mathcal{J}[\mathcal{D}, \mathcal{R}, \mathcal{B}] \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}[\mathcal{D}, \mathcal{R}, \mathcal{B}], \mathcal{T}, \mathcal{N})$ is straightforward by the definition of BZP and BP procedures. $\square$

Appendix G. Proofs for Section 5.4

In order to prove Lemma 5.18, we show the following two technical propositions.

Proposition G.1. *Let $n \geqslant 2$ be a natural number, $\alpha_1 = B_1 \sqsubseteq B'_1, \ldots, \alpha_n = B_n \sqsubseteq B'_n$ DL-Lite$_{core}$ PIs, $\beta = B \sqsubseteq \neg B'$ a DL-Lite$_{core}$ NI, and $g_0, \ldots, g_n, f$ a sequence of ground atoms such that $g_{i-1} \to g_i$ for $1 \leqslant i \leqslant n$ instantiates α_i, and $g_n \to \neg f$ instantiates β. If $B_1 \sqsubseteq \neg B' \notin \mathsf{cl}(\{\alpha_1, \ldots, \alpha_n, \beta\})$, then at least one of the following conditions holds*

(i) *there is* $1 \leqslant i \leqslant n-1$ *such that* α_i *is* $B_i \sqsubseteq \exists R$ *and* α_{i+1} *is* $\exists R^- \sqsubseteq B'_{i+1}$, *where* $R \in \Sigma(\mathcal{T})$.
(ii) $B'_n = \exists R$ *and* $B = \exists R^-$, *where* $R \in \Sigma(\mathcal{T})$.

Proof. We prove it by induction on n. Assume $n = 1$; thus, g_1 instantiates B'_1 and B, and therefore B'_1 and B share the predicate symbol (concept or role name). Case (i) is not applicable, since there is only one α_i. Assume that Case (ii) does not hold. Then, one of the following options holds: (1) either B'_1 is (of the form) $\exists R$ and B is $\exists R$, or (2) B'_1 is A and B is A. From either case we conclude that $B_1 \sqsubseteq \neg B' \in \mathsf{cl}(\{\alpha_1, \beta\})$, which contradicts the assumption of the proposition.

Assume $n > 1$. If Case (i) does not hold, then consider α_i and α_{i+1} for some $i < n$. Since g_i instantiates B'_i and B_{i+1}, they share the predicate symbol, thus one of the following options hold: (1) either B'_i is $\exists R$ and B_{i+1} is $\exists R$ or (2) B'_i is A and B_{i+1} is A. Thus, as in the case above, we conclude that $B_i \sqsubseteq B'_{i+1} \in \mathsf{cl}(\{\alpha_i, \alpha_{i+1}\})$. Applying this argument iteratively to pairs of α_i and α_{i+1} from $i = 1, \dots, n-1$, we obtain that $B_1 \sqsubseteq B'_n \in \{\alpha_1, \dots, \alpha_n\}$. If Case (ii) does not hold, then, using the same argument as in the case of $n = 1$ to $B_1 \sqsubseteq B'_n$ and $B \sqsubseteq B'$, we obtain that $B_1 \sqsubseteq \neg B' \in \{\alpha_1, \dots, \alpha_n, \beta\}$, which contradicts the assumption of the proposition. □

Proposition G.2. *Let* $(\mathcal{K}, \mathcal{N})$, *where* $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, *be a simple DL-Lite*$_{core}$*-evolution setting,* $\mathcal{I} \models \mathcal{K}$, *and* $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$. *If* $\mathcal{N} \models_{\mathcal{T}} \exists R(a)$, $\mathcal{N} \not\models_{\mathcal{T}} R(a, b)$, *and there is an NI* $\alpha \in \mathsf{cl}(\mathcal{T})$ *such that* $\mathcal{I} \cup \{R(a, b)\} \not\models \alpha$, *then* $R(a, b) \notin \mathcal{J}$.

Proof. Assume $R(a, b) \in \mathcal{J}$, then consider $\mathcal{J}' = \mathcal{J} \setminus \{R(a, b)\}$. We now show that $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ and $\mathcal{I} \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J}$, thus contradicting the fact that $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$. Since $\mathcal{N} \not\models_{\mathcal{T}} R(a, b)$, it clearly holds that $\mathcal{J}' \models \mathcal{N}$. Since $\mathcal{N} \models_{\mathcal{T}} \exists R(a)$ and the evolution setting is simple, there is c such that $\mathcal{N} \models_{\mathcal{T}} R(a, c)$ Thus, $R(a, c) \in \mathcal{J}'$ and therefore $\mathcal{J}'$ satisfies PIs of $\mathsf{cl}(\mathcal{T})$. Since $\mathcal{J}' \subseteq \mathcal{J}$ and $\mathcal{J}$ satisfies NIs of $\mathsf{cl}(\mathcal{T})$, so does $\mathcal{J}'$. We conclude that $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$.

Finally, observe that the fact that $\mathcal{I} \cup \{R(a, b)\} \not\models \alpha$ implies that $R(a, b) \notin \mathcal{I}$. Taking into account that $R(a, b) \notin \mathcal{J}'$, we conclude that $R(a, b) \notin \mathcal{I} \ominus \mathcal{J}'$. At the same time $R(a, b) \in \mathcal{J}$ and therefore $R(a, b) \in \mathcal{I} \ominus \mathcal{J}$. Thus, $\mathcal{I} \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J}$ holds and we conclude the proof. □

Proof of Lemma 5.18. Let $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ and $\mathcal{S} = \mathsf{Align}_{\mathcal{T}}((\mathcal{I} \setminus \mathcal{B}_{\mathcal{I}}), \mathcal{N})$. Assume there exists a model $\mathcal{J}' \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$ such that $\mathcal{S} \nsubseteq \mathcal{J}'$. Consider $\mathcal{J}'' = \mathcal{J}' \cup \mathcal{S}$. We will show that $\mathcal{J}'' \models \mathcal{T} \cup \mathcal{N}$ and $\mathcal{I} \ominus \mathcal{J}'' \subsetneq \mathcal{I} \ominus \mathcal{J}'$ which yields a contradiction with $\mathcal{J}' \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$.

To see that $\mathcal{I} \ominus \mathcal{J}'' \subsetneq \mathcal{I} \ominus \mathcal{J}'$ observe the following:

$$\begin{aligned}
\mathcal{I} \ominus \mathcal{J}'' &= \big(\mathcal{I} \setminus \mathcal{J}''\big) \cup \big(\mathcal{J}'' \setminus \mathcal{I}\big) \\
&= \big(\big(\mathcal{I} \setminus \mathcal{J}'\big) \setminus \mathcal{S}\big) \cup \big(\big(\mathcal{J}' \setminus \mathcal{I}\big) \cup (\mathcal{S} \setminus \mathcal{I})\big) \\
&\subsetneq_{\text{due to } \mathcal{S} \subseteq \mathcal{I} \text{ and } \mathcal{S} \nsubseteq \mathcal{J}'} \big(\mathcal{I} \setminus \mathcal{J}'\big) \cup \big(\mathcal{J}' \setminus \mathcal{I}\big) \cup (\mathcal{S} \setminus \mathcal{I}) \\
&=_{\text{due to } \mathcal{S} \subseteq \mathcal{I}} \big(\mathcal{I} \setminus \mathcal{J}'\big) \cup \big(\mathcal{J}' \setminus \mathcal{I}\big) = \mathcal{I} \ominus \mathcal{J}'.
\end{aligned}$$

It remains to show that $\mathcal{J}'' \models \mathcal{T} \cup \mathcal{N}$. The fact that $\mathcal{J}'' \models \mathcal{N}$ follows trivially from the fact that $\mathcal{J}' \in \mathsf{Mod}(\mathcal{N})$, $\mathcal{J}' \subseteq \mathcal{J}''$ and $\mathcal{N}$ does not contain negative MAs. We now prove that $\mathcal{J}'' \models \mathcal{T}$ by showing that both $\mathcal{S}$ and $\mathcal{J}'$ are models of $\mathcal{T}$ and then by applying Proposition 3.3. Obviously, $\mathcal{J}' \models \mathcal{T}$ holds by the definition of $\mathcal{J}'$. Observe that by the definition of alignment:

$$\mathcal{S} = (\mathcal{I} \setminus \mathcal{B}_{\mathcal{I}}) \setminus \bigcup_{g \in \mathcal{I} \setminus \mathcal{B}_{\mathcal{I}} \text{ s.t. } \{g\} \cup \mathcal{N} \models_{\mathcal{T}} \bot} \mathsf{root}_{\mathcal{T}}(g).$$

Since $\mathcal{I} \models \mathcal{T}$, one can show that $\mathcal{S} \models \mathcal{T}$ by applying Proposition 3.6 a necessary (probably infinite) number of times: first to $\mathcal{B}_{\mathcal{I}}$ and then to each $g \in \mathcal{I} \setminus \mathcal{B}_{\mathcal{I}}$ s.t. $\{g\} \cup \mathcal{N} \models_{\mathcal{T}} \bot$.

Since we proved that $\mathcal{S} \models \mathcal{T}$ and $\mathcal{J}' \models \mathcal{T}$ we can apply Proposition 3.3, that is, $\mathcal{S} \cup \mathcal{J}' \models \mathcal{T}$ if for every $f \in \mathcal{S}$ and $g \in \mathcal{J}'$ it holds: $\{f, g\} \not\models_{\mathcal{T}} \bot$. Assume this is not the case, and there are $f \in \mathcal{S}$ and $g \in \mathcal{J}'$ such that $\{f, g\} \models_{\mathcal{T}} \bot$. Let

- $\mathcal{G}$ be the set of atoms g of $\mathcal{J}'$ such that $\{f, g\} \models_{\mathcal{T}} \bot$ for some $f \in \mathcal{S}$ and $\mathsf{root}^{\mathcal{J}'}_{\mathcal{T}}(g) \cap \mathcal{N} = \emptyset$, and
- $\mathcal{H}$ be the set of atoms g of $\mathcal{J}'$ such that $\{f, g\} \models_{\mathcal{T}} \bot$ for some $f \in \mathcal{S}$ and $\mathsf{root}^{\mathcal{J}'}_{\mathcal{T}}(g) \cap \mathcal{N} \neq \emptyset$.

By our assumption, $\mathcal{H} \cup \mathcal{G} \neq \emptyset$. Note that it is enough to consider only the case when f is a unary atom. Indeed, if f is binary, i.e., if $f = R(a, b)$, then, due to the fact that in *DL-Lite*$_{core}$ disjointness is allowed between basic concepts only, $\{R(a, b), g\} \models_{\mathcal{T}} \bot$ holds if and only if either $\{\exists R(a), g\} \models_{\mathcal{T}} \bot$ or $\{\exists R^-(b), g\} \models_{\mathcal{T}} \bot$. If the first case holds, then we can introduce a fresh concept name $A_{\exists R}$ and extend $\mathcal{I}$ by assigning $A^{\mathcal{I}}_{\exists R} = (\exists R)^{\mathcal{I}}$ to be the interpretation of $R^{\mathcal{I}}$ projected on the first coordinate. Then, both the original $\mathcal{I}$ and the extended one will behave equivalently w.r.t. to the proposition.

We first show that $\mathcal{H} = \emptyset$. Assume this is not the case and there is $g \in \mathcal{H}$. Let $g' \in \mathsf{root}^{\mathcal{J}'}_{\mathcal{T}}(g) \cap \mathcal{N}$. By the definition of $\mathsf{root}_{\mathcal{T}}$ for models, there is a sequence of PIs $\alpha_1 = B_1 \sqsubseteq B'_1, \dots, \alpha_n = B_n \sqsubseteq B'_n$ in $\mathsf{cl}(\mathcal{T})$ and of atoms $g_0, \dots, g_n$ in $\mathsf{root}^{\mathcal{J}'}_{\mathcal{T}}(g)$ such that $g_0 = g'$, $g_n = g$ and $g_{i-1} \to g_i$ for $1 \leqslant i \leqslant n$ instantiates α_i. From $\{f, g\} \models_{\mathcal{T}} \bot$ and Lemma 12 of [9] it follows that there is an NI $\beta = B \sqsubseteq \neg B'$ in $\mathsf{cl}(\mathcal{T})$ such that $g_n \to \neg f$ instantiates β. We now show that $B_1 \sqsubseteq \neg B' \notin \mathsf{cl}(\{\alpha_1, \ldots, \alpha_n\})$ holds and then apply Proposition G.1. Assume $B_1 \sqsubseteq \neg B' \in \mathsf{cl}(\{\alpha_1, \ldots, \alpha_n\})$, then it holds that $\mathcal{T} \models B_1 \sqsubseteq \neg B'$. Moreover, $g_0 \to \neg f$ instantiates $B_1 \sqsubseteq \neg B'$ and therefore, $\{f, g_0\} \models_{\{B_1 \sqsubseteq \neg B'\}} \bot$. Combining $\{f, g_0\} \models_{\{B_1 \sqsubseteq \neg B'\}} \bot$ and $\mathcal{T} \models B_1 \sqsubseteq \neg B'$, we conclude that $\{f, g_0\} \models_{\mathcal{T}} \bot$. Taking into account that $g_0 \in \mathcal{N}$, we finally conclude that $\{f\} \cup \mathcal{N} \models_{\mathcal{T}} \bot$. Therefore, $f \notin \mathcal{S}$ which gives a contradiction with $f \in \mathcal{S}$. Thus, we can apply Proposition G.1.

Assume that Case (ii) of Proposition G.1 holds, that is $\alpha_n = B_n \sqsubseteq \exists R$ and $\beta = \exists R^- \sqsubseteq \neg B'$. In particular, this means that g is of the form $R(x, y)$. Since the evolution setting is simple, $\mathcal{T} \not\models \exists R_1 \sqsubseteq \exists R$ for any role R_1. Combining this with $\alpha_n = B_n \sqsubseteq \exists R$, we obtain that B_n and all B_i and B'_i occurring in α_i for $1 \leqslant i \leqslant n-1$ are atomic concepts, say A_i and A'_i, respectively. Indeed, let B'_k be of the form $\exists R_1^-$ and has the highest index among B_is with this property. Then, $\mathcal{T} \models \exists R_1^- \sqsubseteq \exists R$, which contradicts the fact that $\mathcal{K}$, $\mathcal{N}$ is a simple setting. This implies that $\{\alpha_1, \ldots, \alpha_n\} \models_{\mathcal{T}} A_1 \sqsubseteq \exists R$. Combining this with the our assumption that g' instantiates A_1 and $g = R(x, y)$, we obtain that $g' = A(x)$ and $A_1(x) \to R(x, y)$ instantiates $A_1 \sqsubseteq \exists R$. Since $g' \in \mathcal{N}$, we conclude that $\mathcal{N} \models_{\mathcal{T}} \exists R(x)$. Recall that $\{f, R(x, y)\} \models_{\mathcal{T}} \bot$ and $f \in \mathcal{I}$ (since $f \in \mathcal{S}$ and $\mathcal{S} \subseteq \mathcal{I}$), thus $\mathcal{I} \cup \{R(x, y)\}$ does not satisfy at least one NI of $\mathsf{cl}(\mathcal{T})$. Now if $\mathcal{N} \not\models_{\mathcal{T}} R(x, y)$ holds, then we are in the conditions of Proposition G.2 and can conclude that $R(x, y) \notin \mathcal{J}'$, which contradicts the fact that $R(x, y) = g \in \mathcal{J}'$. Therefore, $\mathcal{N} \models_{\mathcal{T}} R(x, y)$. Combining this with $\{f, R(x, y)\} \models_{\mathcal{T}} \bot$, we obtain that $\{f\} \cup \mathcal{N} \models_{\mathcal{T}} \bot$. Since f is unary we conclude that $f \in \mathcal{B}_{\mathcal{I}}$ which contradicts the fact that $f \in \mathcal{S}$.

Assume that Case (i) of Proposition G.1 holds but Case (ii) does not. Then, let k be the maximal index satisfying that α_k and α_{k+1} are respectively of the form $B_k \sqsubseteq \exists R$ and $\exists R^- \sqsubseteq B'_{k+1}$. If $k = n-1$, then $\alpha_n = \exists R^- \sqsubseteq B'_n$. Moreover, since Case (ii) of Proposition G.1 does *not* hold and in the evolution settings the entailment $\mathcal{T} \models \exists R^- \sqsubseteq \exists R'$ is not possible for any role R', we have that $B_i = A_i$ and $B'_i = A'_i$ (in fact, it even holds that $B_i = A_i$ and $B'_i = A_{i+1}$) for $1 \leqslant i \leqslant n-2$, $B_{n-1} = A_{n-1}$, $B'_n = A$, $B = A$ and $B' = A'$, where all A_j, A'_j and A, A' are from $\Sigma(\mathcal{T} \cup \mathcal{N})$. Thus,

$$\alpha_{n-1} = A_{n-1} \sqsubseteq \exists R, \qquad \alpha_n = \exists R^- \sqsubseteq A, \qquad g_0 = A_1(x), \qquad g_{n-1} = R(x, y), \qquad g_n = A(y) \quad \text{and} \quad f = A'(y).$$

If $\mathcal{N} \models_{\mathcal{T}} R(x, y)$, then $\mathcal{N} \models_{\mathcal{T}} A'(y)$, and we obtain that $\{f\} \cup \mathcal{N} \models_{\mathcal{T}} \bot$, which contradicts $f \in \mathcal{S}$. If $\mathcal{N} \not\models_{\mathcal{T}} R(x, y)$, then due to $R(x, y) \models_{\{\alpha_n\}} A(y)$ and the fact that $\{A(y), A'(y)\}$ violates β, we conclude that $\{R(x, y), A'(y)\}$ violates $\exists R^- \sqsubseteq \neg A' \in \mathsf{cl}(\mathcal{T})$. Thus, $\mathcal{I} \cup \{R(x, y)\}$ violates $\exists R^- \sqsubseteq \neg A'$ and we can apply Proposition G.2 to conclude that $R(x, y) \notin \mathcal{J}'$, which contradicts the fact that $R(x, y) \in \mathsf{root}^{\mathcal{J}'}_{\mathcal{T}}(g)$. If $1 \leqslant k < n-1$, then analogously to the previous case we can show that $B'_k = \exists R$ $B_{k+1} = \exists R^-$, for each $1 \leqslant i \leqslant k-2$ and $k+1 \leqslant i \leqslant n$ it holds that $B_i = A_i$ and $B'_i = A'_i$ (in fact, it even holds that $B'_i = A_{i+1}$) and also $B = A'_n$ and $B' = A'$, where all A_j, A'_j, A' and R are from $\Sigma(\mathcal{T} \cup \mathcal{N})$. Moreover,

$$\alpha_k = A_k \sqsubseteq \exists R, \qquad \alpha_{k+1} = \exists R^- \sqsubseteq A_{k+2}, \qquad g_0 = A_1(x), \qquad g_k = R(x, y), \qquad g_n = A_n(y) \quad \text{and} \quad f = A'(y).$$

Thus, applying the same reasoning as above we obtain a contradiction either with $f \in \mathcal{S}$ or $g \in \mathcal{J}'$. We conclude that $\mathcal{H} = \emptyset$ and $\mathcal{G} \neq \emptyset$.

Now consider

$$\hat{\mathcal{J}}' = \mathcal{J}' \setminus \bigcup_{g \in \mathcal{G}} \mathsf{root}_{\mathcal{T}}(g) \cup \bigcup_{h \in \mathsf{root}^{\mathcal{J}'}_{\mathcal{T}}(g) \text{ s.t. } g \in \mathcal{G},\, h \in \mathcal{S}} \mathcal{S}[h].$$

We now show that $\hat{\mathcal{J}}' \ominus \mathcal{I} \subsetneq \mathcal{J}' \ominus \mathcal{I}$ and $\hat{\mathcal{J}}' \models \mathcal{T} \cup \mathcal{N}$, which contradicts the fact that $\mathcal{J}' \in \mathcal{I} \diamond \mathcal{N}$ under $\mathbf{L}^a_{\subseteq}$. The inclusion $\hat{\mathcal{J}}' \ominus \mathcal{I} \subseteq \mathcal{J}' \ominus \mathcal{I}$ follows from the fact that each $\mathcal{S}[h] \subseteq \mathcal{I}$. The inclusion is strict since $\mathcal{G} \neq \emptyset$, $\mathcal{G} \subseteq \mathcal{J}'$, and $\mathcal{G} \cap \mathcal{I} = \emptyset$. Since for each $g \in \mathcal{G}$ it holds that $\mathsf{root}^{\mathcal{J}'}_{\mathcal{T}}(g) \cap \mathcal{N} = \emptyset$, we have that $\hat{\mathcal{J}}' \models \mathcal{N}$. To see that $\hat{\mathcal{J}}' \models \mathcal{T}$, observe that $\mathcal{J}' \setminus \bigcup_{g \in \mathcal{G}} \mathsf{root}_{\mathcal{T}}(g) \models \mathcal{T}$ due to Proposition 3.6, and clearly $\bigcup_{h \in \mathsf{root}^{\mathcal{J}'}_{\mathcal{T}}(g) \text{ s.t. } h \in \mathcal{S}} \mathcal{S}[h] \models \mathcal{T}$. Therefore, we can apply Proposition 3.3: assume there is $g' \in \mathcal{J}' \setminus \bigcup_{g \in \mathcal{G}} \mathsf{root}_{\mathcal{T}}(g)$ and $f' \in \mathcal{S}[h]$ for some h such that $\{g', f'\} \models_{\mathcal{T}} \bot$. Since $g' \in \mathcal{J}'$ and $f' \in \mathcal{S}$, one of the two options should hold: either $g' \in \mathcal{G}$ or $g' \in \mathcal{H}$. The first option is impossible since $g' \in \mathcal{J}' \setminus \bigcup_{g \in \mathcal{G}} \mathsf{root}_{\mathcal{T}}(g)$, and therefore $g' \notin \mathcal{G}$. The second option is also impossible since $\mathcal{H} = \emptyset$. Thus, $\hat{\mathcal{J}}' \models \mathcal{T}$, we obtain a contradiction with $\mathcal{J} \in \mathcal{I} \diamond \mathcal{N}$ under $\mathbf{L}^a_{\subseteq}$, hence $\mathcal{G} = \emptyset$ and we conclude the proof. □

In order to prove Lemma 5.20, we need the following technical property.

Proposition G.3. *Let $\mathcal{E} = (\mathcal{K}, \mathcal{N})$ be a simple DL-Lite$_{core}$-evolution setting, where $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$. Let $\mathcal{I} \models \mathcal{K}$ and $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$. Then,*

(i) *if $\mathcal{N} \not\models_{\mathcal{T}} A(a)$ and there is an NI $\alpha \in \mathsf{cl}(\mathcal{T})$ such that $\mathcal{I} \cup \{A(a)\} \not\models \alpha$, then $A(a) \notin \mathcal{J}$;*
(ii) *if $\mathcal{N} \not\models_{\mathcal{T}} \exists R(a)$, $\mathcal{N} \not\models_{\mathcal{T}} \exists R^-(b)$ and there is an NI $\alpha \in \mathsf{cl}(\mathcal{T})$ s.t. $\mathcal{I} \cup \{R(a, b)\} \not\models \alpha$, then $R(a, b) \notin \mathcal{J}$.*

Proof. Analogous to the proof of Proposition G.2. □

Proof of Lemma 5.20. Case (i): Let $\mathcal{I} \models \mathcal{K}$ be such that $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$. Assume that $D(c) \notin \mathcal{J}$, and

$$\{R(x,c), A(x)\} \nsubseteq \mathcal{J} \quad \text{for each } x \in \Delta,\ \text{each } R \in \mathsf{TR} \text{ and each atomic concept } A \text{ such that}$$
$$D(c) \in \mathsf{DjnAts}[\mathcal{K}, \mathcal{N}](R) \text{ and } A \in \mathtt{ISubCon}[\mathcal{T}](\exists R). \tag{G.1}$$

Observe that the condition $\{R(x,c), A(x)\} \nsubseteq \mathcal{J}$ is satisfied when $R(x,c) \in \mathcal{J}$ and $A(x) \notin \mathcal{J}$. Let

$$\mathcal{D} = \{R(x,c) \mid x \in \Delta,\ R(x,c) \in \mathcal{J},\ D(c) \in \mathsf{DjnAts}[\mathcal{K}, \mathcal{N}](R)\}.$$

Assume $\mathcal{D} \neq \emptyset$. Consider $\mathsf{root}^{\mathcal{J}}_{\mathcal{T}}(R(x,c))$. Due to the assumption in Eq. (G.1), there are no unary atoms in $\mathsf{root}^{\mathcal{J}}_{\mathcal{T}}(R(x,c))$. Moreover, since $(\mathcal{K}, \mathcal{N})$ is a simple evolution setting there are no binary atoms in $\mathsf{root}^{\mathcal{J}}_{\mathcal{T}}(R(x,c))$ besides $R(x,c)$. Thus, $\mathsf{root}^{\mathcal{J}}_{\mathcal{T}}(R(x,c)) = \{R(x,c)\}$. Consider a model

$$\mathcal{J}' = \mathcal{J} \setminus \bigcup_{R(x,c) \in \mathcal{D}} \{R(x,c)\}.$$

We now show that $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ and $\mathcal{I} \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J}$, which contradicts the fact that $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$. Observe that $\mathcal{J}' \models \mathcal{N}$. Assume this is not the case and there is an MA g such that $\mathcal{N} \models g$ and $\mathcal{J}' \not\models g$. We have two cases here.

- Assume that g is a positive MA. Since $\mathcal{J} \models \mathcal{N}$, we conclude that $g \in \bigcup_{R(x,c) \in \mathcal{D}} \{R(x,c)\}$. Therefore, $g = R(x,c)$ for some $R(x,c) \in \mathcal{D}$, we have that $R(x,c) \in \mathcal{N}$. Combining this with the fact that $\{D(c), \exists R^-(c)\} \models_{\mathcal{T}} \bot$ we conclude that $\{D(c)\} \cup \mathcal{N} \models_{\mathcal{T}} \bot$. This contradicts the fact that $\mathcal{N} \not\models_{\mathcal{T}} D(c)$. Thus, $\mathcal{J}' \models \mathcal{N}$.
- Assume that g is a negative MA. Since $\mathcal{J} \models \mathcal{N}$ and $\mathcal{J}' \subseteq \mathcal{J}$, it trivially holds that $\mathcal{J}' \models \mathcal{N}$.

Due to Proposition 3.6, $\mathcal{J}' \models \mathcal{T}$. Therefore, $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$. By construction of $\mathcal{J}'$, we have that every $R(x,c)$ from $\mathcal{D}$ is *not* in $\mathcal{I}$ and $\mathcal{J}'$, while it is in $\mathcal{J}$. Therefore, $\mathcal{I} \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J}$ and we conclude the proof.

Assume $\mathcal{D} = \emptyset$, then consider

$$\mathcal{J}' = \mathcal{J} \cup \mathcal{I}[D(c)].^{10}$$

Again, we now show that $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ and $\mathcal{I} \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J}$, which contradicts the fact that $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$. The inclusion $\mathcal{I} \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J}$ follows from $\mathcal{I}[D(c)] \subseteq \mathcal{I}$, and $\mathcal{I}[D(c)] \subseteq \mathcal{J}'$, and $D(c) \notin \mathcal{J}$. Then, $\mathcal{J}' \models \mathcal{N}$ follows from $\mathcal{J} \models \mathcal{N}$. It remains to show that $\mathcal{J}' \models \mathcal{T}$ holds, and we proceed to a proof of this entailment.

Clearly, both $\mathcal{J}$ and $\mathcal{I}[D(c)]$ satisfy $\mathcal{T}$. Due to Proposition 3.3, to finish the proof of $\mathcal{J}' \models \mathcal{T}$, it remains to show that for every $g_1 \in \mathcal{J}$ and $g_2 \in \mathcal{I}[D(c)]$ it holds that $\{g_1, g_2\} \not\models_{\mathcal{T}} \bot$. Assume this is not the case and there are $g_1 \in \mathcal{J}$ and $g_2 \in \mathcal{I}[D(c)]$ such that $\{g_1, g_2\} \models_{\mathcal{T}} \bot$. Then, there is an NI α in $\mathsf{cl}(\mathcal{T})$ such that $g_1 \to \neg g_2$ is an instantiation of the first-order interpretation of α. Observe that $g_2 \neq D(c)$. Indeed, if $g_2 = D(c)$, then, since $D(c)$ is a unary atom, α is of the form $D \sqsubseteq \neg B$.

(i) If $B = A'$ for some atomic concept A', then $g_1 = A'(c) \in \mathcal{J}$. If $\mathcal{N} \models_{\mathcal{T}} A'(c)$, then $\mathcal{N} \models_{\mathcal{T}} \neg D(c)$, which contradicts the fact that $\mathcal{N} \not\models_{\mathcal{T}} \neg D(c)$. If $\mathcal{N} \not\models_{\mathcal{T}} A'(c)$, then, due to Case (i) of Proposition G.3 and the facts that $D(c) \in \mathcal{I}$ and $\{D(c), A'(c)\} \models_{\mathcal{T}} \bot$, we conclude that $A'(c) \notin \mathcal{J}$, which gives a contradiction.

(ii) If $B = \exists R_1$ for some role R_1, then $g_1 = R_1(c,y) \in \mathcal{J}$ for some $y \in \Delta$. Assume that $\mathcal{N} \models_{\mathcal{T}} R_1(c,y)$, then $\mathcal{N} \models_{\mathcal{T}} \neg D(c)$, which gives a contradiction. If $\mathcal{N} \not\models_{\mathcal{T}} R_1(c,y)$, and also $\mathcal{N} \not\models_{\mathcal{T}} \exists R_1(c)$, $\mathcal{N} \not\models_{\mathcal{T}} \exists R_1(y)$, then, due to Case (ii) of Proposition G.3 and the facts that $D(c) \in \mathcal{I}$ and $\{D(c), R'(c,y)\} \models_{\mathcal{T}} \bot$, we conclude that $R'(c,y) \notin \mathcal{J}$ and again obtain a contradiction. If $\mathcal{N} \not\models_{\mathcal{T}} R_1(c,y)$ and either $\mathcal{N} \models_{\mathcal{T}} \exists R_1(c)$ or $\mathcal{N} \models_{\mathcal{T}} \exists R_1^-(y)$ holds, then, due to Proposition G.2 and the fact that $D(c) \in \mathcal{I}$ and $\{D(c), R'(c,y)\} \models_{\mathcal{T}} \bot$, one can conclude that $R'(c,y) \notin \mathcal{J}'$, thus we obtain a contradiction.

Since $(\mathcal{K}, \mathcal{N})$ is a simple evolution setting, every $g' \in \mathcal{I}[D(c)]$ is unary[11] and $\mathcal{D}(c) \models_{\mathcal{T}} g'$. Thus, we can apply to such g' the same argument as to $D(c)$ above to obtain a contradiction. Thus, due to Proposition 3.3, we conclude that $\mathcal{J} \cup \mathcal{I}[D(c)] \models \mathcal{T}$. This implies that $\mathcal{J} \notin \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$, yields a contradiction, and concludes the proof of Case (i).

Case (ii): Assume there is $D(c) \in \mathcal{J}$ and a unary MA $A(c)$ satisfying $\mathcal{K} \models A(c)$, $\mathcal{T} \models A \sqsubseteq D$, and $A(c) \in \mathsf{Align}_{\mathcal{T}}(\mathcal{I}, \mathcal{N})$, while $A(c) \notin \mathcal{J}$ holds. Let $\mathcal{I}$ be a model of $\mathcal{K}$ such that $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$. Consider

$$\mathcal{J}' = \mathcal{J} \cup \mathcal{I}[A(c)].$$

Again, we now show that $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ and $\mathcal{I} \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J}$, which contradicts the fact that $\mathcal{J} \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}, \mathcal{T}, \mathcal{N})$. Observe that $\mathcal{J}' \models \mathcal{N}$, since $\mathcal{J}$ does so. Since both $\mathcal{J}$ and $\mathcal{I}[A(c)]$ satisfy $\mathcal{T}$, then, due to Proposition 3.3 and Lemma 12 of [9], it

10 Recall that $\mathcal{I}[D(c)]$ is a minimal (w.r.t. set inclusion) submodel of $\mathcal{I}$ containing $D(c)$.

11 In particular, Restriction (ii) of Definition 5.11 implies that there is no role Q such that $\mathcal{T} \models \exists Q \sqsubseteq D$.

suffices to show that for every NI $\alpha \in \mathsf{cl}(\mathcal{T})$ and every $g \in \mathcal{J}$ and $f \in \mathcal{I}[A(c)]$, $\{f, g\}$ satisfies α. Assume there is an NI α in $\mathsf{cl}(\mathcal{T})$, $g \in \mathcal{J}$ and $f \in \mathcal{I}[A(c)]$ such that $\{g, f\} \models_{\{\alpha\}} \bot$. If $\mathcal{N} \models_{\mathcal{T}} g$, then $\mathcal{N} \cup \{f\} \models_{\{\alpha\}} \bot$. Thus, $A(c) \in \mathsf{root}^{\mathcal{I}}_{\mathcal{T}}(f)$ and therefore $A(c) \notin \mathsf{Align}_{\mathcal{T}}(\mathcal{I}, \mathcal{N})$ which gives a contradiction. Assume $\mathcal{N} \not\models g$. If g is unary, then we can apply Case (ii) of Proposition G.3 to obtain a contradiction. If g is binary, then, as in the proof of Case (i) of the current proposition, we can apply Case (ii) of Proposition G.3 or Proposition G.2, and obtain a contradiction. □

Appendix H. Proofs for Section 6

Proof of Lemma 6.4. Let $S = \mathbf{L}^a_{\subseteq}$.

First we show the "if" direction. Suppose that $g \in \mathsf{cl}_{\mathcal{T}}(\mathcal{N}) \cup Y$, where

$$Y = \mathsf{AtAlg}(\mathcal{K}, \mathcal{N}) \setminus \bigcup_{R \in \mathsf{TR}} \bigcup_{A(c) \in \mathsf{DjnAts}[\mathcal{K}, \mathcal{N}](R)} \{\neg \exists R^-(c)\}.$$

If $g \in \mathsf{cl}_{\mathcal{T}}(\mathcal{N})$ then, clearly, by the definition of $\mathcal{K} \diamond_S \mathcal{N}$, g is certain. Suppose that

$$g \in Y \setminus \mathsf{cl}_{\mathcal{T}}(\mathcal{N}) \quad \text{and consequently} \quad g \in \mathsf{AtAlg}(\mathcal{K}, \mathcal{N}). \tag{H.1}$$

Assume g is not certain, that is, there is a model $\mathcal{J}_0 \in \mathcal{K} \diamond_S \mathcal{N}$ such that $\mathcal{J}_0 \models \neg g$. Let $\mathcal{I}_0$ be a model of $\mathcal{K}$ such that $\mathcal{J}_0 \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}_0, \mathcal{T}, \mathcal{N})$. We now exhibit a model $\mathcal{J}_0'$ such that $\mathcal{I}_0 \ominus \mathcal{J}_0' \subsetneq \mathcal{I}_0 \ominus \mathcal{J}_0$ which will give us a contradiction with $\mathcal{J}_0 \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}_0, \mathcal{T}, \mathcal{N})$. Consider the following two cases:

- g is of the form $\neg A(c)$, that is, $A(c) \in \mathcal{J}_0$. Now we show that in this case $\mathcal{N} \parallel_{\mathcal{T}} A(c)$, i.e., $\mathcal{N} \not\models_{\mathcal{T}} \neg A(c)$ and $\mathcal{N} \not\models_{\mathcal{T}} A(c)$. Indeed, (a) $\mathcal{N} \not\models_{\mathcal{T}} \neg A(c)$ holds since $\mathcal{J}_0 \models A(c)$ and $\mathcal{J}_0 \models \mathcal{N}$, and (b) $\mathcal{N} \not\models_{\mathcal{T}} A(c)$: suppose by contradiction that $\mathcal{N} \models_{\mathcal{T}} A(c)$, then $\neg A(c) \notin \mathsf{AtAlg}(\mathcal{K}, \mathcal{N})$ by the definition of AtAlg (note that in this case $\mathcal{N} \cup \{\neg A(c)\} \models_{\mathcal{T}} \bot$); on the other hand, $\neg A(c) \in Y$ by Eq. (H.1), that is, $\neg A(c) \in \mathsf{AtAlg}(\mathcal{K}, \mathcal{N})$ and we obtain a contradiction.
  Now consider the following interpretation: $\mathcal{J}_0' = \mathcal{J}_0 \setminus \mathsf{root}_{\mathcal{T}}(A(c))$. Due to $\mathcal{N} \parallel_{\mathcal{T}} \neg A(c)$, we have that $\mathcal{J}_0' \in \mathsf{Mod}(\mathcal{N})$. Also, $\mathcal{J}_0'$ is in $\mathsf{Mod}(\mathcal{T})$ due to Proposition 3.6. It is easy to check that, by the definition of $\mathcal{J}_0'$ and due to the restrictions of the simple evolution settings (in particular, Restriction (ii) of Definition 5.11 yields that $\mathsf{root}_{\mathcal{T}}(A(c))$ consists only of atomic MAs), the following hold: (a) $\mathcal{J}_0$ and $\mathcal{J}_0'$ differ only on finitely many assertions of $\mathsf{root}_{\mathcal{T}} A(c)$, and (b) $\mathsf{root}_{\mathcal{T}} A(c) \nsubseteq \mathcal{I}_0$ since $\neg A(c) \in \mathsf{AtAlg}(\mathcal{K}, \mathcal{N})$, that is, $\mathcal{K} \models \neg A(c)$, and $\mathcal{I}_0 \models \mathcal{K}$. This leads to $\mathcal{I}_0 \ominus \mathcal{J}_0' \subseteq \mathcal{I}_0 \ominus \mathcal{J}_0$. Then, $A(c) \in (\mathcal{I}_0 \ominus \mathcal{J}_0) \setminus (\mathcal{I}_0 \ominus \mathcal{J}_0')$, which means that the inclusion is strict, i.e., $\mathcal{I}_0 \ominus \mathcal{J}_0' \subsetneq \mathcal{I}_0 \ominus \mathcal{J}_0$. Thus, we obtain a contradiction with $\mathcal{J}_0 \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}_0, \mathcal{T}, \mathcal{N})$.
- g is of the form $\neg \exists R^-(c)$, that is, there exists $\alpha \in \Delta$ such that $R(\alpha, c) \in \mathcal{J}_0$. We now show that $\mathcal{N} \parallel_{\mathcal{T}} \exists R^-(c)$. Indeed, $\mathcal{N} \not\models_{\mathcal{T}} \exists R^-(c)$ holds, otherwise, since $\mathcal{N} \cup \{\neg \exists R^-(c)\} \models_{\mathcal{T}} \bot$ (see the definition of AtAlg), it would hold that $\neg \exists R^-(c) \notin \mathsf{AtAlg}(\mathcal{K}, \mathcal{N})$, which contradicts Eq. (H.1); furthermore, $\mathcal{N} \not\models \neg \exists R^-(c)$ holds since $\mathcal{J}_0 \models R(\alpha, c)$ and $\mathcal{J}_0 \models \mathcal{N}$. Thus, $\mathcal{N} \parallel_{\mathcal{T}} \exists R^-(c)$.
  Observe that $\mathcal{N} \not\models \neg \exists R(\alpha)$ due to $\mathcal{J}_0 \models R(\alpha, c)$ and $\mathcal{J}_0 \models \mathcal{N}$. We are ready to define $\mathcal{J}_0'$. Consider now two following cases: $\mathcal{N} \not\models_{\mathcal{T}} \exists R(\alpha)$ and $\mathcal{N} \models_{\mathcal{T}} \exists R(\alpha)$. In the former case we have that $\mathcal{N} \parallel_{\mathcal{T}} \exists R(\alpha)$ and define $\mathcal{J}_0'$ as follows: $\mathcal{J}_0' = \mathcal{J}_0 \setminus \mathsf{root}_{\mathcal{T}}(\exists R(\alpha)) \setminus \mathsf{root}_{\mathcal{T}}(\exists R^-(c))$. Due to $\mathcal{N} \parallel_{\mathcal{T}} \exists R^-(c)$ and $\mathcal{N} \parallel_{\mathcal{T}} \exists R(\alpha)$, we have that $\mathcal{J}_0' \in \mathsf{Mod}(\mathcal{N})$. Due to Proposition 3.6 we have that $\mathcal{J}_0'$ is in $\mathsf{Mod}(\mathcal{T})$. Thus, taking into account that $(\mathsf{root}_{\mathcal{T}}(\exists R(\alpha)) \cup \mathsf{root}_{\mathcal{T}}(\exists R^-(c))) \setminus \{R(\alpha, c)\}$ consists of unary atoms only (this holds due to Restriction (ii) of Definition 5.11), we obtain that $\mathcal{I}_0 \ominus \mathcal{J}_0' \subseteq \mathcal{I}_0 \ominus \mathcal{J}_0$ and $R(\alpha, c) \in (\mathcal{I}_0 \ominus \mathcal{J}_0) \setminus (\mathcal{I}_0 \ominus \mathcal{J}_0')$, which yields $\mathcal{I}_0 \ominus \mathcal{J}_0' \subsetneq \mathcal{I}_0 \ominus \mathcal{J}_0$ and a contradiction with $\mathcal{J}_0 \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}_0, \mathcal{T}, \mathcal{N})$.
  In the latter case, when $\mathcal{N} \models_{\mathcal{T}} \exists R(\alpha)$, we define $\mathcal{J}_0'$ as follows: $\mathcal{J}_0' = \mathcal{J}_0 \setminus \mathsf{root}_{\mathcal{T}}(\exists R^-(c))$. Due to Restriction (iii) of Definition 5.11, it holds that $\mathcal{N} \models_{\mathcal{T}} R(\alpha, d)$ for some d. Note that $d \neq c$ since $\mathcal{N} \not\models_{\mathcal{T}} \exists R^-(c)$, and therefore $\mathcal{J}_0'$ is a model of $\mathcal{N}$ (as in the previous case, $\mathsf{root}_{\mathcal{T}}(\exists R^-(c)) \setminus \{R(\alpha, c)\}$ consists of only unary atoms). By Proposition 3.6, we have that $\mathcal{J}_0'$ is in $\mathsf{Mod}(\mathcal{T})$. As in the previous case, we obtain $\mathcal{I}_0 \ominus \mathcal{J}_0' \subseteq \mathcal{I}_0 \ominus \mathcal{J}_0$ and $R(\alpha, c) \in (\mathcal{I}_0 \ominus \mathcal{J}_0) \setminus (\mathcal{I}_0 \ominus \mathcal{J}_0')$, which yields $\mathcal{I}_0 \ominus \mathcal{J}_0' \subsetneq \mathcal{I}_0 \ominus \mathcal{J}_0$ a contradiction with $\mathcal{J}_0 \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}_0, \mathcal{T}, \mathcal{N})$.

Therefore, if $g \in \mathsf{cl}_{\mathcal{T}}(\mathcal{N}) \cup Y$, then $\mathcal{K} \diamond_S \mathcal{N} \models g$.

We show now the "only-if" direction. Suppose that $\mathcal{K} \diamond_S \mathcal{N} \models g$, but $g \notin \mathsf{cl}_{\mathcal{T}}(\mathcal{N}) \cup Y$. There are two possible cases: $g \in \mathsf{AtAlg}(\mathcal{K}, \mathcal{N})$ or $g \notin \mathsf{AtAlg}(\mathcal{K}, \mathcal{N})$. In the former case we have that $g \in \mathsf{AtAlg}(\mathcal{K}, \mathcal{N})$ and $g \in \bigcup_{R \in \mathsf{TR}} \bigcup_{B(c) \in \mathsf{DjnAts}[\mathcal{K}, \mathcal{N}](R)} \{\neg \exists R^-(c)\}$, so there exists a concept B such that $\mathcal{T} \models \exists R^- \sqsubseteq \neg B$ and $\mathcal{A} \models_{\mathcal{T}} B(c)$. Observe that the prototype $\mathcal{J}[\{B(c)\}, \langle R \rangle, \langle A \rangle]$ for some $A \in \mathtt{ISubCon}(R)$ is such that it does not satisfy $\neg \exists R^-(c)$. We obtain a contradiction with the assumption that g is certain, and therefore $\mathcal{K} \diamond_S \mathcal{N} \not\models g$. Finally, suppose that $g = \neg f \notin \mathsf{AtAlg}(\mathcal{K}, \mathcal{N})$. First, observe that $\mathcal{N} \parallel_{\mathcal{T}} f$; indeed, (i) since $g \notin \mathsf{cl}_{\mathcal{T}}(\mathcal{N})$, we conclude that $\mathcal{N} \not\models_{\mathcal{T}} \neg f$, (ii) since $\mathcal{K} \diamond_S \mathcal{N} \models \neg f$, we conclude that $\mathcal{N} \not\models_{\mathcal{T}} f$. Second, observe that $\mathcal{A} \parallel_{\mathcal{T}} f$; indeed, (i) suppose that $\mathcal{A} \models_{\mathcal{T}} f$, i.e., $f \in \mathsf{cl}_{\mathcal{T}}(\mathcal{A})$, then, since $\mathcal{N} \not\models_{\mathcal{T}} \neg f$, it holds that $f \in \mathsf{AtAlg}(\mathcal{K}, \mathcal{N})$, which is not the case, and therefore we conclude that $\mathcal{A} \not\models_{\mathcal{T}} f$; (ii) similarly to the previous case, one can show that $\mathcal{A} \not\models_{\mathcal{T}} \neg f$. Recall that $\mathcal{K} \diamond_S \mathcal{N} \models \neg f$. Consider models $\mathcal{J}_0 \in \mathcal{K} \diamond_S \mathcal{N}$ and $\mathcal{I}_0 \in \mathsf{Mod}(\mathcal{K})$ such that $\mathcal{J}_0 \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}_0, \mathcal{T}, \mathcal{N})$. Then, consider a set $Y = \mathsf{chase}_{\mathcal{T}}(f)$, where all the Δ-elements are such that they do not occur in $\mathsf{adom}(\mathcal{K} \cup \mathcal{N})$ and models $\mathcal{I}_0$ and $\mathcal{J}_0$. Using Y we define the following two models: $\mathcal{I}_0' = \mathcal{I}_0 \cup Y$ and $\mathcal{J}_0' = \mathcal{J}_0 \cup Y$. It is easy to check that $\mathcal{I}_0' \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{A})$ and $\mathcal{J}_0 \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$. We are going to show now that $\mathcal{J}_0' \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}_0', \mathcal{T}, \mathcal{N})$, which will lead to a contradiction with the fact that $\mathcal{K} \diamond_S \mathcal{N} \models g$ since $\mathcal{J}_0' \not\models g$. Suppose that $\mathcal{J}_0' \notin \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}_0', \mathcal{T}, \mathcal{N})$, that is, there exists a model $\mathcal{J}_0'' \in \mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$ such that $\mathcal{I}_0' \ominus \mathcal{J}_0'' \subsetneq \mathcal{I}_0' \ominus \mathcal{J}_0'$. Thus, there is an atom $f' \in \mathcal{J}_0'$ such that $f' \in (\mathcal{I}_0' \ominus \mathcal{J}_0') \setminus (\mathcal{I}_0' \ominus \mathcal{J}_0'')$. Note that $f' \notin Y$ since $Y \nsubseteq \mathcal{I}_0' \ominus \mathcal{J}_0'$, and also $\mathcal{J}'' = \mathcal{J}_0'' \setminus (Y \setminus \mathcal{J}_0)$ is in $\mathsf{Mod}(\mathcal{T} \cup \mathcal{N})$. These two observations lead to the fact that $\mathcal{I}_0 \ominus \mathcal{J}'' \subsetneq \mathcal{I}_0 \ominus \mathcal{J}_0$ which contradicts $\mathcal{J}_0 \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}_0, \mathcal{T}, \mathcal{N})$. Therefore, $\mathcal{J}_0' \in \mathsf{loc\_min}^a_{\subseteq}(\mathcal{I}_0', \mathcal{T}, \mathcal{N})$. From $\mathcal{J}_0' \not\models g$ we conclude that g is not certain. $\square$

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