Exchanging OWL 2 QL Knowledge Bases

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Abstract

Knowledge base exchange is an important problem in the area of data exchange and knowledge representation, where one is interested in exchanging information between a source and a target knowledge base connected through a mapping. In this paper, we study this fundamental problem for knowledge bases and mappings expressed in OWL 2 QL, the profile of OWL 2 based on the description logic *DL-Lite*$_{\mathcal{R}}$. More specifically, we consider the problem of computing universal solutions, identified as one of the most desirable translations to be materialized, and the problem of computing UCQ-representations, which optimally capture in a target TBox the information that can be extracted from a source TBox and a mapping by means of unions of conjunctive queries. For the former we provide a novel automata-theoretic technique, and complexity results that range from NP to EXPTIME, while for the latter we show NLOGSPACE-completeness.

1 Introduction

Complex forms of information, maintained in different formats and organized according to different structures, often need to be shared between agents. In recent years, both in the data management and in the knowledge representation communities, several settings have been investigated that address this problem from various perspectives: in *information integration*, uniform access is provided to a collection of data sources by means of an ontology (or global schema) to which the sources are mapped [Lenzerini, 2002]; in *peer-to-peer systems*, a set of peers declaratively linked to each other collectively provide access to the information assets they maintain [Kementsietsidis *et al.*, 2003; Adjiman *et al.*, 2006; Fuxman *et al.*, 2006]; in *ontology matching*, the aim is to understand and derive the correspondences between elements in two ontologies [Euzenat and Shvaiko, 2007; Shvaiko and Euzenat, 2013]; finally, in *data exchange*, the information stored according to a source schema needs to be restructured and translated so as to conform to a target schema [Fagin *et al.*, 2005; Barceló, 2009].

The work we present in this paper is inspired by the latter setting, investigated in databases. We study it, however, under the assumption of incomplete information typical of knowledge representation [Arenas *et al.*, 2011]. Specifically, we investigate the problem of *knowledge base exchange*, where a source knowledge base (KB) is connected to a target KB by means of a declarative mapping specification, and the aim is to exchange knowledge from the source to the target by exploiting the mapping. We rely on a framework for KB exchange based on lightweight Description Logics (DLs) of the *DL-Lite* family [Calvanese *et al.*, 2007], recently proposed in [Arenas *et al.*, 2012a; Arenas *et al.*, 2012b]: both source and target are KBs constituted by a DL TBox, representing implicit information, and an ABox, representing explicit information, and mappings are sets of DL concept and role inclusions. Note that in data and knowledge base exchange, differently from ontology matching, mappings are first-class citizens. In fact, it has been recognized that building schema mappings is an important and complex activity, which requires the designer to have a thorough understanding of the source and how the information therein should be related to the target. Thus, several techniques and tools have been developed to support mapping design, e.g., exploiting lexical information [Fagin *et al.*, 2009]. Here, similar to data exchange, we assume that for building mappings the target signature is given, but no further axioms constraining the target knowledge are available. In fact, such axioms are derived from the source KB and the mapping.

We consider two key problems: *(i)* computing *universal solutions*, which have been identified as one of the most desirable translations to be materialized; *(ii)* UCQ-*representability* of a source TBox by means of a target TBox that captures at best the intensional information that can be extracted from the source according to a mapping using union of conjunctive queries. Determining UCQ-representability is a crucial task, since it allows one to use the obtained target TBox to infer new knowledge in the target, thus reducing the amount of extensional information to be transferred from the source. Moreover, it has been noticed that in many data exchange applications users only extract information from the translated data by using specific queries (usually conjunctive queries), so query-based notions of translation specifically tailored to store enough information to answer such queries have been widely studied in the data exchange area [Madhavan and Halevy, 2003; Fagin *et al.*, 2008; Arenas *et al.*, 2009; Fagin and Kolaitis, 2012; Pichler *et al.*, 2013]. For these two problems, we investigate both the task of checking *membership*, where a candidate universal solution (resp., UCQ-representation) is given and one needs to check its correctness, and *non-emptiness*, where the aim is to determine the existence of a universal solution (resp., UCQ-representation).

We significantly extend previous results in several directions. First of all, we establish results for OWL 2 QL [Motik *et al.*, 2012], one of the profiles of the standard Web Ontology Language OWL 2 [Bao *et al.*, 2012], which is based on the DL *DL-Lite*$_\mathcal{R}$. To do so, we have to overcome the difficulty of dealing with null values in the ABox, since these become necessary in the target to represent universal solutions. Also, for the first time, we address disjointness assertions in the TBox, a construct that is part of OWL 2 QL. The main contribution of our work is then a detailed analysis of the computational complexity of both membership and non-emptiness for universal solutions and UCQ-representability. For the non-emptiness problem of universal solutions, previous known results covered only the simple case of *DL-Lite*$_{RDFS}$, the RDFS fragment of OWL 2 QL, in which no new facts can be inferred, and universal solutions always exist and can be computed in polynomial time via a chase procedure (see [Calvanese *et al.*, 2007]). We show that in our case, instead, the problem is PSPACE-hard, hence significantly more complex, and provide an EXPTIME upper bound based on a novel approach exploiting two-way alternating automata. We provide also NP upper bounds for the simpler case of ABoxes without null values, and for the case of the membership problem. As for UCQ-representability, we adopt the notion of UCQ-*representability* introduced in [Arenas *et al.*, 2012a; Arenas *et al.*, 2012b] and extend it to take into account disjointness of OWL 2 QL. For that case we show NLOGSPACE-completeness of both non-emptiness and membership, improving on the previously known PTIME upper bounds.

The paper is organized as follows. In Section 2, we give preliminary notions on DLs and queries. In Section 3, we define our framework of KB exchange and discuss the problem of computing solutions. In Section 4, we overview our contributions, and then we provide our results on computing universal solutions in Section 5, and on UCQ-representability in Section 6. Finally, in Section 7, we draw some conclusions and outline issues for future work.

2 Preliminaries

The DLs of the *DL-Lite* family [Calvanese *et al.*, 2007] of light-weight DLs are characterized by the fact that standard reasoning can be done in polynomial time. We adapt here *DL-Lite*$_\mathcal{R}$, the DL underlying OWL 2 QL, and present now its syntax and semantics. Let N_C, N_R, N_a, N_ℓ be pairwise disjoint sets of *concept names*, *role names*, *constants*, and *labeled nulls*, respectively. Assume in the following that $A \in N_C$ and $P \in N_R$; in *DL-Lite*$_\mathcal{R}$, B and C are used to denote basic and arbitrary (or complex) concepts, respectively, and R and Q are used to denote basic and arbitrary (or complex) roles, respectively, defined as follows:

$$R ::= P \mid P^- \qquad\qquad B ::= A \mid \exists R$$
$$Q ::= R \mid \neg R \qquad\qquad C ::= B \mid \neg B$$

From now on, for a basic role R, we use R^- to denote P^- when $R = P$, and P when $R = P^-$.

A TBox is a finite set of *concept inclusions* $B \sqsubseteq C$ and *role inclusions* $R \sqsubseteq Q$. We call an inclusion of the form $B_1 \sqsubseteq \neg B_2$ or $R_1 \sqsubseteq \neg R_2$ a *disjointness assertion*. An ABox is a finite set of *membership assertions* $B(a)$, $R(a, b)$, where $a, b \in N_a$. In this paper, we also consider extended ABoxes, which are obtained by allowing labeled nulls in membership assertions. Formally, an *extended ABox* is a finite set of membership assertions $B(u)$ and $R(u, v)$, where $u, v \in (N_a \cup N_\ell)$. Moreover, a(n *extended*) *KB* $\mathcal{K}$ is a pair $\langle \mathcal{T}, \mathcal{A} \rangle$, where $\mathcal{T}$ is a TBox and $\mathcal{A}$ is an (extended) ABox.

A *signature* Σ is a finite set of concept and role names. A KB $\mathcal{K}$ is said to be *defined over* (or simply, *over*) Σ if all the concept and role names occurring in $\mathcal{K}$ belong to Σ (and likewise for TBoxes, ABoxes, concept inclusions, role inclusions and membership assertions). Moreover, an *interpretation* $\mathcal{I}$ *of* Σ is a pair $\langle \Delta^\mathcal{I}, \cdot^\mathcal{I} \rangle$, where $\Delta^\mathcal{I}$ is a non-empty domain and $\cdot^\mathcal{I}$ is an interpretation function such that: (1) $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$, for every concept name $A \in \Sigma$; (2) $P^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$, for every role name $P \in \Sigma$; and (3) $a^\mathcal{I} \in \Delta^\mathcal{I}$, for every constant $a \in N_a$. Function $\cdot^\mathcal{I}$ is extended to also interpret concept and role constructs:

$$\begin{aligned}
(\exists R)^\mathcal{I} &= \{x \in \Delta^\mathcal{I} \mid \exists y \in \Delta^\mathcal{I} \text{ such that } (x, y) \in R^\mathcal{I}\};\\
(P^-)^\mathcal{I} &= \{(y, x) \in \Delta^\mathcal{I} \times \Delta^\mathcal{I} \mid (x, y) \in P^\mathcal{I}\};\\
(\neg B)^\mathcal{I} &= \Delta^\mathcal{I} \setminus B^\mathcal{I}; \qquad (\neg R)^\mathcal{I} = (\Delta^\mathcal{I} \times \Delta^\mathcal{I}) \setminus R^\mathcal{I}.
\end{aligned}$$

Note that, consistently with the semantics of OWL 2 QL, we do *not* make the unique name assumption (UNA), i.e., we allow distinct constants $a, b \in N_a$ to be interpreted as the same object, i.e., $a^\mathcal{I} = b^\mathcal{I}$. Note also that labeled nulls are *not* interpreted by $\mathcal{I}$.

Let $\mathcal{I} = \langle \Delta^\mathcal{I}, \cdot^\mathcal{I} \rangle$ be an interpretation over a signature Σ. Then $\mathcal{I}$ is said to satisfy a concept inclusion $B \sqsubseteq C$ over Σ, denoted by $\mathcal{I} \models B \sqsubseteq C$, if $B^\mathcal{I} \subseteq C^\mathcal{I}$; $\mathcal{I}$ is said to satisfy a role inclusion $R \sqsubseteq Q$ over Σ, denoted by $\mathcal{I} \models R \sqsubseteq Q$, if $R^\mathcal{I} \subseteq Q^\mathcal{I}$; and $\mathcal{I}$ is said to satisfy a TBox $\mathcal{T}$ over Σ, denoted by $\mathcal{I} \models \mathcal{T}$, if $\mathcal{I} \models \alpha$ for every $\alpha \in \mathcal{T}$. Moreover, satisfaction of membership assertions over Σ is defined as follows. A *substitution* over $\mathcal{I}$ is a function $h : (N_a \cup N_\ell) \to \Delta^\mathcal{I}$ such that $h(a) = a^\mathcal{I}$ for every $a \in N_a$. Then $\mathcal{I}$ is said to satisfy an (extended) ABox $\mathcal{A}$, denoted by $\mathcal{I} \models \mathcal{A}$, if there exists a substitution h over $\mathcal{I}$ such that:

- for every $B(u) \in \mathcal{A}$, it holds that $h(u) \in B^\mathcal{I}$; and
- for every $R(u, v) \in \mathcal{A}$, it holds that $(h(u), h(v)) \in R^\mathcal{I}$.

Finally, $\mathcal{I}$ is said to *satisfy* a(n extended) KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$, denoted by $\mathcal{I} \models \mathcal{K}$, if $\mathcal{I} \models \mathcal{T}$ and $\mathcal{I} \models \mathcal{A}$. Such $\mathcal{I}$ is called a *model* of $\mathcal{K}$, and we use $\textsc{Mod}(\mathcal{K})$ to denote the set of all models of $\mathcal{K}$. We say that $\mathcal{K}$ is *consistent* if $\textsc{Mod}(\mathcal{K}) \neq \emptyset$.

As is customary, given an (extended) KB $\mathcal{K}$ over a signature Σ and a membership assertion or an inclusion α over Σ, we use notation $\mathcal{K} \models \alpha$ to indicate that for every interpretation $\mathcal{I}$ of Σ, if $\mathcal{I} \models \mathcal{K}$, then $\mathcal{I} \models \alpha$.

2.1 Queries and certain answers

A k-ary query q over a signature Σ, with $k \geq 0$, is a function that maps every interpretation $\langle \Delta^\mathcal{I}, \cdot^\mathcal{I} \rangle$ of Σ into a k-ary relation $q^\mathcal{I} \subseteq (\Delta^\mathcal{I})^k$. In particular, if $k = 0$, then q is said to be a Boolean query, and $q^{\mathcal{I}}$ is either a relation containing the empty tuple () (representing the value true) or the empty relation (representing the value false). Given a KB $\mathcal{K}$ over Σ, the set of *certain answers* to q over $\mathcal{K}$, denoted by $cert(q,\mathcal{K})$, is defined as:

$$\bigcap_{\mathcal{I}\in\text{MOD}(\mathcal{K})}\{(a_1,\ldots,a_k)\mid \{a_1,\ldots,a_k\}\subseteq N_a \text{ and } (a_1^{\mathcal{I}},\ldots,a_k^{\mathcal{I}})\in q^{\mathcal{I}}\},$$

Notice that the certain answer to a query does *not* contain labeled nulls. Besides, notice that if q is a Boolean query, then $cert(q,\mathcal{K})$ evaluates to true if $q^{\mathcal{I}}$ evaluates to true for every $\mathcal{I}\in\text{MOD}(\mathcal{K})$, and it evaluates to false otherwise.

A *conjunctive query* (CQ) *over a signature* Σ is a formula of the form $q(\vec{x}) = \exists\vec{y}.\,\varphi(\vec{x},\vec{y})$, where $\vec{x},\vec{y}$ are tuples of variables and $\varphi(\vec{x},\vec{y})$ is a conjunction of atoms of the form $A(t)$, with A a concept name in Σ, and $P(t,t')$, with P a role name in Σ, where each of t,t' is either a constant from N_a or a variable from $\vec{x}$ or $\vec{y}$. Given an interpretation $\mathcal{I}=\langle\Delta^{\mathcal{I}},\cdot^{\mathcal{I}}\rangle$ of Σ, the answer of q over $\mathcal{I}$, denoted by $q^{\mathcal{I}}$, is the set of tuples $\vec{a}$ of elements from $\Delta^{\mathcal{I}}$ for which there exist a tuple $\vec{b}$ of elements from $\Delta^{\mathcal{I}}$ such that $\mathcal{I}$ satisfies every conjunct in $\varphi(\vec{a},\vec{b})$. A union of conjunctive queries (UCQ) over a signature Σ is a formula of the form $q(\vec{x}) = \bigvee_{i=1}^{n} q_i(\vec{x})$, where each q_i ($1\leq i\leq n$) is a CQ over Σ, whose semantics is defined as $q^{\mathcal{I}} = \bigcup_{i=1}^{n} q_i^{\mathcal{I}}$.

3 Exchanging OWL 2 QL Knowledge Bases

We generalize now, in Section 3.1, the setting proposed in [Arenas *et al.*, 2011] to OWL 2 QL, and we formalize in Section 3.2 the main problems studied in the rest of the paper.

3.1 A knowledge base exchange framework for OWL 2 QL

Assume that Σ_1,Σ_2 are signatures with no concepts or roles in common. An inclusion $E_1\sqsubseteq E_2$ is said to be *from* Σ_1 *to* Σ_2, if E_1 is a concept or a role over Σ_1 and E_2 is a concept or a role over Σ_2. A mapping is a tuple $\mathcal{M}=(\Sigma_1,\Sigma_2,\mathcal{T}_{12})$, where $\mathcal{T}_{12}$ is a TBox consisting of inclusions from Σ_1 to Σ_2 [Arenas *et al.*, 2012a]. Recall that in this paper, we deal with *DL-Lite*$_{\mathcal{R}}$ TBoxes only, so $\mathcal{T}_{12}$ is assumed to be a set of *DL-Lite*$_{\mathcal{R}}$ concept and role inclusions. The semantics of such a mapping is defined in [Arenas *et al.*, 2012a] in terms of a notion of satisfaction for interpretations, which has to be extended in our case to deal with interpretations not satisfying the UNA (and, more generally, the standard name assumption). More specifically, given interpretations $\mathcal{I}$, $\mathcal{J}$ of Σ_1 and Σ_2, respectively, pair $(\mathcal{I},\mathcal{J})$ *satisfies* TBox $\mathcal{T}_{12}$, denoted by $(\mathcal{I},\mathcal{J})\models\mathcal{T}_{12}$, if *(i)* for every $a\in N_a$, it holds that $a^{\mathcal{I}}=a^{\mathcal{J}}$, *(ii)* for every concept inclusion $B\sqsubseteq C\in\mathcal{T}_{12}$, it holds that $B^{\mathcal{I}}\subseteq C^{\mathcal{J}}$, and *(iii)* for every role inclusion $R\sqsubseteq Q\in\mathcal{T}_{12}$, it holds that $R^{\mathcal{I}}\subseteq Q^{\mathcal{J}}$. Notice that the connection between the information in $\mathcal{I}$ and $\mathcal{J}$ is established through the constants that move from source to target according to the mapping. For this reason, we require constants to be interpreted in the same way in $\mathcal{I}$ and $\mathcal{J}$, i.e., they preserve their meaning when they are transferred. Besides, notice that this is the only restriction imposed on the domains of $\mathcal{I}$ and $\mathcal{J}$ (in particular, we require neither that $\Delta^{\mathcal{I}}=\Delta^{\mathcal{J}}$ nor that $\Delta^{\mathcal{I}}\subseteq\Delta^{\mathcal{J}}$). Finally, $\text{SAT}_{\mathcal{M}}(\mathcal{I})$ is defined as the set of interpretations $\mathcal{J}$ of Σ_2 such that $(\mathcal{I},\mathcal{J})\models\mathcal{T}_{12}$, and given a set $\mathcal{X}$ of interpretations of Σ_1, $\text{SAT}_{\mathcal{M}}(\mathcal{X})$ is defined as $\bigcup_{\mathcal{I}\in\mathcal{X}}\text{SAT}_{\mathcal{M}}(\mathcal{I})$.

The main problem studied in the knowledge exchange area is the problem of translating a KB according to a mapping, which is formalized through several different notions of translation (for a thorough comparison of different notions of solutions see [Arenas *et al.*, 2012a]). The first such notion is the concept of solution, which is formalized as follows. Given a mapping $\mathcal{M}=(\Sigma_1,\Sigma_2,\mathcal{T}_{12})$ and KBs $\mathcal{K}_1$, $\mathcal{K}_2$ over Σ_1 and Σ_2, respectively, $\mathcal{K}_2$ is a *solution* for $\mathcal{K}_1$ under $\mathcal{M}$ if $\text{MOD}(\mathcal{K}_2)\subseteq\text{SAT}_{\mathcal{M}}(\text{MOD}(\mathcal{K}_1))$. Thus, $\mathcal{K}_2$ is a solution for $\mathcal{K}_1$ under $\mathcal{M}$ if every interpretation of $\mathcal{K}_2$ is a valid translation of an interpretation of $\mathcal{K}_1$ according to $\mathcal{M}$. Although natural, this is a mild restriction, which gives rise to the stronger notion of universal solution. Given $\mathcal{M}$, $\mathcal{K}_1$ and $\mathcal{K}_2$ as before, $\mathcal{K}_2$ is a *universal solution* for $\mathcal{K}_1$ under $\mathcal{M}$ if $\text{MOD}(\mathcal{K}_2)=\text{SAT}_{\mathcal{M}}(\text{MOD}(\mathcal{K}_1))$. Thus, $\mathcal{K}_2$ is designed to exactly represent the space of interpretations obtained by translating the interpretations of $\mathcal{K}_1$ under $\mathcal{M}$ [Arenas *et al.*, 2012a]. Below is a simple example demonstrating the notion of universal solutions. This example also illustrates some issues regarding the absence of the UNA, which has to be given up to comply with the OWL 2 QL standard, and regarding the use of disjointness assertions.

Example 3.1 *Assume* $\mathcal{M}=(\{F(\cdot),G(\cdot)\},\{F'(\cdot),G'(\cdot)\},\mathcal{T}_{12})$*, where* $\mathcal{T}_{12}=\{F\sqsubseteq F', G\sqsubseteq G'\}$*, and let* $\mathcal{K}_1=\langle\mathcal{T}_1,\mathcal{A}_1\rangle$*, where* $\mathcal{T}_1=\{\}$ *and* $\mathcal{A}_1=\{F(a),G(b)\}$*. Then the ABox* $\mathcal{A}_2=\{F'(a),G'(b)\}$ *is a universal solution for* $\mathcal{K}_1$ *under* $\mathcal{M}$*.*

Now, if we add a seemingly harmless disjointness assertion $\{F\sqsubseteq\neg G\}$ *to* $\mathcal{T}_1$*, we obtain that* $\mathcal{A}_2$ *is no longer a universal solution (not even a solution) for* $\mathcal{K}_1$ *under* $\mathcal{M}$*. The reason for that is the lack of the UNA on the one hand, and the presence of the disjointness assertion in* $\mathcal{T}_1$ *on the other hand. In fact, the latter forces* a *and* b *to be interpreted differently in the source. Thus, for a model* $\mathcal{J}$ *of* $\mathcal{A}_2$ *such that* $a^{\mathcal{J}}=b^{\mathcal{J}}$ *and* $F'^{\mathcal{J}}=G'^{\mathcal{J}}=\{a^{\mathcal{J}}\}$*, there exists no model* $\mathcal{I}$ *of* $\mathcal{K}_1$ *such that* $(\mathcal{I},\mathcal{J})\models\mathcal{T}_{12}$ *(which would require* $a^{\mathcal{I}}=a^{\mathcal{J}}$ *and* $b^{\mathcal{I}}=b^{\mathcal{J}}$*). In general, there exists no universal solution for* $\mathcal{K}_1$ *under* $\mathcal{M}$*, even though* $\mathcal{K}_1$ *and* $\mathcal{T}_{12}$ *are consistent with each other.*

A second class of translations is obtained in [Arenas *et al.*, 2012a] by observing that solutions and universal solutions are too restrictive for some applications, in particular when one only needs a translation storing enough information to properly answer some queries. For the particular case of UCQ, this gives rise to the notions of UCQ-solution and universal UCQ-solution. Given a mapping $\mathcal{M}=(\Sigma_1,\Sigma_2,\mathcal{T}_{12})$, a KB $\mathcal{K}_1=\langle\mathcal{T}_1,\mathcal{A}_1\rangle$ over Σ_1 and a KB $\mathcal{K}_2$ over Σ_2, $\mathcal{K}_2$ is a UCQ-*solution* for $\mathcal{K}_1$ under $\mathcal{M}$ if for every query $q\in$ UCQ over Σ_2: $cert(q,\langle\mathcal{T}_1\cup\mathcal{T}_{12},\mathcal{A}_1\rangle)\subseteq cert(q,\mathcal{K}_2)$, while $\mathcal{K}_2$ is a *universal* UCQ-*solution* for $\mathcal{K}_1$ under $\mathcal{M}$ if for every query $q\in$ UCQ over Σ_2: $cert(q,\langle\mathcal{T}_1\cup\mathcal{T}_{12},\mathcal{A}_1\rangle)=cert(q,\mathcal{K}_2)$.

Finally, a last class of solutions is obtained in [Arenas *et al.*, 2012a] by considering that users want to translate as much of the knowledge in a TBox as possible, as a lot of effort is put in practice when constructing a TBox. This observation gives rise to the notion of UCQ-representation [Arenas *et al.*, 2012a], which formalizes the idea of translating a source TBox according to a mapping. Next, we present an alternative formalization of this notion, which is appropriate for our setting where disjointness assertions are considered.[1] Assume that $\mathcal{M} = (\Sigma_1, \Sigma_2, \mathcal{T}_{12})$ and $\mathcal{T}_1$, $\mathcal{T}_2$ are TBoxes over Σ_1 and Σ_2, respectively. Then $\mathcal{T}_2$ is a UCQ-*representation* of $\mathcal{T}_1$ under $\mathcal{M}$ if for every query $q \in$ UCQ over Σ_2 and every ABox $\mathcal{A}_1$ over Σ_1 that is consistent with $\mathcal{T}_1$:

$$cert(q, \langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle) = \bigcap_{\substack{\mathcal{A}_2\,:\, \mathcal{A}_2 \text{ is an ABox over } \Sigma_2 \text{ that} \\ \text{is a UCQ-solution for } \mathcal{A}_1 \text{ under } \mathcal{M}}} cert(q, \langle \mathcal{T}_2, \mathcal{A}_2 \rangle). \quad (\dagger)$$

Notice that in the previous definition, $\mathcal{A}_2$ is said to be a UCQ-solution for $\mathcal{A}_1$ under $\mathcal{M}$ if the KB $\langle \emptyset, \mathcal{A}_2 \rangle$ is a UCQ-solution for the KB $\langle \emptyset, \mathcal{A}_1 \rangle$ under $\mathcal{M}$. Let us explain the intuition behind the definition of the notion of UCQ-representation. Assume that $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{M}$ satisfy (†). First, $\mathcal{T}_2$ captures the information in $\mathcal{T}_1$ that is translated by $\mathcal{M}$ and that can be extracted by using a UCQ, as for every ABox $\mathcal{A}_1$ over Σ_1 that is consistent with $\mathcal{T}_1$ and every UCQ q over Σ_2, if we choose an arbitrary UCQ-solution $\mathcal{A}_2$ for $\mathcal{A}_1$ under $\mathcal{M}$, then it holds that $cert(q, \langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle) \subseteq cert(q, \langle \mathcal{T}_2, \mathcal{A}_2 \rangle)$. Notice that $\mathcal{A}_1$ is required to be consistent with $\mathcal{T}_1$ in the previous condition, as we are interested in translating data that make sense according to $\mathcal{T}_1$. Second, $\mathcal{T}_2$ does not include any piece of information that can be extracted by using a UCQ and it is not the result of translating the information in $\mathcal{T}_1$ according to $\mathcal{M}$. In fact, if $\mathcal{A}_1$ is an ABox over Σ_1 that is consistent with $\mathcal{T}_1$ and q is a UCQ over Σ_2, then it could be the case that $cert(q, \langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle) \subsetneq cert(q, \langle \mathcal{T}_2, \mathcal{A}_2^\star \rangle)$ for some UCQ-solution $\mathcal{A}_2^\star$ for $\mathcal{A}_1$ under $\mathcal{M}$. However, the extra tuples extracted by query q are obtained from the extra information in $\mathcal{A}_2^\star$, as if we consider a tuple $\vec{a}$ that belong to $cert(q, \langle \mathcal{T}_2, \mathcal{A}_2 \rangle)$ for every UCQ-solution $\mathcal{A}_2$ for $\mathcal{A}_1$ under $\mathcal{M}$, then it holds that $\vec{a} \in cert(q, \langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle)$.

Example 3.2 *Assume that* $\mathcal{M} = (\{F(\cdot), G(\cdot), H(\cdot), D(\cdot)\}, \{F'(\cdot), G'(\cdot), H'(\cdot)\}, \mathcal{T}_{12})$*, where* $\mathcal{T}_{12} = \{F \sqsubseteq F', G \sqsubseteq G', H \sqsubseteq H'\}$*, and let* $\mathcal{T}_1 = \{F \sqsubseteq G\}$*. As expected, TBox* $\mathcal{T}_2 = \{F' \sqsubseteq G'\}$ *is a* UCQ*-representation of* $\mathcal{T}_1$ *under* $\mathcal{M}$*. Moreover, we can add the inclusion* $D \sqsubseteq \neg H'$ *to* $\mathcal{T}_{12}$*, and* $\mathcal{T}_2$ *will still remain a* UCQ*-representation of* $\mathcal{T}_1$ *under* $\mathcal{M}$*. Notice that in this latter setting, our definition has to deal with some ABoxes* $\mathcal{A}_1$ *that are consistent with* $\mathcal{T}_1$ *but not with* $\mathcal{T}_1 \cup \mathcal{T}_{12}$*, for instance* $\mathcal{A}_1 = \{H(a), D(a)\}$ *for some constant* a*. In those cases, Equation* (†) *is trivially satisfied, since* $\text{MOD}(\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle) = \emptyset$ *and the set of* UCQ*-solutions for* $\mathcal{A}_1$ *under* $\mathcal{M}$ *is empty.*

3.2 On the problem of computing solutions

Arguably, the most important problem in knowledge exchange [Arenas *et al.*, 2011; Arenas *et al.*, 2012a], as well as in data exchange [Fagin *et al.*, 2005; Kolaitis, 2005], is the task of computing a translation of a KB according to a mapping. To study the computational complexity of this task for the different notions of solutions presented in the previous section, we introduce the following decision problems. The *membership* problem for universal solutions (resp. universal UCQ-solutions) has as input a mapping $\mathcal{M} = (\Sigma_1, \Sigma_2, \mathcal{T}_{12})$ and KBs $\mathcal{K}_1, \mathcal{K}_2$ over Σ_1 and Σ_2, respectively. Then the question to answer is whether $\mathcal{K}_2$ is a universal solution (resp. universal UCQ-solution) for $\mathcal{K}_1$ under $\mathcal{M}$. Moreover, the membership problem for UCQ-representations has as input a mapping $\mathcal{M} = (\Sigma_1, \Sigma_2, \mathcal{T}_{12})$ and TBoxes $\mathcal{T}_1$, $\mathcal{T}_2$ over Σ_1 and Σ_2, respectively, and the question to answer is whether $\mathcal{T}_2$ is a UCQ-representation of $\mathcal{T}_1$ under $\mathcal{M}$.

In our study, we cannot leave aside the existential versions of the previous problems, which are directly related with the problem of computing translations of a KB according to a mapping. Formally, the *non-emptiness* problem for universal solutions (resp. universal UCQ-solutions) has as input a mapping $\mathcal{M} = (\Sigma_1, \Sigma_2, \mathcal{T}_{12})$ and a KB $\mathcal{K}_1$ over Σ_1. Then the question to answer is whether there exists a universal solution (resp. universal UCQ-solution) for $\mathcal{K}_1$ under $\mathcal{M}$. Moreover, the non-emptiness problem for UCQ-representations has as input a mapping $\mathcal{M} = (\Sigma_1, \Sigma_2, \mathcal{T}_{12})$ and a TBox $\mathcal{T}_1$ over Σ_1, and the question to answer is whether there exists a UCQ-representation of $\mathcal{T}_1$ under $\mathcal{M}$.

4 Our contributions

In Section 3.2, we have introduced the problems that are studied in this paper. It is important to notice that these problems are defined by considering only KBs (as opposed to extended KBs), as they are the formal counterpart of OWL 2 QL. Nevertheless, as shown in Section 5, there are natural examples of OWL 2 QL specifications and mappings where null values are needed when constructing solutions. Thus, we also study the problems defined in Section 3.2 in the case where translations can be extended KBs. It should be noticed that the notions of solution, universal solution, UCQ-solution, universal UCQ-solution, and UCQ-representation have to be enlarged to consider extended KBs, which is straightforward to do. In particular, given a mapping $\mathcal{M} = (\Sigma_1, \Sigma_2, \mathcal{T}_{12})$ and TBoxes $\mathcal{T}_1$, $\mathcal{T}_2$ over Σ_1 and Σ_2, respectively, $\mathcal{T}_2$ is said to be a UCQ-representation of $\mathcal{T}_1$ under $\mathcal{M}$ in this extended setting if in Equation (†), $\mathcal{A}_2$ is an extended ABox over Σ_2 that is a UCQ-solution for $\mathcal{A}_1$ under $\mathcal{M}$.

The main contribution of this paper is to provide a detailed analysis of the complexity of the membership and non-emptiness problems for the notions of universal solution and UCQ-representation. In Figure 1, we provide a summary of the main results in the paper, which are explained in more detail in Sections 5 and 6. It is important to notice that these results considerably extend the previous known results about these problems [Arenas *et al.*, 2012a; Arenas *et al.*, 2012b]. In the first place, the problem of computing universal solutions was studied in [Arenas *et al.*, 2012a] for the case of *DL-Lite*$_{RDFS}$, a fragment of *DL-Lite*$_\mathcal{R}$ that allows neither for inclusions of the form $B \sqsubseteq \exists R$ nor for disjointness assertions. In that case, it is straightforward to show that every source KB has a universal solution

[1]If disjointness assertions are not allowed, then this new notion can be shown to be equivalent to the original formalization of UCQ-representation proposed in [Arenas *et al.*, 2012a].

Membership	ABoxes	extended ABoxes
Universal solutions	in NP	NP-complete
UCQ-representations	NLOGSPACE-complete	

Non-emptiness	ABoxes	extended ABoxes
Universal solutions	in NP	PSPACE-hard, in EXPTIME
UCQ-representations	NLOGSPACE-complete	

Figure 1: Complexity results obtained in the paper about the membership and non-emptiness problems.

that can be computed by using the chase procedure [Calvanese *et al.*, 2007]. Unfortunately, this result does not provide any information about how to solve the much larger case considered in this paper, where, in particular, the non-emptiness problem is not trivial. In fact, for the case of the notion of universal solution, all the lower and upper bounds provided in Figure 1 are new results, which are not consequences of the results obtained in [Arenas *et al.*, 2012a]. In the second place, a notion of UCQ-representation that is appropriate for the fragment of *DL-Lite*$_{\mathcal{R}}$ not including disjointness assertions was studied in [Arenas *et al.*, 2012a; Arenas *et al.*, 2012b]. In particular, it was shown that the membership and non-emptiness problems for this notion are solvable in polynomial time. In this paper, we considerably strengthen these results: *(i)* by generalizing the definition of the notion of UCQ-representation to be able to deal with OWL 2 QL, that is, with the entire language *DL-Lite*$_{\mathcal{R}}$ (which includes disjointness assertions); and *(ii)* by showing that the membership and non-emptiness problems are both NLOGSPACE-complete in this larger scenario.

It turns out that reasoning about universal UCQ-solutions is much more intricate. In fact, as a second contribution of our paper, we provide a PSPACE lower bound for the complexity of the membership problem for the notion of universal UCQ-solution, which is in sharp contrast with the NP and NLOGSPACE upper bounds for this problem for the case of universal solutions and UCQ-representations, respectively (see Figure 1). Although many questions about universal UCQ-solutions remain open, we think that this is an interesting first result, as universal UCQ-solutions have only been investigated before for the very restricted fragment *DL-Lite*$_{RDFS}$ of *DL-Lite*$_{\mathcal{R}}$ [Arenas *et al.*, 2012a], which is described in the previous paragraph.

5 Computing universal solutions

In this section, we study the membership and non-emptiness problems for universal solutions, in the cases where nulls are not allowed (Section 5.1) and are allowed (Section 5.2) in such solutions. But before going into this, we give an example that shows the shape of universal solutions in *DL-Lite*$_{\mathcal{R}}$.

Example 5.1 *Assume that* $\mathcal{M} = (\{F(\cdot), S(\cdot,\cdot)\}, \{G'(\cdot)\}, \{\exists S^- \sqsubseteq G'\})$*, and let* $\mathcal{K}_1 = \langle \mathcal{T}_1, \mathcal{A}_1\rangle$*, where* $\mathcal{T}_1 = \{F \sqsubseteq \exists S\}$ *and* $\mathcal{A}_1 = \{F(a)\}$*. Then a natural way to construct a universal solution for* $\mathcal{K}_1$ *under* $\mathcal{M}$ *is to 'populate' the target with all implied facts (as it is usually done in data exchange). Thus, the ABox* $\mathcal{A}_2 = \{G'(n)\}$*, where* n *is a labeled null, is a universal solution for* $\mathcal{K}_1$ *under* $\mathcal{M}$ *if nulls are allowed. Notice that here, a universal solution with non-extended ABoxes does not exist: substituting* n *by any constant is too restrictive, ruining universality.*

Example 5.2 *Now, assume* $\mathcal{M} = (\{F(\cdot), S(\cdot,\cdot), T(\cdot,\cdot)\}, \{S'(\cdot,\cdot)\}, \{S \sqsubseteq S', T \sqsubseteq S'\})$*, and* $\mathcal{K}_1 = \langle \mathcal{T}_1, \mathcal{A}_1\rangle$*, where* $\mathcal{T}_1 = \{F \sqsubseteq \exists S, \exists S^- \sqsubseteq \exists S\}$ *and* $\mathcal{A}_1 = \{F(a), T(a,a)\}$*. In this case, we cannot use the same approach as in Example 5.1 to construct a universal solution, as now we would need of an infinite number of labeled nulls to construct such a solution. However, as* S *and* T *are transferred to the same role* S'*, it is possible to use constant* a *to represent all implied facts. In particular, in this case* $\mathcal{A}_2 = \{S'(a,a)\}$ *is a universal solution for* $\mathcal{K}_1$ *under* $\mathcal{M}$*.*

5.1 Universal solutions without null values

We explain here how the NP upper bound for the non-emptiness problem for universal solutions is obtained, when ABoxes are not allowed to contain null values.

Assume given a mapping $\mathcal{M} = (\Sigma_1, \Sigma_2, \mathcal{T}_{12})$ and a KB $\mathcal{K}_1 = \langle \mathcal{T}_1, \mathcal{A}_1\rangle$ over Σ_1. To check whether $\mathcal{K}_1$ has a universal solution under $\mathcal{M}$, we use the following non-deterministic polynomial-time algorithm. First, we construct an ABox $\mathcal{A}_2$ over Σ_2 containing every membership assertion α such that $\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1\rangle \models \alpha$, where α is of the form either $B(a)$ or $R(a,b)$, and a, b are constants mentioned in $\mathcal{A}_1$. Second, we guess an interpretation $\mathcal{I}$ of Σ_1 such that $\mathcal{I} \models \mathcal{K}_1$ and $(\mathcal{I}, \mathcal{U}_{\mathcal{A}_2}) \models \mathcal{T}_{12}$, where $\mathcal{U}_{\mathcal{A}_2}$ is the interpretation of Σ_2 naturally corresponding[2] to $\mathcal{A}_2$. The correctness of the algorithm is a consequence of the facts that:

a) there exists a universal solution for $\mathcal{A}_1$ under $\mathcal{M}$ if and only if $\mathcal{A}_2$ is a solution for $\mathcal{A}_1$ under $\mathcal{M}$; and

b) $\mathcal{A}_2$ is a solution for $\mathcal{A}_1$ under $\mathcal{M}$ if and only if there exists a model $\mathcal{I}$ of $\mathcal{K}_1$ such that $(\mathcal{I}, \mathcal{U}_{\mathcal{A}_2}) \models \mathcal{T}_{12}$.

Moreover, the algorithm can be implemented in a non-deterministic polynomial-time Turing machine given that: *(i)* $\mathcal{A}_2$ can be constructed in polynomial time; *(ii)* if there exists a model $\mathcal{I}$ of $\mathcal{K}_1$ such that $(\mathcal{I}, \mathcal{U}_{\mathcal{A}_2}) \models \mathcal{T}_{12}$, then there exists a model of $\mathcal{K}_1$ of polynomial-size satisfying this condition; and *(iii)* it can be checked in polynomial time whether $\mathcal{I} \models \mathcal{K}_1$ and $(\mathcal{I}, \mathcal{U}_{\mathcal{A}_2}) \models \mathcal{T}_{12}$.

In addition, in this case, the membership problem can be reduced to the non-emptiness problem, thus, we have that:

Theorem 5.3 *The non-emptiness and membership problems for universal solutions are in* NP.

The exact complexity of these problems remains open. In fact, we conjecture that these problems are in PTIME.

We conclude by showing that reasoning about universal UCQ-solutions is harder than reasoning about universal solutions, which can be explained by the fact that TBoxes have

[2]Interpretation $\mathcal{U}_{\mathcal{A}_2}$ can be defined as the Herbrand model of $\mathcal{A}_2$ extended with fresh domain elements to satisfy assertions of the form $\exists R(a)$ in $\mathcal{A}_2$.

bigger impact on the structure of universal UCQ-solutions rather than of universal solutions. In fact, by using a reduction from the validity problem for quantified Boolean formulas, similar to a reduction in [Konev *et al.*, 2011], we are able to prove the following:

Theorem 5.4 *The membership problem for universal* UCQ-*solutions is* PSPACE-*hard.*

5.2 Universal solutions with null values

We start by considering the non-emptiness problem for universal solutions with null values, that is, when extended ABoxes are allowed in universal solutions. As our first result, similar to the reduction above, we show that this problem is PSPACE-hard, and identify the inclusion of inverse roles as one of the main sources of complexity.

To obtain an upper bound for this problem, we use *two-way alternating automata on infinite trees (2ATA)*, which are a generalization of nondeterministic automata on infinite trees [Vardi, 1998] well suited for handling inverse roles in *DL-Lite*$_{\mathcal{R}}$. More precisely, given a KB $\mathcal{K}$, we first show that it is possible to construct the following automata:

- $\mathbb{A}^{can}_{\mathcal{K}}$ is a 2ATA that accepts trees corresponding to the canonical model of $\mathcal{K}$ [3] with nodes arbitrary labeled with a special symbol G;
- $\mathbb{A}^{mod}_{\mathcal{K}}$ is a 2ATA that accepts a tree if its subtree labeled with G corresponds to a tree model $\mathcal{I}$ of $\mathcal{K}$ (that is, a model forming a tree on the labeled nulls); and
- $\mathbb{A}_{fin}$ is a (one-way) non-deterministic automaton that accepts a tree if it has a finite prefix where each node is marked with G, and no other node in the tree is marked with G.

Then to verify whether a KB $\mathcal{K}_1 = \langle \mathcal{T}_1, \mathcal{A}_1 \rangle$ has a universal solution under a mapping $\mathcal{M} = (\Sigma_1, \Sigma_2, \mathcal{T}_{12})$, we solve the non-emptiness problem for an automaton $\mathbb{B}$ defined as the product automaton of $\pi_{\Gamma_\mathcal{K}}(\mathbb{A}^{can}_{\mathcal{K}})$, $\pi_{\Gamma_\mathcal{K}}(\mathbb{A}^{mod}_{\mathcal{K}})$ and $\mathbb{A}_{fin}$, where $\mathcal{K} = \langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle$, $\pi_{\Gamma_\mathcal{K}}(\mathbb{A}^{can}_{\mathcal{K}})$ is the projection of $\mathbb{A}^{can}_{\mathcal{K}}$ on a vocabulary $\Gamma_\mathcal{K}$ not mentioning symbols from Σ_1, and likewise for $\pi_{\Gamma_\mathcal{K}}(\mathbb{A}^{mod}_{\mathcal{K}})$. If the language accepted by $\mathbb{B}$ is empty, then there is no universal solution for $\mathcal{K}_1$ under $\mathcal{M}$, otherwise a universal solution (possibly of exponential size) exists, and we can compute it by extracting the ABox encoded in some tree accepted by $\mathbb{B}$. Summing up, we get:

Theorem 5.5 *If extended ABoxes are allowed in universal solutions, then the non-emptiness problem for universal solutions is* PSPACE-*hard and in* EXPTIME.

Interestingly, the membership problem can be solved more efficiently in this scenario, as now the candidate universal solutions are part of the input. In the following theorem, we pinpoint the exact complexity of this problem.

Theorem 5.6 *If extended ABoxes are allowed in universal solutions, then the membership problem for universal solutions is* NP-*complete.*

[3] If $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$, then this model essentially corresponds to the chase of $\mathcal{A}$ with $\mathcal{T}$ (see [Konev *et al.*, 2011] for a formal definition).

6 Computing UCQ-representations

In Section 5, we show that the complexity of the membership and non-emptiness problems for universal solutions differ depending on whether ABoxes or extended ABoxes are considered. On the other hand, we show in the following proposition that the use of null values in ABoxes does not make any difference in the case of UCQ-representations. In this proposition, given a mapping $\mathcal{M}$ and TBoxes $\mathcal{T}_1$, $\mathcal{T}_2$, we say that $\mathcal{T}_2$ is a UCQ-representation of $\mathcal{T}_1$ under $\mathcal{M}$ *considering extended ABoxes* if $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{M}$ satisfy Equation (†) in Section 3.1, but assuming that $\mathcal{A}_2$ is an extended ABox over Σ_2 that is a UCQ-solution for $\mathcal{A}_1$ under $\mathcal{M}$.

Proposition 6.1 *A TBox $\mathcal{T}_2$ is a* UCQ-*representation of a TBox $\mathcal{T}_1$ under a mapping $\mathcal{M}$ if and only if $\mathcal{T}_2$ is a* UCQ-*representation of $\mathcal{T}_1$ under $\mathcal{M}$ considering extended ABoxes.*

Thus, from now on we study the membership and non-emptiness problems for UCQ-representations assuming that ABoxes can contain null values.

We start by considering the membership problem for UCQ-representations. In this case, one can immediately notice some similarities between this task and the membership problem for universal UCQ-solutions, which was shown to be PSPACE-hard in Theorem 5.4. However, the universal quantification over ABoxes in the definition of the notion of UCQ-representation makes the latter problem computationally simpler, which is illustrated by the following example.

Example 6.2 *Assume that $\mathcal{M} = (\Sigma_1, \Sigma_2, \mathcal{T}_{12})$, where $\Sigma_1 = \{F(\cdot), S_1(\cdot,\cdot), S_2(\cdot,\cdot), T_1(\cdot,\cdot), T_2(\cdot,\cdot)\}$, $\Sigma_2 = \{F'(\cdot), S'(\cdot,\cdot), T'(\cdot,\cdot), G'(\cdot)\}$ and $\mathcal{T}_{12} = \{F \sqsubseteq F', S_1 \sqsubseteq S', S_2 \sqsubseteq S', T_1 \sqsubseteq T', T_2 \sqsubseteq T', \exists T_1^- \sqsubseteq G'\}$. Moreover, assume that $\mathcal{T}_1 = \{F \sqsubseteq \exists S_1, F \sqsubseteq \exists S_2, \exists S_1^- \sqsubseteq \exists T_1, \exists S_2^- \sqsubseteq \exists T_2\}$ and $\mathcal{T}_2 = \{F' \sqsubseteq \exists S', \exists S'^- \sqsubseteq \exists T', \exists T'^- \sqsubseteq G'\}$. If we were to verify whether $\langle \mathcal{T}_2, \{F'(a)\} \rangle$ is a universal* UCQ-*solution for $\langle \mathcal{T}_1, \{F(a)\} \rangle$ under $\mathcal{M}$ (which it is in this case), then we would first need to construct the* path *$\pi = \langle F'(a), S'(a,n), T'(n,m), G'(m) \rangle$ formed by the inclusions in $\mathcal{T}_2$, where n, m are fresh null values, and then we would need to explore the translations according to $\mathcal{M}$ of all paths formed by the inclusions in $\mathcal{T}_1$ to find one that matches π.*

On the other hand, to verify whether $\mathcal{T}_2$ is a UCQ-*representation of $\mathcal{T}_1$ under $\mathcal{M}$, one does not need to execute any "backtracking", as it is sufficient to consider independently a polynomial number of pieces $\mathcal{C}$ taken from the paths formed by the inclusions in $\mathcal{T}_1$, each of them of polynomial size, and then checking whether the translation $\mathcal{C}'$ of $\mathcal{C}$ according to $\mathcal{M}$ matches with the paths formed from $\mathcal{C}'$ by the inclusions in $\mathcal{T}_2$. If any of these pieces does not satisfy this condition, then it can be transformed into a witness that Equation (†) is not satisfied, showing that $\mathcal{T}_2$ is not a* UCQ-*representation of $\mathcal{T}_1$ under $\mathcal{M}$ (as we have a universal quantification over the ABoxes over Σ_1 in the definition of* UCQ-*representations). In fact, one of the pieces considered in this case is $\mathcal{C} = \langle T_2(n,m) \rangle$, where n, m are null values, which does not satisfy the previous condition as the translation $\mathcal{C}'$ of $\mathcal{C}$ according to $\mathcal{M}$ is $\langle T'(n,m) \rangle$, and this does not match with the path $\langle T'(n,m), G'(m) \rangle$ formed from $\mathcal{C}'$ by the inclusions in $\mathcal{T}_2$. This particular case is transformed into an ABox*

$\mathcal{A}_1 = \{T_2(b, c)\}$ *and a query* $q = T'(b, c) \wedge G'(c)$, *where* b, c *are fresh constants, for which we have that Equation* (†) *is not satisfied.*

Notice that disjointness assertions in the mapping may cause $\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle$ to become inconsistent for some source ABoxes $\mathcal{A}_1$ (which will make all possible tuples to be in the answer to every query), therefore additional conditions have to be imposed on $\mathcal{T}_2$. To give more intuition about how the membership problem for UCQ-representations is solved, we give an example showing how one can deal with some of these inconsistency issues.

Example 6.3 *Assume that* $\mathcal{M} = (\Sigma_1, \Sigma_2, \mathcal{T}_{12})$, *where* $\Sigma_1 = \{F(\cdot), G(\cdot), H(\cdot)\}$, $\Sigma_2 = \{F'(\cdot), G'(\cdot), H'(\cdot)\}$ *and* $\mathcal{T}_{12} = \{F \sqsubseteq F', G \sqsubseteq G', H \sqsubseteq H'\}$. *Moreover, assume that* $\mathcal{T}_1 = \{F \sqsubseteq G\}$ *and* $\mathcal{T}_2 = \{F' \sqsubseteq G'\}$. *In this case, it is clear that* $\mathcal{T}_2$ *is a* UCQ*-representation of* $\mathcal{T}_1$ *under* $\mathcal{M}$. *However, if we add inclusion* $H \sqsubseteq \neg G'$ *to* $\mathcal{T}_{12}$, *then* $\mathcal{T}_2$ *is no longer a* UCQ*-representation of* $\mathcal{T}_1$ *under* $\mathcal{M}$. *To see why this is the case, consider an ABox* $\mathcal{A}_1 = \{F(a), H(a)\}$, *which is consistent with* $\mathcal{T}_1$, *and a query* $q = F'(b)$, *where* b *is a fresh constant. Then we have that* $cert(q, \langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle) = \{()\}$ *as KB* $\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle$ *is inconsistent, while* $cert(q, \langle \mathcal{T}_2, \mathcal{A}_2 \rangle) = \emptyset$ *for* UCQ*-solution* $\mathcal{A}_2 = \{F'(a), H'(a)\}$ *for* $\mathcal{A}_1$ *under* $\mathcal{M}$. *Thus, we conclude that Equation* (†) *is violated in this case.*

One can deal with the issue raised in the previous example by checking that on every pair (B, B') of $\mathcal{T}_1$-consistent basic concepts over Σ_1,[4] it holds that: (B, B') is $(\mathcal{T}_1 \cup \mathcal{T}_{12})$-consistent if and only if (B, B') is $(\mathcal{T}_{12} \cup \mathcal{T}_2)$-consistent, and likewise for every pair of basic roles over Σ_1. This condition guarantees that for every ABox $\mathcal{A}_1$ over Σ_1 that is consistent with $\mathcal{T}_1$, it holds that: $\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle$ is consistent if and only if there exists an extended ABox $\mathcal{A}_2$ over Σ_2 such that $\mathcal{A}_2$ is a UCQ-solution for $\mathcal{A}_1$ under $\mathcal{M}$ and $\langle \mathcal{T}_2, \mathcal{A}_2 \rangle$ is consistent. Thus, the previous condition ensures that the sets on the left- and right-hand side of Equation (†) coincide whenever the intersection on either of these sides is taken over an empty set.

The following theorem, which requires of a lengthy and non-trivial proof, shows that there exists an efficient algorithm for the membership problem for UCQ-representations that can deal with all the aforementioned issues.

Theorem 6.4 *The membership problem for* UCQ*-representations is* NLOGSPACE*-complete.*

We conclude by pointing out that the non-emptiness problem for UCQ-representations can also be solved efficiently. We give an intuition of how this can be done in the following example, where we say that $\mathcal{T}_1$ is UCQ*-representable under* $\mathcal{M}$ if there exists a UCQ-representation $\mathcal{T}_2$ of $\mathcal{T}_1$ under $\mathcal{M}$.

Example 6.5 *Assume that* $\mathcal{M} = (\Sigma_1, \Sigma_2, \mathcal{T}_{12})$, *where* $\Sigma_1 = \{F(\cdot), G(\cdot), H(\cdot)\}$, $\Sigma_2 = \{F'(\cdot), G'(\cdot)\}$ *and* $\mathcal{T}_{12} = \{F \sqsubseteq F', G \sqsubseteq G', H \sqsubseteq F'\}$. *Moreover, assume that* $\mathcal{T}_1 = \{F \sqsubseteq G\}$. *Then it follows that* $\mathcal{T}_1 \cup \mathcal{T}_{12} \models F \sqsubseteq G'$, *and in order for* $\mathcal{T}_1$ *to be* UCQ*-representable under* $\mathcal{M}$, *the following condition must be satisfied:*

($\star$) *there exists a concept* B' *over* Σ_2 *s.t.* $\mathcal{T}_{12} \models F \sqsubseteq B'$, *and for each concept* B *over* Σ_1 *with* $\mathcal{T}_1 \cup \mathcal{T}_{12} \models B \sqsubseteq B'$ *it follows that* $\mathcal{T}_1 \cup \mathcal{T}_{12} \models B \sqsubseteq G'$.

The idea is then to add the inclusion $B' \sqsubseteq G'$ *to a* UCQ*-representation* $\mathcal{T}_2$ *so that* $\mathcal{T}_{12} \cup \mathcal{T}_2 \models F \sqsubseteq G'$ *as well. In our case, concept* F' *satisfies the condition* $\mathcal{T}_{12} \models F \sqsubseteq F'$, *but it does not satisfy the second requirement as* $\mathcal{T}_1 \cup \mathcal{T}_{12} \models H \sqsubseteq F'$ *and* $\mathcal{T}_1 \cup \mathcal{T}_{12} \not\models H \sqsubseteq G'$. *In fact,* $F' \sqsubseteq G'$ *cannot be added to* $\mathcal{T}_2$ *as it would result in* $\mathcal{T}_{12} \cup \mathcal{T}_2 \models H \sqsubseteq G'$, *hence in Equation* (†), *the inclusion from right to left would be violated. There is no way to reflect the inclusion* $F \sqsubseteq G'$ *in the target, so in this case* $\mathcal{T}_1$ *is not* UCQ*-representable under* $\mathcal{M}$.

The proof of the following result requires of some involved extensions of the techniques used to prove Theorem 6.4.

Theorem 6.6 *The non-emptiness problem for* UCQ*-representations is* NLOGSPACE*-complete.*

The techniques used to prove Theorem 6.6, which is sketched in the example below.

Example 6.7 *Consider* $\mathcal{M}$ *and* $\mathcal{T}_1$ *from Example 6.5, but assuming that* $\mathcal{T}_{12}$ *does not contain the inclusion* $H \sqsubseteq F'$. *Again,* $\mathcal{T}_1 \cup \mathcal{T}_{12} \models F \sqsubseteq G'$, *but now condition* ($\star$) *is satisfied. Then, an algorithm for computing a representation essentially needs to take any* B' *given by condition* ($\star$) *and add the inclusion* $B' \sqsubseteq F'$ *to* $\mathcal{T}_2$. *In this case,* $\mathcal{T}_2 = \{F' \sqsubseteq G'\}$ *is a* UCQ*-representation of* $\mathcal{T}_1$ *under* $\mathcal{M}$.

7 Conclusions

In this paper, we have studied the problem of KB exchange for OWL 2 QL, improving on previously known results with respect to both the expressiveness of the ontology language and the understanding of the computational properties of the problem. Our investigation leaves open several issues, which we intend to address in the future. First, it would be good to have characterizations of classes of source KBs and mappings for which universal (UCQ-)solutions are guaranteed to exist. As for the computation of universal solutions, while we have pinned-down the complexity of membership for extended ABoxes as NP-complete, an exact bound for the other case is still missing. Moreover, it is easy to see that allowing for inequalities between terms (e.g., $a \neq b$ in Example 3.1) and for negated atoms in the (target) ABox would allow one to obtain more universal solutions, but a full understanding of this case is still missing. Finally, we intend to investigate the challenging problem of computing universal UCQ-solutions, adopting also here an automata-based approach.

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[4] A pair $(B, B)'$ is $\mathcal{T}$-consistent for a TBox $\mathcal{T}$, if the KB $\langle \mathcal{T}, \{B(a), B'(a)\} \rangle$ is consistent, where a is an arbitrary constant.

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A Definitions and Preliminary Results

Let Σ be a *DL-Lite*$_\mathcal{R}$ signature; a concept name A (role name P) is said to be over Σ, if $A \in \Sigma$ ($P \in \Sigma$). A basic role R is said to be over Σ, if, either it is a role name over Σ, or $R = P^-$ for a role P over Σ; a basic concept B is said to be over Σ, if either it is a concept name, which is over Σ, or $B = \exists R$ and R is a basic role over Σ. We naturally extend these definitions to TBoxes, ABoxes, KBs, and queries; so we can refer to Σ-TBoxes or TBoxes over Σ, and analogiously for ABoxes, KBs, and queries.

Define relation $\sqsubseteq^\mathcal{R}_\mathcal{T}$ to be the reflexive and transitive closure of the following relation on the set of all basic roles over N_R:

$$\{(R_1, R_2) \mid R_1 \sqsubseteq R_2 \in \mathcal{T} \text{ or } R_1^- \sqsubseteq R_2^- \in \mathcal{T}\},$$

and let $\sqsubseteq^\mathcal{C}_\mathcal{T}$ be the reflexive and transitive closure of the following relation on the set of all basic concepts over N_C:

$$\{(B_1, B_2) \mid B_1 \sqsubseteq B_2 \in \mathcal{T}\} \cup \{(\exists R_1, \exists R_2) \mid R_1 \sqsubseteq^\mathcal{R}_\mathcal{T} R_2\}.$$

Then define the relation $\vdash$ between $\mathcal{K}$ and the *DL-Lite*$_\mathcal{R}$ membership assertions over Σ as:

$$\{(\mathcal{K}, B(a)) \mid \text{there exists a basic concept } B' \text{ s.t. } \mathcal{A} \models B'(a) \text{ and } B' \sqsubseteq^\mathcal{C}_\mathcal{T} B\} \cup$$
$$\{(\mathcal{K}, R(a,b)) \mid \text{there exists a basic role } R' \text{ s.t. } \mathcal{A} \models R'(a,b) \text{ and } R' \sqsubseteq^\mathcal{R}_\mathcal{T} R\}.$$

Notice that for consistent $\mathcal{K}$, for every membership assertion α it holds that $\mathcal{K} \vdash \alpha$ if and only if $\mathcal{K} \models \alpha$. Moreover, for every basic role R over N_R, define $[R]$ as $\{S \mid R \sqsubseteq^\mathcal{R}_\mathcal{T} S \text{ and } S \sqsubseteq^\mathcal{R}_\mathcal{T} R\}$, and then let $\leq_\mathcal{T}$ be a partial order on the set $\{[R] \mid R \text{ is a basic role over } N_R\}$ defined as $[R] \leq_\mathcal{T} [S]$ if $R \sqsubseteq^\mathcal{R}_\mathcal{T} S$. For each set $[R]$, where R is a basic role, consider an element $w_{[R]}$, *witness for* $[R]$. Now, define a *generating relationship* $\rightsquigarrow_\mathcal{K}$ between the set $N_a \cup \{w_{[R]} \mid R \text{ is a basic role}\}$ and the set $\{w_{[R]} \mid R \text{ is a basic role}\}$, as follows:

- $a \rightsquigarrow_\mathcal{K} w_{[R]}$, if (1) $\mathcal{K} \vdash \exists R(a)$; (2) $\mathcal{K} \not\vdash R(a,b)$ for every $b \in N_a$; (3) $[R'] = [R]$ for every $[R']$ such that $[R'] \leq_\mathcal{T} [R]$ and $\mathcal{K} \vdash \exists R'(a)$.
- $w_{[S]} \rightsquigarrow_\mathcal{K} w_{[R]}$, if (1) $\mathcal{T} \vdash \exists S^- \sqsubseteq \exists R$; (2) $[S^-] \neq [R]$; (3) $[R'] = [R]$ for every $[R']$ such that $[R'] \leq_\mathcal{T} [R]$ and $\mathcal{T} \vdash \exists S^- \sqsubseteq \exists R'$.

Denote by $\mathsf{path}(\mathcal{K})$ the set of all $\mathcal{K}$-paths, where a $\mathcal{K}$-path is a sequence $a \cdot w_{[R_1]} \cdot \ldots \cdot w_{[R_n]}$ (sometimes we simply write $a w_{[R_1]} \ldots w_{[R_n]}$) such that $a \in N_a$, $a \rightsquigarrow_\mathcal{K} w_{[R_1]}$ and $w_{[R_i]} \rightsquigarrow_\mathcal{K} w_{[R_{i+1}]}$ for every $i \in \{1, \ldots, n-1\}$. Moreover, for every $\sigma \in \mathsf{path}(\mathcal{K})$, denote by $\mathsf{tail}(\sigma)$ the last element in σ.

With all the previous notation, we can finally define the *canonical model* $\mathcal{U}_\mathcal{K}$. The domain $\Delta^{\mathcal{U}_\mathcal{K}}$ of $\mathcal{U}_\mathcal{K}$ is defined as $\mathsf{path}(\mathcal{K})$, and $a^{\mathcal{U}_\mathcal{K}} = a$ for every $a \in N_a$. Moreover, for every concept A:

$$A^{\mathcal{U}_\mathcal{K}} = \{\sigma \in \mathsf{path}(\mathcal{K}) \mid \mathcal{K} \vdash A(\mathsf{tail}(\sigma)) \text{ or } \mathsf{tail}(\sigma) = w_{[R]} \text{ and } \mathcal{T} \vdash \exists R^- \sqsubseteq A\},$$

and for every role P, we have that $P^{\mathcal{U}_\mathcal{K}}$ is defined as follows:

$$\begin{aligned}\{(\sigma_1, \sigma_2) \in \mathsf{path}(\mathcal{K}) \times \mathsf{path}(\mathcal{K}) \mid \mathcal{K} \vdash P(\mathsf{tail}(\sigma_1), \mathsf{tail}(\sigma_2));\ \text{or}& \\ \sigma_2 = \sigma_1 \cdot w_{[R]},\ \mathsf{tail}(\sigma_1) \rightsquigarrow_\mathcal{K} w_{[R]} \text{ and } [R] \leq_\mathcal{T} [P];\ \text{or}& \\ \sigma_1 = \sigma_2 \cdot w_{[R]},\ \mathsf{tail}(\sigma_2) \rightsquigarrow_\mathcal{K} w_{[R]} \text{ and } [R] \leq_\mathcal{T} [P^-]\}.&\end{aligned}$$

Notice that $\mathcal{U}_\mathcal{K}$ defined above can be treated (by ignoring sets $N^{\mathcal{U}_\mathcal{K}}$ for some concepts and role names N) as a Σ-interpretation, for any Σ. Denote also by $\mathsf{Ind}(\mathcal{A})$ the set of constants occuring in $\mathcal{A}$.

Let us point out the similarity of our definition of $\mathcal{U}_\mathcal{K}$ with the definition of the *canonical model* $\mathcal{M}_\mathcal{K}$ defined in [Konev *et al.*, 2011]. When $\mathcal{K}$ is consistent, many results proved there for $\mathcal{M}_\mathcal{K}$ apply to $\mathcal{U}_\mathcal{K}$. In particular, from the proof of Theorem 5 in [Konev *et al.*, 2011] we can immediately conclude:

Claim A.1 *If $\mathcal{K}$ is consistent, $\mathcal{U}_\mathcal{K}$ is a model of $\mathcal{K}$.*

We are going to introduce the notions of Σ-types and Σ-homomorphisms, heavily employed in the proofs. For an interpretation $\mathcal{I}$ and a signature Σ, the Σ-*types* $\mathbf{t}^\mathcal{I}_\Sigma(x)$ and $\mathbf{r}^\mathcal{I}_\Sigma(x,y)$ for $x, y \in \Delta^\mathcal{I}$ are given by

$$\mathbf{t}^\mathcal{I}_\Sigma(x) = \{B \text{ - basic concept over } \Sigma \mid x \in B^\mathcal{I}\},$$
$$\mathbf{r}^\mathcal{I}_\Sigma(x,y) = \{R \text{ - basic role over } \Sigma \mid (x,y) \in R^\mathcal{I}\}.$$

We also use $\mathbf{t}^{\mathcal{I}}(x)$ and $\mathbf{r}^{\mathcal{I}}(x,y)$ to refer to the types over the signature of all *DL-Lite* concepts and roles. A Σ-*homomorphism* from an interpretation $\mathcal{I}$ to $\mathcal{I}'$ is a function $h : \Delta^{\mathcal{I}} \mapsto \Delta^{\mathcal{I}'}$ such that $h(a^{\mathcal{I}}) = a^{\mathcal{I}'}$, for all individual names a interpreted in $\mathcal{I}$, $\mathbf{t}^{\mathcal{I}}_{\Sigma}(x) \subseteq \mathbf{t}^{\mathcal{I}'}_{\Sigma}(h(x))$ and $\mathbf{r}^{\mathcal{I}}_{\Sigma}(x,y) \subseteq \mathbf{r}^{\mathcal{I}'}_{\Sigma}(h(x),h(y))$ for all $x, y \in \Delta^{\mathcal{I}}$. We say that $\mathcal{I}$ is (*finitely*) Σ-*homomorphically embeddable* into $\mathcal{I}'$ if, for every (finite) subinterpretation $\mathcal{I}_1$ of $\mathcal{I}$, there exists a Σ-homomorphism from $\mathcal{I}_1$ to $\mathcal{I}'$. If Σ is a set of all *DL-Lite* concepts and roles, we call Σ-homomorphism simply *homomorphism*.

The claim below from the proof of Theorem 5 in [Konev *et al.*, 2011] establishes the relation between $\mathcal{U}_{\mathcal{K}}$ and the models of $\mathcal{K}$.

Claim A.2 *For every model $\mathcal{I} \models \mathcal{K}$, there exists a homomorphism from $\mathcal{U}_{\mathcal{K}}$ to $\mathcal{I}$.*

Another result follows from Theorem 5 in [Konev *et al.*, 2011]:

Claim A.3 *For each consistent KB $\mathcal{K}$, every UCQ $q(\vec{x})$ and tuple $\vec{a} \subseteq N_a$, it holds $\mathcal{K} \models q[\vec{a}]$ iff $\mathcal{U}_{\mathcal{K}} \models q[\vec{a}]$.*

It is important to notice that the notion of certain answers can be characterized through the notion of canonical model. Finally, for a signature Σ and two KBs $\mathcal{K}_1 = \langle \mathcal{T}_1, \mathcal{A}_1 \rangle$ and $\mathcal{K}_2 = \langle \mathcal{T}_2, \mathcal{A}_2 \rangle$, we say that $\mathcal{K}_1$ Σ-*query entails* $\mathcal{K}_2$ if, for all Σ-queries $q(\vec{x})$ and all $\vec{a} \subseteq N_a$, $\mathcal{K}_2 \models q[\vec{a}]$ implies $\mathcal{K}_1 \models q[\vec{a}]$. The KBs $\mathcal{K}_1$ and $\mathcal{K}_2$ are said to be Σ-*query equivalent* if $\mathcal{K}_1$ Σ-query entails $\mathcal{K}_2$ and vice versa. The following is a consequence of Theorem 7 in [Konev *et al.*, 2011]:

Claim A.4 *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be consistent KBs. Then $\mathcal{K}_1$ Σ-query entails $\mathcal{K}_2$ iff $\mathcal{U}_{\mathcal{K}_2}$ is finitely Σ-homomorphically embeddable into $\mathcal{U}_{\mathcal{K}_1}$.*

B Proofs in Section 5

B.1 Definitions and Preliminary Results: Characterization of Universal Solutions

First, we define the notion of canonical model for extended ABoxes. Let $\mathcal{A}$ be an extended ABox. Without loss of generality, assume that $\mathcal{A}$ does not contain assertions of the form $\exists R(x)$. Then the *canonical model* of $\mathcal{A}$, denoted $\mathcal{V}_{\mathcal{A}}$ is defined as follows: $\Delta^{\mathcal{V}_{\mathcal{A}}} = \mathsf{Null}(\mathcal{A}) \cup N_a$, where $\mathsf{Null}(\mathcal{A})$ is the set of labeled nulls mentioned in $\mathcal{A}$, $a^{\mathcal{V}_{\mathcal{A}}} = a$ for each $a \in N_a$, $A^{\mathcal{V}_{\mathcal{A}}} = \{x \in \Delta^{\mathcal{V}_{\mathcal{A}}} \mid A(x) \in \mathcal{A}\}$ for each atomic concept A, and $P^{\mathcal{V}_{\mathcal{A}}} = \{(x,y) \in \Delta^{\mathcal{V}_{\mathcal{A}}} \times \Delta^{\mathcal{V}_{\mathcal{A}}} \mid P(x,y) \in \mathcal{A}\}$ for each atomic role P. Let h be a function from $N_a \cup N_l \to \Delta^{\mathcal{V}_{\mathcal{A}_2}}$ such that $h(a) = a$ for every $a \in N_a$ and $h(x) = x$ for every $x \in N_l$. Then

Lemma B.1 *$\mathcal{V}_{\mathcal{A}_2}$ is a model of $\mathcal{A}_2$ with substitution h.*

Lemma B.2 *For every model $\mathcal{I} \models \mathcal{A}_2$, there exists a homomorphism from $\mathcal{V}_{\mathcal{A}_2}$ to $\mathcal{I}$.*

Proof. Let $\mathcal{I}$ be a model of $\mathcal{A}_2$ with a substitution h'. Then h' is the desired homomorphism from $\mathcal{V}_{\mathcal{A}_2}$ to $\mathcal{I}$. $\square$

Given an extended ABox $\mathcal{A}$, we denote by $\Delta^{\mathcal{A}}$ the set of all constants and nulls mentioned in $\mathcal{A}$, $\Delta^{\mathcal{A}} = \mathsf{Ind}(\mathcal{A}) \cup \mathsf{Null}(\mathcal{A})$. Moreover, given an interpretation $\mathcal{I}$, the *size* of $\mathcal{I}$, denoted $|\mathcal{I}|$, is the sum of cardinalities of interpretations of all predicates (the domain is not included as it is always infinite).

Let us denote by *DL-Lite*$^{pos}_{\mathcal{R}}$ the positive fragment of *DL-Lite*$_{\mathcal{R}}$. More precisely, a *DL-Lite*$^{pos}_{\mathcal{R}}$ TBox is a finite set of concept inclusions $B_1 \sqsubseteq B_2$, where B_1, B_2 are basic concepts, and role inclusions $R_1 \sqsubseteq R_2$, where R_1, R_2 are basic roles, and a *DL-Lite*$^{pos}_{\mathcal{R}}$ KB $\mathcal{K}$ is a pair $\langle \mathcal{T}, \mathcal{A} \rangle$, where $\mathcal{T}$ is a *DL-Lite*$^{pos}_{\mathcal{R}}$ TBox and $\mathcal{A}$ is an (extended) *DL-Lite*$_{\mathcal{R}}$ ABox (without inequalities).

Lemma B.3 *Let $\mathcal{M} = (\Sigma_1, \Sigma_2, \mathcal{T}_{12})$ be a DL-Lite$^{pos}_{\mathcal{R}}$ mapping, $\mathcal{K}_1 = \langle \mathcal{T}_1, \mathcal{A}_1 \rangle$ a DL-Lite$^{pos}_{\mathcal{R}}$ KBs over Σ_1, and $\mathcal{A}_2$ an (extended, without inequalities, without negation) ABox over Σ_2. Then, $\mathcal{A}_2$ is a universal solution (with extended ABoxes) for $\mathcal{K}_1$ under $\mathcal{M}$ iff $\mathcal{V}_{\mathcal{A}_2}$ is Σ_2-homomorphically equivalent to $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$.*

Proof. ($\Rightarrow$) Let $\mathcal{A}_2$ be a universal solution for $\mathcal{K}_1$ under $\mathcal{M}$. Then $\mathcal{V}_{\mathcal{A}_2}$ is Σ_2-homomorphically equivalent to $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$: since $\mathcal{A}_2$ is a solution, there exists $\mathcal{I}$ a model of $\mathcal{K}_1$ such that $(\mathcal{I}, \mathcal{V}_{\mathcal{A}_2}) \models \mathcal{T}_{12}$. Then $\mathcal{I} \cup \mathcal{V}_{\mathcal{A}_2}$ is a model of $\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle$, therefore there is a homomorphism h from $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$ to $\mathcal{I} \cup \mathcal{V}_{\mathcal{A}_2}$. As Σ_1 and Σ_2 are disjoint signatures it follows that h is a Σ_2-homomorphism from $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$ to $\mathcal{V}_{\mathcal{A}_2}$. On the other hand, as $\mathcal{A}_2$ is a universal solution, $\mathcal{J}$, the interpretation of Σ_2 obtained from $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$ is a model of $\mathcal{A}_2$ with a substitution h'. This h' is exactly a homomorphism from $\mathcal{V}_{\mathcal{A}_2}$ to $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$.

($\Leftarrow$) Assume $\mathcal{V}_{\mathcal{A}_2}$ is Σ_2-homomorphically equivalent to $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$. We show that $\mathcal{A}_2$ is a universal solution for $\mathcal{K}_1$ under $\mathcal{M}$.

First, $\mathcal{A}_2$ is a solution for $\mathcal{K}_1$ under $\mathcal{M}$. Let $\mathcal{J}$ be a model of $\mathcal{A}_2$, and h_1 a homomorphism from $\mathcal{V}_{\langle\emptyset,\mathcal{A}_2\rangle}$ to $\mathcal{J}$. Furthermore, let h be a Σ_2-homomorphism from $\mathcal{U}_{\langle\mathcal{T}_1\cup\mathcal{T}_{12},\mathcal{A}_1\rangle}$ to $\mathcal{V}_{\mathcal{A}_2}$. Then $h' = h_1 \circ h$ is a Σ_2-homomorphism from $\mathcal{U}_{\langle\mathcal{T}_1\cup\mathcal{T}_{12},\mathcal{A}_1\rangle}$ to $\mathcal{J}$. Let $\mathcal{I}$ be the interpretation of Σ_1 defined as the image of h' applied to $\mathcal{U}_{\mathcal{K}_1}$, $\mathcal{I} = h'(\mathcal{U}_{\mathcal{K}_1})$. The it is easy to see that $\mathcal{I}$ is a model of $\mathcal{K}_1$ and $(\mathcal{I},\mathcal{J}) \models \mathcal{M}$ as $\mathcal{K}_1$ and $\mathcal{M}$ contain only positive information. Indeed, $\mathcal{A}_2$ is a solution for $\mathcal{K}_1$ under $\mathcal{M}$.

Second, $\mathcal{A}_2$ is a universal solution. Let $\mathcal{I}$ be a model of $\mathcal{K}_1$ and $\mathcal{J}$ an interpretation of Σ_2 such that $(\mathcal{I},\mathcal{J}) \models \mathcal{M}$. Then, since $\mathcal{U}_{\langle\mathcal{T}_1\cup\mathcal{T}_{12},\mathcal{A}_1\rangle}$ is the canonical model of $\mathcal{K}_1 \cup \mathcal{T}_{12}$, there exists a homomorphism h from $\mathcal{U}_{\langle\mathcal{T}_1\cup\mathcal{T}_{12},\mathcal{A}_1\rangle}$ to $\mathcal{I}\cup\mathcal{J}$ ($\mathcal{I}\cup\mathcal{J}$ is a model of $\mathcal{K}_1 \cup \mathcal{T}_{12}$). In turn, there is a homomorphism h_1 from $\mathcal{V}_{\mathcal{A}_2}$ to $\mathcal{U}_{\langle\mathcal{T}_1\cup\mathcal{T}_{12},\mathcal{A}_1\rangle}$, therefore $h' = h \circ h_1$ is a homomorphism from $\mathcal{V}_{\mathcal{A}_2}$ to $\mathcal{I}\cup\mathcal{J}$, and a Σ_2-homomorphism from $\mathcal{V}_{\mathcal{A}_2}$ to $\mathcal{J}$. Hence, $\mathcal{J}$ is a model of $\mathcal{A}_2$: take h' as the substitution for the labeled nulls. By definition of universal solution, $\mathcal{A}_2$ is a universal solution for $\mathcal{K}_1$ under $\mathcal{M}$. $\square$

Definition B.4 *Let $\mathcal{M} = (\Sigma_1, \Sigma_2, \mathcal{T}_{12})$ be a DL-Lite$_\mathcal{R}$ mapping, and $\mathcal{K}_1 = \langle\mathcal{T}_1,\mathcal{A}_1\rangle$ a DL-Lite$_\mathcal{R}$ KB over Σ_1. Then, we say that $\mathcal{K}_1$ and $\mathcal{M}$ are Σ_2-*positive *if*

(a) *for each $b \in B^{\mathcal{U}_{\langle\mathcal{T}_1\cup\mathcal{T}_{12},\mathcal{A}_1\rangle}}$ and $c \in C^{\mathcal{U}_{\langle\mathcal{T}_1\cup\mathcal{T}_{12},\mathcal{A}_1\rangle}}$ with $\mathcal{T}_1 \models B \sqcap C \sqsubseteq \bot$, it is not the case that*

$$b \in \mathsf{InTarget} \quad \textit{and} \quad c \in \mathsf{InTarget},$$

(b) *for each $(b_1, b_2) \in R^{\mathcal{U}_{\langle\mathcal{T}_1\cup\mathcal{T}_{12},\mathcal{A}_1\rangle}}$ and $(c_1, c_2) \in Q^{\mathcal{U}_{\langle\mathcal{T}_1\cup\mathcal{T}_{12},\mathcal{A}_1\rangle}}$ with $\mathcal{T}_1 \models R \sqcap Q \sqsubseteq \bot$ for basic roles R, Q, it is not the case that*

$$b_i \in \mathsf{InTarget} \quad \textit{and} \quad c_i \in \mathsf{InTarget} \quad \textit{for } i = 1, 2,$$

(c) *for each $(a, b) \in R^{\mathcal{U}_{\langle\mathcal{T}_1\cup\mathcal{T}_{12},\mathcal{A}_1\rangle}}$ and $(a, c) \in Q^{\mathcal{U}_{\langle\mathcal{T}_1\cup\mathcal{T}_{12},\mathcal{A}_1\rangle}}$ with $\mathcal{T}_1 \models R \sqcap Q \sqsubseteq \bot$ for basic roles R, Q, it is not the case that*

$$b \in \mathsf{InTarget} \quad \textit{and} \quad c \in \mathsf{InTarget},$$

where

$$\mathsf{InTarget} = \{x \in \Delta^{\mathcal{U}_{\langle\mathcal{T}_1\cup\mathcal{T}_{12},\mathcal{A}_1\rangle}} \mid \mathbf{t}_{\Sigma_2}^{\mathcal{U}_{\langle\mathcal{T}_1\cup\mathcal{T}_{12},\mathcal{A}_1\rangle}}(x) \neq \emptyset\} \cup N_a$$

(d) *for each $B \sqsubseteq \neg B' \in \mathcal{T}_{12}$, $B^{\mathcal{U}_{\langle\mathcal{T}_1\cup\mathcal{T}_{12},\mathcal{A}_1\rangle}} = \emptyset$ and for each $R \sqsubseteq \neg R' \in \mathcal{T}_{12}$, $R^{\mathcal{U}_{\langle\mathcal{T}_1\cup\mathcal{T}_{12},\mathcal{A}_1\rangle}} = \emptyset$.*

In the following, given a TBox $\mathcal{T}$, we denote by $\mathcal{T}^{pos}$ the subset of $\mathcal{T}$ without disjointness assertions, and given a KB $\mathcal{K} = \langle\mathcal{T},\mathcal{A}\rangle$, we denote by $\mathcal{K}^{pos}$ the KB $\langle\mathcal{T}^{pos},\mathcal{A}\rangle$. Moreover, if $\mathcal{M} = (\Sigma_1, \Sigma_2, \mathcal{T}_{12})$ is a *DL-Lite$_\mathcal{R}$* mapping, then $\mathcal{M}^{pos}$ denotes the mapping $(\Sigma_1, \Sigma_2, \mathcal{T}_{12}^{pos})$.

Lemma B.5 *Let $\mathcal{M} = (\Sigma_1, \Sigma_2, \mathcal{T}_{12})$ be a DL-Lite$_\mathcal{R}$ mapping, $\mathcal{K}_1 = \langle\mathcal{T}_1,\mathcal{A}_1\rangle$ a DL-Lite$_\mathcal{R}$ KBs over Σ_1, and $\mathcal{A}_2$ an (extended, without inequalities, without negation) ABox over Σ_2. Then, $\mathcal{A}_2$ is a universal solution (with extended ABoxes) for $\mathcal{K}_1$ under $\mathcal{M}$ iff*

1. *$\mathcal{K}_1$ and $\mathcal{M}$ are Σ_2-positive,*
2. *$\mathcal{A}_2$ is a universal solution for $\mathcal{K}_1^{pos}$ under $\mathcal{M}^{pos}$.*

Proof. ($\Rightarrow$) Let $\mathcal{A}_2$ be a universal solution for $\mathcal{K}_1$ under $\mathcal{M}$. Then $\mathcal{A}_2$ is a universal solution for $\mathcal{K}_1^{pos}$ under $\mathcal{M}^{pos}$.

For the sake of contradiction, assume that $\mathcal{K}_1$ and $\mathcal{M}$ are not Σ_2-positive, and e.g., (**a**) does not hold, i.e., there is a disjointness constraint in $\mathcal{T}_1$ of the form $B \sqcap C \sqsubseteq \bot$, such that $b \in B^{\mathcal{U}_{\langle\mathcal{T}_1\cup\mathcal{T}_{12},\mathcal{A}_1\rangle}}$ and $c \in C^{\mathcal{U}_{\langle\mathcal{T}_1\cup\mathcal{T}_{12},\mathcal{A}_1\rangle}}$, and

$$\mathbf{t}_{\Sigma_2}^{\mathcal{U}_{\langle\mathcal{T}_1\cup\mathcal{T}_{12},\mathcal{A}_1\rangle}}(b) \neq \emptyset \quad \text{or} \quad b \in N_a,$$
$$\mathbf{t}_{\Sigma_2}^{\mathcal{U}_{\langle\mathcal{T}_1\cup\mathcal{T}_{12},\mathcal{A}_1\rangle}}(c) \neq \emptyset \quad \text{or} \quad c \in N_a.$$

Let h be a Σ_2-homomorphism from $\mathcal{U}_{\langle\mathcal{T}_1\cup\mathcal{T}_{12},\mathcal{A}_1\rangle}$ to $\mathcal{V}_{\mathcal{A}_2}$ (it exists by Lemma B.3). Then it follows that

$$\mathbf{t}_{\Sigma_2}^{\mathcal{V}_{\mathcal{A}_2}}(h(b)) \neq \emptyset \quad \text{or} \quad b \in N_a$$
$$\mathbf{t}_{\Sigma_2}^{\mathcal{V}_{\mathcal{A}_2}}(h(c)) \neq \emptyset \quad \text{or} \quad c \in N_a$$

Take a minimal model $\mathcal{J}$ of $\mathcal{A}_2$ with a substitution h' such that $h'(h(b)) = h'(h(c))$. Assume that both b and c are constants (i.e., $b^\mathcal{J} = c^\mathcal{J}$). Then, obviously there exists no model $\mathcal{I}$ of Σ_1 such that $\mathcal{I} \models \mathcal{K}_1$ and $(\mathcal{I},\mathcal{J}) \models \mathcal{T}_{12}$: in every such $\mathcal{I}$, $b^\mathcal{I}$ must be equal to $c^\mathcal{I}$ which contradicts $B \sqcap C \sqsubseteq \bot$, and $b^\mathcal{I} \in B^\mathcal{I}$ and $c^\mathcal{I} \in C^\mathcal{I}$. Now, assume that at least b is not a constant and $\mathsf{tail}(b) = w_{[R]}$ for some role R over Σ_1 (hence, $b \in (\exists R^-)^{\mathcal{U}_{\langle\mathcal{T}_1\cup\mathcal{T}_{12},\mathcal{A}_1\rangle}}$ and $\mathcal{T}_1 \models \exists R^- \sqsubseteq B$). Let $B' \in \mathbf{t}_{\Sigma_2}^{\mathcal{U}_{\langle\mathcal{T}_1\cup\mathcal{T}_{12},\mathcal{A}_1\rangle}}(b)$, then by construction of the canonical model, $\mathcal{T}_1 \cup \mathcal{T}_{12} \models \exists R^- \sqsubseteq B'$, by homomorphism, $B'(h(b)) \in \mathcal{A}_2$, $h'(h(b)) \in B'^{\mathcal{J}}$, and since $\mathcal{J}$ is a minimal model, $B'^{\mathcal{J}}$ is minimal. As $\mathcal{A}_2$ is a universal solution, let $\mathcal{I}$ be a model of $\mathcal{K}_1$ such that $(\mathcal{I}, \mathcal{J})$ satisfy $\mathcal{T}_{12}$. Then $(\exists R^-)^{\mathcal{I}}$ is not empty, and by minimality of $B'^{\mathcal{J}}$, it must be the case that $h'(h(b)) \in (\exists R^-)^{\mathcal{I}}$, hence $h'(h(b)) \in B^{\mathcal{I}}$. By a similar argument, it can be shown that $h'(h(c))$ must be in $C^{\mathcal{I}}$. As we took $\mathcal{J}$ such that $h'(h(b)) = h'(h(c))$, it contradicts that $\mathcal{I}$ is a model of $B \sqcap C \sqsubseteq \bot$. Contradiction with $\mathcal{A}_2$ being a universal solution. Similar to (**a**) we can derive a contradiction if assume that (**b**) or (**c**) does not hold.

Finally, assume (**d**) does not hold, i.e., $B \sqsubseteq \neg B' \in \mathcal{T}_{12}$ and $B^{\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}} \neq \emptyset$. Note that $\mathcal{A}_2$ is an extended ABox, i.e., it contains only assertions of the form $A(u)$, $P(u, v)$ for $u, v \in \mathcal{N}_a \cup N_l$. Take a model $\mathcal{J}$ of $\mathcal{A}_2$ such that $B'^{\mathcal{J}} = \Delta^{\mathcal{J}}$. Such $\mathcal{J}$ exists as $\mathcal{A}_2$ contains only positive facts. Since $\mathcal{A}_2$ is a universal solution, there exist a model $\mathcal{I}$ of $\mathcal{K}_1$ such that $(\mathcal{I}, \mathcal{J}) \models \mathcal{T}_{12}$. Then, $B^{\mathcal{I}} \neq \emptyset$, and it is easy to see that $(\mathcal{I}, \mathcal{J}) \not\models B \sqsubseteq \neg B'$ because $B^{\mathcal{I}} \not\subseteq \Delta^{\mathcal{J}} \setminus B'^{\mathcal{J}} = \emptyset$. In every case we derive a contradiction, hence $\mathcal{K}_1$ and $\mathcal{M}$ are Σ_2-positive.

($\Leftarrow$) Assume conditions 1-2 are satisfied. We show that $\mathcal{A}_2$ is a universal solution for $\mathcal{K}_1$ under $\mathcal{M}$.

First, $\mathcal{A}_2$ is a solution for $\mathcal{K}_1$ under $\mathcal{M}$. Let $\mathcal{J}$ be a model of $\mathcal{A}_2$, then there exists $\mathcal{I}$ a model of $\mathcal{K}_1^{pos}$ such that $(\mathcal{I}, \mathcal{J}) \models \mathcal{T}_{12}^{pos}$. Let h be a homomorphism from $\mathcal{U}_{\mathcal{K}_1}$ to $\mathcal{I}$, and w.l.o.g., $\mathcal{I} = h(\mathcal{U}_{\mathcal{K}_1})$. Define a new function $h' : \Delta^{\mathcal{U}_{\mathcal{K}_1}} \to \Delta \cup \Delta^{\mathcal{I}}$, where Δ is an infinite set of domain elements disjoint from $\Delta^{\mathcal{I}}$, as follows:

- $h'(x) = h(x)$ if $\mathbf{t}_{\Sigma_2}^{\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}}(x) \neq \emptyset$ or $x \in N_a$.
- $h'(x) = d_x$, a fresh domain element from Δ, otherwise.

We show that interpretation $\mathcal{I}'$ defined as the image of h' applied to $\mathcal{U}_{\mathcal{K}}$, is a model of $\mathcal{K}_1$ and $(\mathcal{I}', \mathcal{J}) \models \mathcal{M}$. Clearly, $\mathcal{I}'$ is a model of the positive inclusions in $\mathcal{T}_1$ and $(\mathcal{I}', \mathcal{J})$ satisfy the positive inclusions from $\mathcal{T}_{12}$. Let $\mathcal{T}_1 \models B \sqcap C \sqsubseteq \bot$ for basic concepts B, C. By contradiction, assume $\mathcal{I}' \not\models B \sqcap C \sqsubseteq \bot$, i.e., for some $d \in \Delta^{\mathcal{I}'}$, $d \in B^{\mathcal{I}'} \cap C^{\mathcal{I}'}$. We defined $\mathcal{I}'$ as the image of h' on $\mathcal{U}_{\mathcal{K}_1}$, hence there must exist $b, c \in \Delta^{\mathcal{U}_{\mathcal{K}_1}}$ such that $b \in B^{\mathcal{U}_{\mathcal{K}_1}}$, $c \in C^{\mathcal{U}_{\mathcal{K}_1}}$, and $h'(b) = h'(c) = d$. Then it cannot be the case that $\big[\mathbf{t}_{\Sigma_2}^{\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}}(b) \neq \emptyset$ or b is a constant$\,\big]$, and $\big[\,\mathbf{t}_{\Sigma_2}^{\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}}(c) \neq \emptyset$ or c is a constant$\,\big]$ as it contradicts (**a**) in the definition of $\mathcal{K}_1$ and $\mathcal{M}$ are Σ_2-positive. Assume b is a null and $\mathbf{t}_{\Sigma_2}^{\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}}(b) = \emptyset$. Then by definition of h', $h'(b) = d_b \in \Delta$ (and $d = d_b$). In either case c is a constant, or $\mathbf{t}_{\Sigma_2}^{\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}}(c) \neq \emptyset$, or $\mathbf{t}_{\Sigma_2}^{\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}}(c) = \emptyset$, we obtain contradiction with $h'(b) = d_b = h'(c)$ (remember, Δ and $\Delta^{\mathcal{I}}$ are disjoint). Contradiction rises from the assumption $\mathcal{I} \not\models B \sqcap C \sqsubseteq \bot$. Next, assume $\mathcal{T}_1 \models R \sqcap Q \sqsubseteq \bot$ for roles R, Q, and $\mathcal{I}' \not\models R \sqcap Q \sqsubseteq \bot$, i.e., for some $d_1, d_2 \in \Delta^{\mathcal{I}'}$, $(d_1, d_2) \in R^{\mathcal{I}'} \cap Q^{\mathcal{I}'}$. We defined $\mathcal{I}'$ as the image of h' on $\mathcal{U}_{\mathcal{K}_1}$, hence there must exist $b_1, b_2, c_1, c_2 \in \Delta^{\mathcal{U}_{\mathcal{K}_1}}$ such that $(b_1, b_2) \in R^{\mathcal{U}_{\mathcal{K}_1}}$, $(c_1, c_2) \in Q^{\mathcal{U}_{\mathcal{K}_1}}$, and $h'(b_i) = h'(c_i) = d_i$ for $i = 1, 2$. Then it cannot be the case that $\big[\mathbf{t}_{\Sigma_2}^{\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}}(b_i) \neq \emptyset$ or b_i is a constant$\,\big]$, and $\big[\,\mathbf{t}_{\Sigma_2}^{\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}}(c_i) \neq \emptyset$ or c_i is a constant$\,\big]$ as it contradicts (**a**) in the definition of $\mathcal{K}_1$ and $\mathcal{M}$ are Σ_2-positive. Consider the following cases:

- b_1 is a null and $\mathbf{t}_{\Sigma_2}^{\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}}(b_1) = \emptyset$. Then by definition of h', $h'(b_1) = d_{b_1} \in \Delta$ (and $d_1 = d_{b_1}$).
  - c_1 is a null and $\mathbf{t}_{\Sigma_2}^{\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}}(c_1) = \emptyset$, then $h'(c_1) = d_{c_1} = d_1$, hence $c_1 = b_1$ and $(b_1, b_2) \in R^{\mathcal{U}_{\mathcal{K}_1}}$, $(b_1, c_2) \in Q^{\mathcal{U}_{\mathcal{K}_1}}$. By (**c**) in the definition of $\mathcal{K}_1$ and $\mathcal{M}$ are Σ_2-positive, it cannot be the case that $\big[\mathbf{t}_{\Sigma_2}^{\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}}(b_2) \neq \emptyset$ or b_2 is a constant$\,\big]$, and $\big[\,\mathbf{t}_{\Sigma_2}^{\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}}(c_2) \neq \emptyset$ or c_2 is a constant$\,\big]$. Assume b_2 is a null and $\mathbf{t}_{\Sigma_2}^{\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}}(b_2) = \emptyset$. Then $h'(b_2) = d_{b_2} \in \Delta$ and in either case c_2 is a constant, or $\mathbf{t}_{\Sigma_2}^{\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}}(c_2) \neq \emptyset$, or $\mathbf{t}_{\Sigma_2}^{\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}}(c_2) = \emptyset$, we obtain contradiction with $h'(b_2) = d_{b_2} = h'(c_2)$
  - otherwise we obtain contradiction with $h'(b_1) = d_{b_1} = h'(c_1)$

The cases b_2 or c_i are nulls with the empty Σ_2-type are covered by swapping R and Q or by taking their inverses. Finally, assume $B \sqsubseteq \neg C \in \mathcal{T}_{12}$ and $(\mathcal{I}', \mathcal{J}) \not\models \mathcal{T}_{12}$, i.e., for some $d \in B^{\mathcal{I}'}$, $d \notin \Delta^{\mathcal{J}} \setminus C^{\mathcal{J}}$. Then there must exist $b \in B^{\mathcal{U}_{\mathcal{K}_1}}$ such that $h'(b) = d$. Contradiction with (**d**). Therefore, indeed, $\mathcal{I}$ is a model of $\mathcal{K}_1$ and $(\mathcal{I}, \mathcal{J}) \models \mathcal{T}_{12}$. This concludes the proof $\mathcal{A}_2$ is a solution for $\mathcal{K}_1$ under $\mathcal{M}$.

Second, $\mathcal{A}_2$ is a universal solution. Let $\mathcal{I}$ be a model of $\mathcal{K}_1$ and $\mathcal{J}$ an interpretation of Σ_2 such that $(\mathcal{I}, \mathcal{J}) \models \mathcal{T}_{12}$. Then, $\mathcal{I}$ is a model of $\mathcal{K}_1^{pos}$ and $(\mathcal{I}, \mathcal{J}) \models \mathcal{T}_{12}^{pos}$, and as $\mathcal{A}_2$ is a universal solution for $\mathcal{K}_1^{pos}$ under $\mathcal{M}^{pos}$, it follows that $\mathcal{J}$ is a model $\mathcal{A}_2$. $\square$

The following lemma establishes shows that Σ_2-positiveness can be checked in polynomial time.

Lemma B.6 *Let $\mathcal{M} = (\Sigma_1, \Sigma_2, \mathcal{T}_{12})$ be a mapping, and $\mathcal{K}_1 = \langle \mathcal{T}_1, \mathcal{A}_1 \rangle$ a KB over Σ_1. Then it can be decided in polynomial time whether $\mathcal{K}_1$ and $\mathcal{M}$ are Σ_2-positive.*

Proof. We check (**a**) as follows:

- for each concept disjointness axiom $B_1 \sqcap B_2 \sqsubseteq \bot \in \mathcal{T}_1$, check for $i = 1, 2$ if $\mathcal{K}_1 \models B_i(b_i)$ for some $b_i \in \mathsf{Ind}(\mathcal{A}_1)$ or there exists a $\mathcal{K}_1$-path $x = a \cdot w_{[S_1]} \dots w_{[S_n]}$ such that $B_i \in \mathbf{t}^{\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}}(x)$ and $\mathbf{t}_{\Sigma_2}^{\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}}(x) \neq \emptyset$. If yes, then (**a**) does not hold, otherwise it holds.

We check (**b**) as follows:

- for each role disjointness axiom $R \sqcap Q \sqsubseteq \bot \in \mathcal{T}_1$, check for $i = 1, 2, 3, 4$ if $\mathcal{K}_1 \models B_i(b_i)$ for some $b_i \in \mathsf{Ind}(\mathcal{A}_1)$ or there exists a $\mathcal{K}_1$-path $x = a \cdot w_{[S_1]} \dots w_{[S_n]}$ such that $B_i \in \mathbf{t}^{\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}}(x)$ and $\mathbf{t}_{\Sigma_2}^{\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}}(x) \neq \emptyset$, where $B_1 = \exists R$, $B_2 = \exists R^-$, $B_3 = \exists S$, $B_4 = \exists S^-$. If yes, then (**b**) does not hold, otherwise it holds.

We check (**c**) as follows:

- for each role disjointness axiom $R_1 \sqcap R_2 \sqsubseteq \bot \in \mathcal{T}_1$, check if there exists a $\mathcal{K}_1$-path $x = a \cdot w_{[S_1]} \dots w_{[S_n]}$ such that $\exists R_1, \exists R_2 \in \mathbf{t}^{\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}}(x)$, then check for $i = 1, 2$ if $\mathcal{K}_1 \models R_i(x, b_i)$ for some $b_i \in \mathsf{Ind}(\mathcal{A}_1)$ or there exists a $\mathcal{K}_1$-path $y_i = a' \cdot w_{[Q_1]} \dots w_{[Q'_n]}$ such that $R_i \in \mathbf{r}^{\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}}(x, y_i)$ and $\mathbf{t}_{\Sigma_2}^{\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}}(y_i) \neq \emptyset$. If yes, then (**c**) does not hold, otherwise it holds.

Note that in the previous three checks, it is sufficient to look for paths where n is bounded by the number of roles in $\mathcal{K}_1$, moreover in the last check $|n - n'| = 1$.

We check (**d**) as follows:

- for each concept disjointness axiom $B \sqsubseteq \neg B' \in \mathcal{T}_{12}$, check if $\mathcal{K}_1$ implies that B is necessarily non-empty. If yes, then (**d**) does not hold, otherwise
- for each role disjointness axiom $R \sqsubseteq \neg R' \in \mathcal{T}_{12}$, check if $\mathcal{K}_1$ implies that R is necessarily non-empty. If yes, then (**d**) does not hold, otherwise it holds.

It is straightforward to see that each of the checks can be done in polynomial time as the standard reasoning in *DL-Lite*$_{\mathcal{R}}$ is in NLOGSPACE. □

Lemma B.7 *Let $\mathcal{M} = (\Sigma_1, \Sigma_2, \mathcal{T}_{12})$ be a mapping, and $\mathcal{K}_1 = \langle \mathcal{T}_1, \mathcal{A}_1 \rangle$ a KB over Σ_1 such that $\mathcal{K}_1$ and $\mathcal{M}$ are Σ_2-positive. Then, a universal solution (with extended ABoxes) for $\mathcal{K}_1$ under $\mathcal{M}$ exists iff $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$ is Σ_2-homomorphically embeddable into a finite subset of itself.*

Proof. ($\Leftarrow$) Let ABox $\mathcal{A}_2$ be an ABox over Σ_2 such that $\mathcal{V}_{\mathcal{A}_2}$ is a finite subset of $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$ and there exists a Σ_2-homomorphism h from $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$ to $\mathcal{V}_{\mathcal{A}_2}$. Then, $\mathcal{U}_{\langle \emptyset, \mathcal{A}_2 \rangle}$ is trivially homomorphically embeddable into $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$. Hence by Lemma B.5, $\mathcal{A}_2$ is a universal solution for $\mathcal{K}_1$ under $\mathcal{M}$.

($\Rightarrow$) Let $\mathcal{A}_2$ be a universal solution for $\mathcal{K}_1$ under $\mathcal{M}$. Then $\mathcal{V}_{\mathcal{A}_2}$ is Σ_2-homomorphically equivalent to $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$ by Lemma B.5. Let h be a homomorphism from $\mathcal{V}_{\mathcal{A}_2}$ to $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$, and $h(\mathcal{V}_{\mathcal{A}_2})$ the image of h. Then, $h(\mathcal{V}_{\mathcal{A}_2})$ is a finite subset of $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$, moreover it is homomorphically equivalent to $\mathcal{V}_{\mathcal{A}_2}$ and to $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$. Therefore, it follows that $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$ is Σ_2-homomorphically embeddable to a finite subset of itself. □

B.2 Definitions and Preliminary Results: The Automata Construction for Theorem 5.5

Definition of alternating two-way automatas

Infinite trees are represented as prefix closed (infinite) sets of words over $\mathbb{N}$ (the set of positive natural numbers). Formally, an infinite tree is a set of words $T \subseteq \mathbb{N}^*$, such that if $x \cdot c \in T$, where $x \in \mathbb{N}^*$ and $c \in \mathbb{N}$, then also $x \in T$. The elements of T are called nodes, the empty word ϵ is the root of T, and for every $x \in T$, the nodes $x \cdot c$, with $c \in \mathbb{N}$, are the successors of x. By convention we take $x \cdot 0 = x$, and $x \cdot i \cdot -1 = x$. The branching degree $d(x)$ of a node x denotes the number of successors of x. If the branching degree of all nodes of a tree is bounded by k, we say that the tree has branching degree k. An infinite path P of T is a prefix closed set $P \subseteq T$ such that for every $i \geq 0$ there exists a unique node $x \in P$ with $|x| = i$. A labeled tree over an alphabet Σ is a pair (T, V), where T is a tree and $V : T \to \Sigma$ maps each node of T to an element of Σ.

Alternating automata on infinite trees are a generalization of nondeterministic automata on infinite trees, introduced in [9]. They allow for an elegant reduction of decision problems for temporal and program logics [3, 1]. Let $\mathcal{B}(I)$ be the set of positive boolean formulae over I, built inductively by applying $\wedge$ and

$\vee$ starting from true, false, and elements of I. For a set $J \subseteq I$ and a formula $\phi \in \mathcal{B}(I)$, we say that J satisfies ϕ if and only if, assigning true to the elements in J and false to those in $I \setminus J$, makes ϕ true. For a positive integer k, let $[k] = \{-1, 0, 1, \ldots, k\}$. A *two-way alternating tree automaton (2ATA)* running over infinite trees with branching degree k, is a tuple $\mathbb{A} = \langle \Sigma, Q, \delta, q_0, F \rangle$, where Σ is the input alphabet, Q is a finite set of states, $\delta : Q \times \Sigma \to \mathcal{B}([k] \times Q)$ is the transition function, $q_0 \in Q$ is the initial state, and F specifies the acceptance condition.

The transition function maps a state $q \in Q$ and an input letter $\sigma \in \Sigma$ to a positive boolean formula over $[k] \times Q$. Intuitively, if $\delta(q, \sigma) = \phi$, then each pair (c, q') appearing in ϕ corresponds to a new copy of the automaton going to the direction suggested by c and starting in state q'. For example, if $k = 2$ and $\delta(q_1, \sigma) = ((1, q_2) \wedge (1, q_3)) \vee ((-1, q_1) \wedge (0, q_3))$, when the automaton is in the state q_1 and is reading the node x labeled by the letter σ, it proceeds either by sending off two copies, in the states q_2 and q_3 respectively, to the first successor of x (i.e., $x \cdot 1$), or by sending off one copy in the state q_1 to the predecessor of x (i.e., $x \cdot -1$) and one copy in the state q_3 to x itself (i.e., $x \cdot 0$).

A run of a 2ATA $\mathbb{A}$ over a labeled tree (T, V) is a labeled tree $(T_{\mathbf{r}}, \mathbf{r})$ in which every node is labeled by an element of $T \times Q$. A node in $T_{\mathbf{r}}$ labeled by (x, q) describes a copy of A that is in the state q and reads the node x of T. The labels of adjacent nodes have to satisfy the transition function of $\mathbb{A}$. Formally, a run $(T_{\mathbf{r}}, \mathbf{r})$ is a $T \times Q$-labeled tree satisfying:

- $\epsilon \in T_{\mathbf{r}}$ and $\mathbf{r}(\epsilon) = (\epsilon, q_0)$.
- Let $y \in T_{\mathbf{r}}$, with $\mathbf{r}(y) = (x, q)$ and $\delta(q, V(x)) = \phi$. Then there is a (possibly empty) set $S = \{(c_1, q_1), \ldots, (c_n, q_n)\} \subseteq [k] \times Q$ such that:
  - S satisfies ϕ and
  - for all $1 \leq i \leq n$, we have that $y \cdot i \in T_{\mathbf{r}}$, $x \cdot c_i$ is defined ($x \cdot c_i \in T$), and $\mathbf{r}(y \cdot i) = (x \cdot c_i, q_i)$.

A run $(T_{\mathbf{r}}, \mathbf{r})$ is accepting if all its infinite paths satisfy the acceptance condition. Given an infinite path $P \in T_{\mathbf{r}}$, let $\mathit{inf}(P) \subseteq Q$ be the set of states that appear infinitely often in P (as second components of node labels). We consider here Büchi acceptance conditions. A Büchi condition over a state set Q is a subset F of Q, and an infinite path P satisfies F if $\mathit{inf}(P) \cap F \neq \emptyset$.

The non-emptiness problem for 2ATAs consists in determining, for a given 2ATA, whether the set of trees it accepts is nonempty. It is known that this problem can be solved in exponential time in the number of states of the input automaton $\mathbb{A}$, but in linear time in the size of the alphabet as well as in the size of the transition function of $\mathbb{A}$.

The automata construction

Now, we are going to construct two 2ATA automatas and a one-way non-deterministic automata to use them as a mechanism to decide the non-emptiness problem for universal solutions. More specifically, let Σ_1, Σ_2 be signatures with no concepts or roles in common, and $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ a KB over $\Sigma_1 \cup \Sigma_2$, $\mathbf{N} = \{a_1, \ldots, a_n\}$ be the set of individuals in $\mathcal{A}_1$, $\mathbf{B}$ be the set of basic concepts and $\mathbf{R}$ be the set of basic roles over the signature of $\mathcal{K}$ (that is, over $\Sigma_1 \cup \Sigma_2$). Finally, assume that r, G are special characters not mentioned in $\mathbf{N} \cup \mathbf{B} \cup \mathbf{R}$, and let $\mathbf{P} = \{P_{ij} \mid P$ is an atomic role over the signature of $\mathcal{K}$ and $1 \leq i, j \leq n\}$. Then assuming that $\Sigma_{\mathcal{K}} = 2^{\mathbf{N} \cup \mathbf{B} \cup \mathbf{R} \cup \mathbf{P} \cup \{r, G\}}$ and $\Gamma_{\mathcal{K}} = \{\sigma \in \Sigma_{\mathcal{K}} \mid r \in \sigma$, $\sigma \cap \mathbf{N} \neq \emptyset$, or every basic concept and every basic role in σ is over $\Sigma_2\}$, we construct the following automata:

- $\mathbb{A}^{can}_{\mathcal{K}}$: The alphabet of this automaton is $\Sigma_{\mathcal{K}}$, and it accepts trees that are essentially the tree corresponding to the canonical model of $\mathcal{K}$, but with nodes arbitrary labeled with the special character G.
- $\mathbb{A}^{mod}_{\mathcal{K}}$: The alphabet of this automaton is $\Sigma_{\mathcal{K}}$, and it accepts a tree if its subtree labeled with G corresponds to a tree model $\mathcal{I}$ of $\mathcal{K}$ (tree models are models which from trees on the labeled nulls).
- $\mathbb{A}_{\mathit{fin}}$: The alphabet of this automaton is $\Gamma_{\mathcal{K}}$, and it accepts a tree if it has a finite prefix where each node is marked with the special symbol G, and no other node in the tree is marked with G.

Automaton $\mathbb{A}^{can}_{\mathcal{K}}$ for the canonical model of $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$

$\mathbb{A}^{can}_{\mathcal{K}}$ is a two way alternating tree automaton (2ATA) that accepts the tree corresponding to the canonical model of the *DL-Lite*$_{\mathcal{R}}$ KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$, with nodes arbitrarily labeled with a special character G. Formally, $\mathbb{A}^{can}_{\mathcal{K}} = \langle \Sigma_{\mathcal{K}}, Q_{can}, \delta_{can}, q_0, F_{can} \rangle$, where

$$Q_{can} \quad = \quad \{q_0, q_s, q^*_{\neg r}, q_d\} \cup \{q^*_X, q^*_{\neg X} \mid X \in \mathbf{N} \cup \mathbf{B} \cup \mathbf{R} \cup \mathbf{P}\} \cup \{q_{\exists R}, q_R \mid R \in \mathbf{R}\},$$

and the transition function δ_{can} is defined as follows. Assume without loss of generality that the number of basic roles over the signature of $\mathcal{K}$ is equal to n (this can always be done by adding the required assertions to the ABox), and let $f : \mathbf{R} \to \{1, \ldots, n\}$ be a one-to-one function. Then $\delta_{can} : Q_{can} \times \Sigma_{\mathcal{K}} \to \mathcal{B}([n] \times Q_{can})$ is defined as:

1. For each $\sigma \in \Sigma_{\mathcal{K}}$ such that $r \in \sigma$, $\delta_{can}(q_0, \sigma)$ is defined as:

$$\bigwedge_{i=1}^{n} \Bigg[(i, q_s) \wedge (i, q^*_{\neg r}) \wedge (i, q^*_{a_i}) \wedge \Bigg(\bigwedge_{j \in \{1,\ldots,n\} \,:\, j \neq i} (i, q^*_{\neg a_j}) \Bigg) \wedge$$
$$\bigwedge_{j=1}^{n} \Bigg(\bigwedge_{P \in \mathbf{P} \,:\, \mathcal{K} \models P(a_i, a_j)} (0, q^*_{P_{ij}}) \wedge \bigwedge_{P \in \mathbf{P} \,:\, \mathcal{K} \not\models P(a_i, a_j)} (0, q^*_{\neg P_{ij}}) \Bigg) \wedge$$
$$\Bigg(\bigwedge_{B \in \mathbf{B} \,:\, \mathcal{K} \models B(a_i)} (i, q^*_B) \Bigg) \wedge \Bigg(\bigwedge_{B \in \mathbf{B} \,:\, \mathcal{K} \not\models B(a_i)} (i, q^*_{\neg B}) \Bigg) \wedge$$
$$\Bigg(\bigwedge_{\substack{R \in \mathbf{R} \,:\, R \text{ is } \leq_{\mathcal{T}}\text{-minimal s.t.} \\ \mathcal{K} \models \exists R(a_i) \text{ and} \\ \bigwedge_{j=1}^{n} \mathcal{K} \not\models R(a_i, a_j)}} (i, q_{\exists R}) \Bigg) \wedge \Bigg(\bigwedge_{\substack{R \in \mathbf{R} \,:\, \mathcal{K} \not\models \exists R(a_i), \text{ or} \\ \bigvee_{j=1}^{n} \mathcal{K} \models R(a_i, a_j), \text{ or} \\ R \text{ is not } \leq_{\mathcal{T}}\text{-minimal}}} (i, q^{ng}_{\exists R}) \Bigg) \Bigg]$$

2. For each $\sigma \in \Sigma_{\mathcal{K}}$:

$$\delta_{can}(q_s, \sigma) \quad = \quad \bigwedge_{i=1}^{n} \Bigg[(i, q_s) \ \wedge \ (i, q^*_{\neg r}) \ \wedge \ \bigwedge_{j=1}^{n} (i, q^*_{\neg a_j}) \ \wedge \ \Bigg((i, q_d) \ \vee \ \bigvee_{R \in \mathbf{R}} (i, q^*_R) \Bigg) \Bigg]$$

3. For each $\sigma \in \Sigma_{\mathcal{K}}$:

$$\delta_{can}(q_d, \sigma) \quad = \quad \bigwedge_{R \in \mathbf{R}} (0, q^*_{\neg R}) \wedge \bigwedge_{i=1}^{n} (i, q_d)$$

4. For each $\sigma \in \Sigma_{\mathcal{K}}$ and each basic role $[R]$ from $\mathbf{R}$:

$$\delta_{can}(q^{ng}_{\exists R}, \sigma) = \bigwedge_{R' \in \mathbf{R}} (f(R), q^*_{\neg R'})$$

5. For each $\sigma \in \Sigma_{\mathcal{K}}$ and each basic role $[R]$ from $\mathbf{R}$:

$$\delta_{can}(q_{\exists R}, \sigma) = (f(R), q_R)$$

6. For each $\sigma \in \Sigma_{\mathcal{K}}$ such that $\sigma \cap \mathbf{N} = \emptyset$ and each basic role $[R]$ from $\mathbf{R}$, $\delta_{can}(q_R, \sigma)$ is defined as

$$\Bigg(\bigwedge_{R' \in \mathbf{R} \,:\, \mathcal{K} \models R \sqsubseteq R'} (0, q^*_{R'}) \Bigg) \wedge \Bigg(\bigwedge_{R' \in \mathbf{R} \,:\, \mathcal{K} \not\models R \sqsubseteq R'} (0, q^*_{\neg R'}) \Bigg) \wedge$$
$$\Bigg(\bigwedge_{B \in \mathbf{B} \,:\, \mathcal{K} \models \exists R^- \sqsubseteq B} (0, q^*_B) \Bigg) \wedge \Bigg(\bigwedge_{B \in \mathbf{B} \,:\, \mathcal{K} \not\models \exists R^- \sqsubseteq B} (0, q^*_{\neg B}) \Bigg) \wedge$$
$$\Bigg(\bigwedge_{\substack{S \in \mathbf{R} \,:\, S \text{ is } \leq_{\mathcal{T}}\text{-minimal s.t.} \\ \mathcal{K} \models \exists R^- \sqsubseteq \exists S,\, [R^-] \neq [S]}} (0, q_{\exists S}) \Bigg) \wedge \Bigg(\bigwedge_{\substack{S \in \mathbf{R} \,:\, \mathcal{K} \not\models \exists R^- \sqsubseteq \exists S, \text{ or } [R^-] = [S], \\ \text{or } S \text{ is not } \leq_{\mathcal{T}}\text{-minimal}}} (0, q^{ng}_{\exists S}) \Bigg)$$

7. For each $\sigma \in \Sigma_{\mathcal{K}}$:

$$\delta_{can}(q^*_{\neg r}, \sigma) = \begin{cases} \textit{true} & \text{if } r \notin \sigma \\ \textit{false} & \text{otherwise} \end{cases}$$

8. For each $\sigma \in \Sigma_{\mathcal{K}}$ and each $X \in \mathbf{B} \cup \mathbf{R} \cup \mathbf{N} \cup \mathbf{P}$:

$$\delta(q^*_X, \sigma) = \begin{cases} \textit{true} & \text{if } X \in \sigma \\ \textit{false} & \text{otherwise} \end{cases} \qquad \delta_{can}(q^*_{\neg X}, \sigma) = \begin{cases} \textit{true} & \text{if } X \notin \sigma \\ \textit{false} & \text{otherwise} \end{cases}$$

Finally, the acceptance condition is $F_{can} = Q_{can}$.

To represent the canonical model $\mathcal{U}_{\mathcal{K}}$ of $\mathcal{K}$ as a labeled tree, we label each individual x with the set of concepts B such that $x \in B^{\mathcal{U}_{\mathcal{K}}}$. We also add a basic role R to the label of x whenever $(x', x) \in R^{\mathcal{U}_{\mathcal{K}}}$

and x is not an individual. Moreover, we make sure this tree is an infinite full n-ary tree, where n is the number of individuals in $\mathsf{Ind}(\mathcal{A})$ and basic roles in $\mathbf{R}$. Thus, let n^* be the set of sequences of numbers from 1 to n of the form $n^* = \{i_1 \cdot i_2 \cdot \cdots \cdot i_m \mid 1 \leq i_j \leq n, m \geq j \geq 0\}$, the sequence of length 0 is denoted by ϵ.

Recall that we have a numbering of individuals $\{a_1, \ldots, a_n\} = \mathsf{Ind}(\mathcal{A})$, and each role $R \in \mathbf{R}$ can be identified through the number $f(R) \in \{1, \ldots, n\}$. Therefore, the elements of $\Delta^{\mathcal{U}}$ can be seen as sequences of natural numbers, namely a sequence $a_i \cdot w_{[R_1]} \cdots \cdot w_{[R_m]}$ corresponds to the numeric sequence $i \cdot f(R_1) \cdot \cdots \cdot f(R_m)$. However, for better readability we use the original notation as $a_i \cdot w_{[R_1]} \cdot \cdots \cdot w_{[R_m]}$. Note, that $\Delta^{\mathcal{U}} \subseteq n^*$.

In the following, we assume $\mathcal{K}$ is fixed and for simplicity we use $\mathcal{U}$ instead of $\mathcal{U}_{\mathcal{K}}$.

The tree encoding of the canonical model $\mathcal{U}$ of $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ is the $\Sigma_{\mathcal{K}}$-labeled tree $T_{\mathcal{U}} = (n^*, V^{\mathcal{U}})$, such that

- $V^{\mathcal{U}}(\epsilon) = \{r\} \cup \{P_{ij} \mid (a_i, a_j) \in P^{\mathcal{U}}, P \text{ is an atomic role}\}$,
- for each $x \in \Delta^{\mathcal{U}}$:

$$\begin{array}{rcl} V^{\mathcal{U}}(x) & = & \{B \mid x \in B^{\mathcal{U}}\} \cup \\ & & \{S \mid (x', x) \in S^{\mathcal{U}} \text{ and } x = x' \cdot w_{[R]} \text{ for some role } R \text{ s.t. } [R] \leq_{\mathcal{T}} [S]\} \cup \\ & & \{a \mid a \in \mathsf{Ind}(\mathcal{A}) \text{ and } x = a\}. \end{array}$$

Conversely, we can see any $\Sigma_{\mathcal{K}}$-labeled tree as a representation of an interpretation of $\mathcal{K}$, provided that each individual name occurs in the label of only one node, a child of the root. Informally, the domain of this interpretation are the nodes of the tree reachable from the root through a sequence of roles, except the root itself. The extensions of individuals, concepts and roles are determined by the node labels.

Given a $\Sigma_{\mathcal{K}}$-labeled tree (T, V), we call a node c an *individual node* if $a \in V(c)$ for some $a \in \mathsf{Ind}(\mathcal{A})$, and we call c an a-node if we want to make the precise a explicit. We say that T is *individual unique* if for each $a \in \mathsf{Ind}(\mathcal{A})$ there is exactly one a-node, a child of the root of T.

An individual unique $\Sigma_{\mathcal{K}}$-labeled tree (T, V), *represents* the interpretation $\mathcal{I}_T$ defined as follows. For each role name P, let:

$$\begin{array}{rcl} R_p & = & \{(x, x \cdot i) \mid P \in V(x \cdot i)\} \cup \{(x \cdot i, x) \mid P^- \in V(x \cdot i)\} \cup \\ & & \{(c, c') \mid a_i \in V(c), a_j \in V(c') \text{ and } P_{ij} \in V(\epsilon)\} \end{array}$$

and

$$\Delta^{\mathcal{I}_T} = \{x \mid (i, x) \in \bigcup_{P \in \mathbf{R}} (R_P \cup R_P^-)^*, i \in \{1, \ldots, n\}\},$$

where R_P^- denotes the inverse of relation R_P. Then the interpretation $\mathcal{I}_T = (\Delta^{\mathcal{I}_T}, \cdot^{\mathcal{I}_T})$ is defined as:

$$\begin{array}{rcll} a_i^{\mathcal{I}_T} & = & c \text{ such that } a_i \in V(c), & \text{for each } a_i \in \mathsf{Ind}(\mathcal{A}) \\ A^{\mathcal{I}_T} & = & \Delta^{\mathcal{I}_T} \cap \{x \mid A \in V(x)\}, & \text{for each atomic concept } A \in \mathbf{B} \text{ and} \\ P^{\mathcal{I}_T} & = & (\Delta^{\mathcal{I}_T} \times \Delta^{\mathcal{I}_T}) \cap R_P, & \text{for each atomic role } P \in \mathbf{R} \end{array}$$

Proposition B.8 *The following hold for* $\mathbb{A}_{\mathcal{K}}^{can}$*:*

- $T_{\mathcal{U}} \in \mathcal{L}(\mathbb{A}_{\mathcal{K}}^{can})$.
- *for each* $(T, V) \in \mathcal{L}(\mathbb{A}_{\mathcal{K}}^{can})$*,* (T, V) *is individual unique and* $\mathcal{I}_T$ *is isomorphic to* $\mathcal{U}$*, the canonical model of* $\mathcal{K}$*.*

Proof. For the first item, assume $T_{\mathcal{U}} = (n^*, V^{\mathcal{U}})$ is the tree encoding of the universal model $\mathcal{U}$ of $\mathcal{K}$. We show that a full run of $\mathbb{A}_{\mathcal{K}}^{can}$ over $T_{\mathcal{U}}$ exists.

The run $(T_{\mathbf{r}}, \mathbf{r})$ is built starting from the root ϵ, and setting $\mathbf{r}(\epsilon) = (\epsilon, q_0)$. Then, to correctly execute the initial transition, the root has children as follows:

- for each $a_k \in \mathsf{Ind}(\mathcal{A})$
  - a child k_s with $\mathbf{r}(k_s) = (a_k, q_s)$,
  - a child $k^*_{\neg r}$ with $\mathbf{r}(k^*_{\neg r}) = (a_k, q^*_{\neg r})$,
  - a child $k^*_{a_k}$ with $\mathbf{r}(k^*_{a_k}) = (a_k, q^*_{a_k})$,
  - a child $k^*_{\neg a_j}$ for each $j \neq k$ with $\mathbf{r}(k^*_{\neg a_j}) = (a_k, q^*_{\neg a_j})$,
  - a child k^*_B for each $B \in \mathbf{B}$ such that $a_k \in B^{\mathcal{U}}$, with $\mathbf{r}(k^*_B) = (a_k, q^*_B)$,
  - a child $k^*_{\neg B}$ for each $B \in \mathbf{B}$ such that $a_k \notin B^{\mathcal{U}}$, with $\mathbf{r}(k^*_{\neg B}) = (a_k, q^*_{\neg B})$,

- a child $k_{\exists R}$ for each $\leq_{\mathcal{T}}$-minimal role R s.t. $\mathcal{U} \models \exists R(a_i)$ and $\mathcal{U} \not\models R(a_i, a_j)$ for each $j \in \{1, \ldots, n\}$, with $\mathbf{r}(k_{\exists R}) = (a_k, q_{\exists R})$,
- a child $k_{\exists R}^{ng}$ for each role R s.t. $\mathcal{U} \not\models \exists R(a_i)$, or $\mathcal{U} \models R(a_i, a_j)$ for some $j \in \{1, \ldots, n\}$, or R is not $\leq_{\mathcal{T}}$-minimal, with $\mathbf{r}(k_{\exists R}^{ng}) = (a_k, q_{\exists R}^{ng})$,

- a child k^*_{P,a_k,a_j} for each $a_k, a_j \in \mathsf{Ind}(\mathcal{A})$ and each atomic role P such that $(a_k, a_j) \in P^{\mathcal{U}}$, with $\mathbf{r}(k^*_{P,a_k,a_j}) = (\epsilon, q^*_{P_{kj}})$,
- a child $k^*_{\neg P,a_k,a_j}$ for each $a_k, a_j \in \mathsf{Ind}(\mathcal{A})$ and each atomic role P such that $(a_k, a_j) \notin P^{\mathcal{U}}$, with $\mathbf{r}(k^*_{\neg P,a_k,a_j}) = (\epsilon, q^*_{\neg P_{kj}})$,

Note that nodes $y \in T_{\mathbf{r}}$ with $\mathbf{r}(y) = (x, q^*_{\ldots})$ are leafs of the tree $T_{\mathbf{r}}$, as by the transition function δ_{can}, all the states of the form $q^*_{\ldots}$ in Q_{can} can be satisfied with the empty assignment.

Other nodes, however, can have children. They are defined inductively as follows.

2. Let y be a node in $T_{\mathbf{r}}$ such that $\mathbf{r}(y) = (x, q_s)$ for some $x \in n^*$. Moreover, let $i \in \{1, \ldots, n\}$. Then y has
   - a child $y \cdot i_s$ with $\mathbf{r}(y \cdot i_s) = (x \cdot i, q_s)$,
   - a child $y \cdot i^*_{\neg r}$ with $\mathbf{r}(y \cdot i^*_{\neg r}) = (x \cdot i, q^*_{\neg r})$,
   - a child $y \cdot i^*_{\neg a_j}$ for each $j \in \{1, \ldots, n\}$ with $\mathbf{r}(y \cdot i^*_{\neg a_j}) = (x \cdot i, q^*_{\neg a_j})$,
   - if $x \in \Delta^{\mathcal{U}}$ and for $R \in \mathbf{R}$ s.t. $f(R) = i$, $x \cdot w_{[R]} \in \Delta^{\mathcal{U}}$,
     - a child $y \cdot i^*_R$ with $\mathbf{r}(y \cdot i^*_R) = (x \cdot w_{[R]}, q^*_R)$,
   - otherwise
     - a child $y \cdot i_d$ with $\mathbf{r}(y \cdot i_d) = (x \cdot i, q_d)$,
3. Let y be a node in $T_{\mathbf{r}}$ such that $\mathbf{r}(y) = (x, q_d)$ for some $x \in n^*$. Then y has
   - a child $y \cdot i_d$ for each $i \in \{1, \ldots, n\}$, with $\mathbf{r}(y \cdot i_d) = (x \cdot i, q_d)$,
   - a child $y \cdot 0^*_{\neg R}$ for each $R \in \mathbf{R}$, with $\mathbf{r}(y \cdot 0^*_{\neg R}) = (x, q^*_{\neg R})$,
4. Let y be a node in $T_{\mathbf{r}}$ such that $\mathbf{r}(y) = (x, q^{ng}_{\exists R})$ for some $x \in \Delta^{\mathcal{U}}$ and $R \in \mathbf{R}$. Then y has
   - a child $y \cdot f(R)^*_{\neg R'}$ for each $R' \in \mathbf{R}$, with $\mathbf{r}(y \cdot f(R)^*_{\neg R'}) = (x \cdot f(R), q^*_{\neg R'})$,
5. Let y be a node in $T_{\mathbf{r}}$ such that $\mathbf{r}(y) = (x, q_{\exists R})$ for some $x \in \Delta^{\mathcal{U}}$ and $R \in \mathbf{R}$. Then $x \cdot w_{[R]} \in \Delta^{\mathcal{U}}$ and y has
   - a child $y \cdot f(R)_R$ with $\mathbf{r}(y \cdot f(R)_R) = (x \cdot w_{[R]}, q_R)$,
6. Let y be a node in $T_{\mathbf{r}}$ such that $\mathbf{r}(y) = (x, q_R)$ for some $x \in \Delta^{\mathcal{U}}$ and $R \in \mathbf{R}$. Then y has
   - a child $y \cdot 0^*_{R'}$ for each $R' \in \mathbf{R}$ s.t. $\mathcal{K} \models R \sqsubseteq R'$, with $\mathbf{r}(y \cdot 0^*_{R'}) = (x, q^*_{R'})$,
   - a child $y \cdot 0^*_{\neg R'}$ for each $R' \in \mathbf{R}$ s.t. $\mathcal{K} \not\models R \sqsubseteq R'$, with $\mathbf{r}(y \cdot 0^*_{\neg R'}) = (x, q^*_{\neg R'})$,
   - a child $y \cdot 0^*_B$ for each $B \in \mathbf{B}$ s.t. $\mathcal{K} \models \exists R^- \sqsubseteq B$, with $\mathbf{r}(y \cdot 0^*_B) = (x, q^*_B)$,
   - a child $y \cdot 0^*_{\neg B}$ for each $B \in \mathbf{B}$ s.t. $\mathcal{K} \not\models \exists R^- \sqsubseteq B$, with $\mathbf{r}(y \cdot 0^*_{\neg B}) = (x, q^*_{\neg B})$,
   - a child $y \cdot 0_{\exists S}$ for each $\leq_{\mathcal{T}}$-minimal role S s.t. $\mathcal{K} \models \exists R^- \sqsubseteq \exists S$ and $[R^-] \neq [S]$, with $\mathbf{r}(y \cdot 0_{\exists S}) = (x, q_{\exists S})$,
   - a child $y \cdot 0^{ng}_{\exists S}$ for each role S s.t. $\mathcal{K} \not\models \exists R^- \sqsubseteq \exists S$, or $[R^-] = [S]$, or S is not $\leq_{\mathcal{T}}$-minimal, with $\mathbf{r}(y \cdot 0^{ng}_{\exists S}) = (x, q^{ng}_{\exists S})$.

Each node of $T_{\mathbf{r}}$ defined as described above satisfies the transition function δ_{can}.

It is easy to see that this run is accepting, as for each infinite path P of $T_{\mathbf{r}}$, either $q_s \in \mathit{inf}(P)$, or $q_s \in \mathit{inf}(P)$, or $q_R \in \mathit{inf}(P)$ for some R. Hence, $T_{\mathcal{U}} \in \mathcal{L}(\mathbb{A}^{can}_{\mathcal{K}})$.

To show the second item, let $(T, V) \in \mathcal{L}(\mathbb{A}^{can}_{\mathcal{K}})$ and $(T_{\mathbf{r}}, \mathbf{r})$ an accepting run of (T, V). First, assume T is not individual unique, that is,

- there exists an a-node x in T, such that x is not a child of the root, or
- there exist two nodes i and j in T such that $a \in V(i)$ and $a \in V(j)$.

In the former case, let x' be the parent of x, $x' \neq \epsilon$, then there exists a node $y' \in T_{\mathbf{r}}$ with $\mathbf{r}(y') = (x', q_s)$ and a node $y \in \mathcal{T}_{\mathbf{r}}$ with $\mathbf{r}(y) = (x, q^\star_{\neg a})$, which contradicts that $(\mathcal{T}_{\mathbf{r}}, \mathbf{r})$ is an accepting run of (T, V) as $a \in V(x)$. In the latter case, assume a is equal to a_i. Then we get contradiction with $\delta_{can}(q_0, \sigma)$.

Hence, T is individual unique. Let $\mathcal{I}_T$ be the interpretation represented by T. We show that $\mathcal{I}_T$ is isomorphic to $\mathcal{U}$, by constructing a function h from $\Delta^{\mathcal{I}_T}$ to $\Delta^{\mathcal{U}}$ and showing that it is a one-to-one and onto homomorphism. We construct h by induction on the length of the sequence $x \in \Delta^{\mathcal{I}_T}$.

Initially, as T is individual unique, we set for each $i \in \{1, \ldots, n\}$, $h(i) = a_i$, where $a_i \in V(i)$. Note that by definition of $\mathcal{U}$, $a_i \in \Delta^{\mathcal{U}}$ and by definition of $\mathcal{I}_T$, $i \in \Delta^{\mathcal{I}_T}$. Then the following holds for $i, j \in \{1, \ldots, n\}$.

1. for an atomic role P, $(i,j) \in P^{\mathcal{I}_T}$ iff $(a_i, a_j) \in P^{\mathcal{U}}$: let $(i,j) \in P^{\mathcal{I}_T}$, by definition of $\mathcal{I}_T$ it follows that $P_{ij} \in V(\epsilon)$. Assume $\mathcal{K} \not\models P(a_i, a_j)$, then $(0, q^*_{\neg P_{ij}}) \in \delta_{can}(q_0, V(\epsilon))$ and in $T_{\mathbf{r}}$ there exists a node y, s.t. $\mathbf{r}(y) = (\epsilon, q^*_{\neg P_{ij}})$, hence y does not satisfy the condition on a run. Contradiction with $(T_{\mathbf{r}}, \mathbf{r})$ being accepting. Therefore, indeed $\mathcal{K} \models P(a_i, a_j)$ and $(a_i, a_j) \in P^{\mathcal{U}}$. Similarly for the other direction.
2. for a basic concept B, $i \in B^{\mathcal{I}_T}$ iff $a_i \in B^{\mathcal{U}}$: let $i \in B^{\mathcal{I}_T}$, by definition of $\mathcal{I}_T$ it follows that $B \in V(i)$. Assume $\mathcal{K} \not\models B(a_i)$, then $(i, q^*_{\neg B}) \in \delta_{can}(q_0, V(\epsilon))$ and there exists $y \in T_{\mathbf{r}}$ with $\mathbf{r}(y) = (i, q^*_{\neg B})$. We get contradiction as y does not satisfy the condition on a run. Therefore, indeed $\mathcal{K} \models B(a_i)$ and $a_i \in B^{\mathcal{U}}$. Similarly for the other direction.

For the inductive step we prove two auxiliary claims.

Claim B.9 (1) *Let $i \cdot f(R) \in \Delta^{\mathcal{I}_T}$ for some $i \in \{1, \ldots, n\}$. Then $\mathcal{K} \models \exists R(a_i)$, $\mathcal{K} \not\models R(a_i, a_j)$ for each $j \in \{1, \ldots, n\}$ and R is a $\leq_{\mathcal{T}}$-minimal such role.*

Proof. Assume $\mathcal{K} \not\models \exists R(a_i)$, or $\mathcal{K} \models R(a_i, a_j)$ for some $j \in \{1, \ldots, n\}$, or R is not a $\leq_{\mathcal{T}}$-minimal such role. Then by definition of $\delta_{can}(q_0, V(\epsilon))$ and of a run, there exists a node $y = \epsilon \cdot i^{ng}_{\exists R}$ in $T_{\mathbf{r}}$ such that $\mathbf{r}(y) = (i, q^{ng}_{\exists R})$ and by $\delta_{can}(q^{ng}_{\exists R}, V(i))$ it is required that $R' \notin V(x \cdot f(R))$ for each $R' \in \mathbf{R}$. It means that $i \cdot f(R)$ is not connected to i through any role. Contradiction with $i \cdot f(R)$ being in $\Delta^{\mathcal{I}_T}$. □ □

Claim B.10 (2) *Let $x \cdot f(R) \in \Delta^{\mathcal{I}_T}$, $\mathsf{len}(x) \geq 2$ and there exists $y \in T_{\mathbf{r}}$ with $\mathbf{r}(y) = (x, q_S)$. Then $\mathcal{K} \models \exists S^- \sqsubseteq \exists R$, $[S^-] \neq [R]$ and R is a $\leq_{\mathcal{T}}$-minimal such role.*

Proof. For the sake of contradiction assume $\mathcal{K} \not\models \exists S^- \sqsubseteq \exists R$. Then by definition of $\delta_{can}(q_S, V(x))$ and of a run, there exists a node $y'' = y \cdot 0^{ng}_{\exists R}$ in $T_{\mathbf{r}}$ such that $\mathbf{r}(y'') = (x, q^{ng}_{\exists R})$ and by $\delta_{can}(q^{ng}_{\exists R}, V(x))$ it is required that $R' \notin V(x \cdot f(R))$ for each $R' \in \mathbf{R}$. It means that $x \cdot f(R)$ is not connected to x through any role. Contradiction with $x \cdot f(R)$ being in $\Delta^{\mathcal{I}_T}$.

By the same argument it can be shown that $[S^-] \neq [R]$ and R is $\leq_{\mathcal{T}}$-minimal. □ □

Let $x \in \Delta^{\mathcal{I}_T}$, $h(x)$ is defined and $h(x) \in \Delta^{\mathcal{U}}$. Moreover, if $\mathsf{len}(x) \geq 2$, let $\mathsf{tail}(x) = f(S)$ and $\mathsf{tail}(h(x)) = w_{[S]}$ for some role S, and there exist a node $y \in T_{\mathbf{r}}$ such that $\mathbf{r}(y) = (x, q_S)$. Then

1. for each $h(x) \cdot w_{[R]} \in \Delta^{\mathcal{U}}$, $x \cdot f(R)$ is in $\Delta^{\mathcal{I}_T}$.
2. for each $x \cdot f(R) \in \Delta^{\mathcal{I}_T}$, $h(x) \cdot w_{[R]}$ is in $\Delta^{\mathcal{U}}$.

Let $h(x) \cdot w_{[R]} \in \Delta^{\mathcal{U}}$. Then R is $\leq_{\mathcal{T}}$-minimal such that $\mathcal{K} \models \exists S^- \sqsubseteq \exists R$ if $\mathsf{tail}(h(x)) = w_{[S]}$, or $\mathcal{K} \models \exists R(a_i)$ and $\mathcal{K} \not\models R(a_i, a_j)$ for $j \in \{1, \ldots, n\}$ if $h(x) = a_i$. By definition of δ_{can}, there exist a node y' in $T_{\mathbf{r}}$ with $\mathbf{r}(y') = (x, q_{\exists R})$. Since $T_{\mathbf{r}}$ is a run, it follows that there exist a node $y'' = y' \cdot f(R)_R$ in $T_{\mathbf{r}}$ with $\mathbf{r}(y'') = (x \cdot f(R), q_R)$, and $x \cdot f(R) \in T$. Therefore, $R \in V(x \cdot f(R))$ and by definition of $\mathcal{I}_T$, $x \cdot f(R) \in \Delta^{\mathcal{I}_T}$.

Let $x \cdot f(R) \in \Delta^{\mathcal{I}_T}$. Then by Claim (1) and (2), $\mathsf{tail}(h(x)) \rightsquigarrow_{\mathcal{K}} w_{[R]}$, hence $h(x) \cdot w_{[R]} \in \Delta^{\mathcal{U}}$. Moreover, we also obtain that there exists y'' in $T_{\mathbf{r}}$ such that $\mathbf{r}(y'') = (x \cdot f(R), q_R)$.

Thus, we can set $h(x \cdot f(R))$ to $h(x) \cdot w_{[R]}$. Obviously, h is one-to-one and onto. To verify that h is a homomorphism it remains to show

- for each role R', $(x, x \cdot f(R)) \in R'^{\mathcal{I}_T}$ iff $(h(x), h(x) \cdot w_{[R]}) \in R'^{\mathcal{U}}$, and
- for each basic concept B, $x \cdot f(R) \in B^{\mathcal{I}_T}$ iff $h(x) \cdot w_{[R]} \in B^{\mathcal{U}}$.

Let $(x, x \cdot f(R)) \in R'^{\mathcal{I}_T}$ for some role R'. By contradiction assume $(h(x), h(x) \cdot w_{[R]}) \notin R'^{\mathcal{U}}$, this implies that $\mathcal{K} \not\models R \sqsubseteq R'$. Hence, $(0, q^*_{\neg R'}) \in \delta_{can}(q_R, V(x \cdot f(R)))$, and in $T_{\mathbf{r}}$ there is a node $y''' = y'' \cdot 0^*_{\neg R'}$ with $\mathbf{r}(y''') = (x \cdot f(R), q^*_{\neg R'})$. We get a contradiction with $T_{\mathbf{r}}$ being a run as by definition of $\mathcal{I}_T$, $R' \in V(i \cdot f(R))$. Similarly for the other direction.

Finally, let $x \cdot f(R) \in A^{\mathcal{I}_T}$ for some concept A, and assume $h(x) \cdot w_{[R]} \notin A^{\mathcal{U}}$. The latter implies that $\mathcal{K} \not\models \exists R^- \sqsubseteq A$. Hence, $(0, q^*_{\neg A}) \in \delta_{can}(q_R, V(x \cdot f(R)))$, and in $T_{\mathbf{r}}$ there is a node $y''' = y'' \cdot 0^*_{\neg A}$ with $\mathbf{r}(y''') = (x \cdot f(R), q^*_{\neg A})$. We get a contradiction with $T_{\mathbf{r}}$ being a run as by definition of $\mathcal{I}_T$, $A \in V(x \cdot f(R))$. Similarly for the other direction. □

Automaton $\mathbb{A}^{mod}_{\mathcal{K}}$ for a model of $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$

$\mathbb{A}^{mod}_{\mathcal{K}}$ is a 2ATA on infinite trees that accepts a tree if its subtree labeled with G corresponds to a tree model $\mathcal{I}$ of $\mathcal{K}$. Formally, $\mathbb{A}^{mod}_{\mathcal{K}}$ is defined as the tuple $\langle \Sigma_{\mathcal{K}}, Q_{mod}, \delta_{mod}, q_0, F_{mod} \rangle$, where

$$Q_{mod} \quad = \quad \{q_0\} \cup \{q_X \mid X \in \mathbf{N} \cup \mathbf{B} \cup \mathbf{R} \cup \mathbf{P}\},$$

$F_{mod} = Q_{mod}$ and transition function $\delta_{mod} : Q_{mod} \times \Sigma_{\mathcal{K}} \to \mathcal{B}([n] \times Q_{mod})$ is defined as follows:

1. For each $\sigma \in \Sigma_{\mathcal{K}}$ such that $\{r, G\} \subseteq \sigma$, $\delta_{mod}(q_0, \sigma)$ is defined as:

$$\bigwedge_{i=1}^{n} \left[(i, q_{a_i}) \wedge \left(\bigwedge_{A \in \mathbf{B} \,:\, \mathcal{K} \models A(a_i)} (i, q_A) \right) \wedge \bigwedge_{j=1}^{n} \left(\bigwedge_{P \in \mathbf{R} \,:\, \mathcal{K} \models P(a_i, a_j)} (0, q_{P_{ij}}) \right) \right]$$

2. For each $\sigma \in \Sigma_{\mathcal{K}}$ such that $\{r, G\} \subseteq \sigma$ and each $P_{ij} \in \mathbf{P}$:

$$\delta_{mod}(q_{P_{ij}}, \sigma) \quad = \quad (i, q_{\exists P}) \wedge (j, q_{\exists P^-})$$

3. For each $\sigma \in \Sigma_{\mathcal{K}}$ such that $\sigma \cap \mathbf{N} = \{a_i\}$ and each atomic role P in the signature of $\mathcal{K}$:

$$\begin{aligned} \delta_{mod}(q_{\exists P}, \sigma) &= \left(\bigvee_{j=1}^{n} (j, q_P) \right) \vee \left(\bigvee_{j=1}^{n} (-1, q_{P_{ij}}) \right) \\ \delta_{mod}(q_{\exists P^-}, \sigma) &= \left(\bigvee_{j=1}^{n} (j, q_{P^-}) \right) \vee \left(\bigvee_{j=1}^{n} (-1, q_{P_{ji}}) \right) \end{aligned}$$

4. For each $\sigma \in \Sigma_{\mathcal{K}}$ such that $\sigma \cap \mathbf{N} = \emptyset$ and each basic role $R \in \mathbf{R}$,

$$\delta_{mod}(q_{\exists R}, \sigma) = (0, q_{R^-}) \vee \left(\bigvee_{i=1}^{n} (i, q_R) \right)$$

5. For each $\sigma \in \Sigma_{\mathcal{K}}$ such that $\sigma \cap \mathbf{N} = \emptyset$ and each basic role $R \in \mathbf{R}$:

$$\delta_{mod}(q_R, \sigma) = \left(\bigwedge_{R' \in \mathbf{R} \,:\, \mathcal{K} \models R \sqsubseteq R'} (0, q_{R'}) \right) \wedge (0, q_{\exists R^-})$$

6. For each $\sigma \in \Sigma_{\mathcal{K}}$ and each $B \in \mathbf{B}$:

$$\delta_{mod}(q_B, \sigma) = \bigwedge_{B' \in \mathbf{B} \,:\, \mathcal{K} \models B \sqsubseteq B'} (0, q_{B'})$$

7. For each $\sigma \in \Sigma_{\mathcal{K}}$ and each $X \in \mathbf{B} \cup \mathbf{R} \cup \mathbf{N} \cup \mathbf{P}$:

$$\delta_{mod}(q_X, \sigma) = \begin{cases} \textit{true} & \text{if } G \in \sigma \text{ and } X \in \sigma \\ \textit{false} & \text{otherwise} \end{cases}$$

If there are several entries of δ_{mod} for the same $q \in Q_{mod}$ and $\sigma \in \Sigma_{mod}$, $\delta_{mod}(q, \sigma) = \phi_1, \ldots, \delta_{mod}(q, \sigma) = \phi_m$, then we assume that $\delta_{mod}(q, \sigma) = \bigwedge_{i=1}^{m} \phi_i$.

Given a model $\mathcal{I}$, a *path* π from x to x', $x, x' \in \Delta^{\mathcal{I}}$, is a sequence of the form $(x = x_1, x_2, \ldots, x_m, x_{m+1} = x')$, $m \geq 0$, such that $x_i \in \Delta^{\mathcal{I}}$ and $(x_i, x_{i+1}) \in R_i^{\mathcal{I}}$ for some R_i, and m is the length of π. A model $\mathcal{I}$ of $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ is said to be a *tree model* if for each $x \in \Delta^{\mathcal{I}} \setminus \mathsf{Ind}(\mathcal{A})$ there exists a unique shortest path from x to $\mathsf{Ind}(\mathcal{A})$. The *depth* of an object x in a tree model $\mathcal{I}$, denoted $\mathsf{dep}(x)$, is the length of the shortest path from x to $\mathsf{Ind}(\mathcal{A})$. It is said that x' is a successor of x, $x' \in \mathsf{succ}(x)$ if x belongs to the path from x' to $\mathsf{Ind}(\mathcal{A})$ and $\mathsf{dep}(x') = \mathsf{dep}(x) + 1$.

Note that given a tree-model $\mathcal{I}$ of $\mathcal{K}$ with branching degree n, each domain element of $\mathcal{I}$ can be seen as an element of n^*. For $x' \in \Delta^{\mathcal{I}}$ with $\mathsf{dep}(x') = m \geq 0$, we assume a one-to-one numbering $g_{m,x'}(x)$ of each $x \in \mathsf{succ}(x')$, such that $1 \leq g_{m,x'}(x) \leq n$. Then $x \in \Delta^{\mathcal{I}}$ corresponds to

- i if $x = a_i$,
- $x' \cdot i$, where $\mathsf{dep}(x') = m \geq 0$, $x \in \mathsf{succ}(x')$ and $g_{m,x'}(x) = i$.

Then, $i \cdot -1$ denotes the empty sequence ϵ. Conversely, each sequence of natural numbers $x \in n^*$ can be seen as an element of $\Delta^{\mathcal{I}}$.

The *G-tree encoding* of a tree-model $\mathcal{I}$ of $\mathcal{K}$ with branching degree n is the $\Sigma_{\mathcal{K}}$-labeled tree $T_{\mathcal{I},G} = (n^*, V^{\mathcal{I},G})$, such that

- $V^{\mathcal{I},G}(\epsilon) = \{r, G\} \cup \{P_{ij} \mid (a_i, a_j) \in P^{\mathcal{I}}, P \text{ is an atomic role}\}$,
- for each $x \in \Delta^{\mathcal{I}}$:

$$\begin{array}{rcl} V^{\mathcal{I},G}(x) & = & \{G\} \cup \{B \mid x \in B^{\mathcal{U}}\} \cup \\ & & \{S \mid (x', x) \in S^{\mathcal{U}} \text{ and } \mathsf{dep}(x) > \mathsf{dep}(x')\} \cup \\ & & \{a \mid a \in \mathsf{Ind}(\mathcal{A}) \text{ and } x = a\}. \end{array}$$

Given a labeled tree (T, V), the *restriction* of T on G is a set T_G such that $T_G \subseteq T$ and for each $x \in T$: $x \in T_G$ iff $G \in V(x)$.

Given a labeled tree (T, V) and a run $(T_{\mathbf{r}}, \mathbf{r})$, the *interpretation represented* by T and $T_{\mathbf{r}}$, denoted, $\mathcal{I}_{T,T_{\mathbf{r}}}$, is defined similarly to $\mathcal{I}_T$:

$$\begin{array}{rcl} \Delta^{\mathcal{I}_{T,T_{\mathbf{r}}}} & = & \Delta^{\mathcal{I}_T}, \\ a_i^{\mathcal{I}_{T,T_{\mathbf{r}}}} & = & a_i^{\mathcal{I}_T}, \\ A^{\mathcal{I}_{T,T_{\mathbf{r}}}} & = & \Delta^{\mathcal{I}_T} \cap \{x \mid A \in V(x) \text{ and there exists } y \in T_{\mathbf{r}} \text{ with } \mathbf{r}(y) = (x, q_A)\}, \\ & & \text{for each atomic concept } A \in \mathbf{B} \text{ and} \\ P^{\mathcal{I}_{T,T_{\mathbf{r}}}} & = & (\Delta^{\mathcal{I}_T} \times \Delta^{\mathcal{I}_T}) \cap \\ & & \{(x, x') \in R_P \mid \text{ there exists } y \in T_{\mathbf{r}} \text{ s.t. } \mathbf{r}(y) = (x', q_P) \text{ or } \mathbf{r}(y) = (x, q_{P^-})\}, \\ & & \text{for each atomic role } P \in \mathbf{R}. \end{array}$$

Proposition B.11 *The following hold for $\mathbb{A}_{\mathcal{K}}^{mod}$:*

- *Let $\mathcal{I}$ be a tree model of $\mathcal{K}$ with branching degree n. Then $T_{\mathcal{I},G} \in \mathcal{L}(\mathbb{A}_{\mathcal{K}}^{mod})$.*
- *for each $(T, V) \in \mathcal{L}(\mathbb{A}_{\mathcal{K}}^{mod})$, if T_G is an individual unique tree and $(T_{\mathbf{r}}, \mathbf{r})$ is a corresponding run, then $\mathcal{I}_{T_G,T_{\mathbf{r}}}$ is a model of $\mathcal{K}$.*

Proof. For the first item, assume $T_{\mathcal{I},G} = (n^*, V^{\mathcal{I},G})$ is the tree encoding of a model $\mathcal{I}$ of $\mathcal{K}$. We show that a full run of $\mathbb{A}_{\mathcal{K}}^{mod}$ over $T_{\mathcal{I},G}$ exists.

The run $(T_{\mathbf{r}}, \mathbf{r})$ is built starting from the root ϵ, and setting $\mathbf{r}(\epsilon) = (\epsilon, q_0)$. Then, to correctly execute the initial transition, the root has children as follows:

- for each $a_k \in \mathsf{Ind}(\mathcal{A})$
  - a child $k^*_{a_k}$ with $\mathbf{r}(k^*_{a_k}) = (a_k, q^*_{a_k})$,
  - a child k^*_B for each $B \in \mathbf{B}$ such that $a_k \in B^{\mathcal{I}}$, with $\mathbf{r}(k^*_B) = (a_k, q^*_B)$,
- a child k^*_{P,a_k,a_j} for each $a_k, a_j \in \mathsf{Ind}(\mathcal{A})$ and each atomic role P such that $(a_k, a_j) \in P^{\mathcal{I}}$, with $\mathbf{r}(k^*_{P,a_k,a_j}) = (\epsilon, q^*_{P_{kj}})$,

Then the successor relationship in $T_{\mathbf{r}}$ is defined inductively as follows.

2. Let y be a node in $T_{\mathbf{r}}$ such that $\mathbf{r}(y) = (x, q_{P_{ij}})$ for $x = \epsilon$ and $P \in \mathbf{R}$. Then y has
   - a child $y \cdot i_{\exists P}$ with $\mathbf{r}(y \cdot i_{\exists P}) = (x \cdot i, q_{\exists P})$,
   - a child $y \cdot j_{\exists P^-}$ with $\mathbf{r}(y \cdot j_{\exists P^-}) = (x \cdot j, q_{\exists P^-})$,
3. Let y be a node in $T_{\mathbf{r}}$ such that $\mathbf{r}(y) = (x, q_{\exists R})$ for some $x \in \Delta^{\mathcal{I}}$, $V^{\mathcal{I},G}(x) \cap \mathbf{N} = \{a_i\}$, $R \in \mathbf{R}$, and R_{ij} denotes P_{ij} if $R = P$ and P_{ji} if $R = P^-$. Then y has
   - if $R \in V^{\mathcal{I},G}(x \cdot j)$
     - a child $y \cdot j_R$ with $\mathbf{r}(y \cdot j_R) = (x \cdot j, q_R)$,
   - if $R_{ij} \in V^{\mathcal{I},G}(x \cdot -1)$
     - a child $y \cdot -1_{R_{ij}}$ with $\mathbf{r}(y \cdot -1_{R_{ij}}) = (x \cdot -1, q_{R_{ij}})$,
4. Let y be a node in $T_{\mathbf{r}}$ such that $\mathbf{r}(y) = (x, q_{\exists R})$ for some $x \in \Delta^{\mathcal{I}}$, $V^{\mathcal{I},G}(x) \cap \mathbf{N} = \emptyset$ and $R \in \mathbf{R}$. Then y has
   - if $R \in V^{\mathcal{I},G}(x \cdot i)$
     - a child $y \cdot i_R$ with $\mathbf{r}(y \cdot i_R) = (x \cdot i, q_R)$,

- if $R^- \in V^{\mathcal{I},G}(x)$
  - a child $y \cdot 0_{R^-}$ with $\mathbf{r}(y \cdot 0_R) = (x, q_R)$,

5. Let y be a node in $T_\mathbf{r}$ such that $\mathbf{r}(y) = (x, q_R)$ for some $x \in \Delta^\mathcal{I}$ and $R \in \mathbf{R}$. Then y has
   - a child $y \cdot 0^*_{R'}$ for each $R' \in \mathbf{R}$ s.t. $\mathcal{K} \models R \sqsubseteq R'$, with $\mathbf{r}(y \cdot 0^*_{R'}) = (x, q^*_{R'})$,
   - a child $y \cdot 0_{\exists R^-}$, with $\mathbf{r}(y \cdot 0_{\exists R^-}) = (x, q_{\exists R^-})$,
6. Let y be a node in $T_\mathbf{r}$ such that $\mathbf{r}(y) = (x, q_B)$ for some $x \in \Delta^\mathcal{I}$ and $B \in \mathbf{B}$. Then y has
   - a child $y \cdot 0^*_{B'}$ for each $B' \in \mathbf{B}$ s.t. $\mathcal{K} \models B \sqsubseteq B'$, with $\mathbf{r}(y \cdot 0^*_{B'}) = (x, q^*_{B'})$,

Since $\mathcal{I}$ is a model of $\mathcal{K}$, $T_\mathbf{r}$ satisfies the transition function δ_{mod}. In particular, in the rules 3 and 4 in the inductive definition of $T_\mathbf{r}$, there will exists a node $x' \in \Delta^\mathcal{I}$ such that $(x, x') \in R^\mathcal{I}$, hence at least one of conditions will be satisfied.

It is easy to see that this run is accepting, as for each infinite path P of $T_\mathbf{r}$, $q_R \in \mathit{inf}(P)$ for some R. Hence, $T_{\mathcal{I},G} \in \mathcal{L}(\mathbb{A}^{mod}_\mathcal{K})$.

To show the second item, let $(T, V) \in \mathcal{L}(\mathbb{A}^{can}_\mathcal{K})$ and $(T_\mathbf{r}, \mathbf{r})$ an accepting run of (T, V). Moreover, let T_G be a tree (i.e., prefix closed) and individual unique. Then $\mathcal{I}_{T_G,T_\mathbf{r}}$ is defined and it can be shown that $\mathcal{I}_{T_G,T_\mathbf{r}}$ a model of $\mathcal{K}$:

1. for each $i \in \{1, \ldots, n\}$, $\mathcal{K} \models B(a_i)$ implies $a_i \in B^{\mathcal{I}_{T_G,T_\mathbf{r}}}$,
2. for each $i, j \in \{1, \ldots, n\}$, $\mathcal{K} \models P(a_i, a_j)$ implies $(a_i, a_j) \in P^{\mathcal{I}_{T_G,T_\mathbf{r}}}$,
3. if $x \in B^{\mathcal{I}_{T_G,T_\mathbf{r}}}$, then $x \in B'^{\mathcal{I}_{T_G,T_\mathbf{r}}}$ for each B' s.t. $\mathcal{K} \models B \sqsubseteq B'$,
4. if $(x, x') \in R^{\mathcal{I}_{T_G,T_\mathbf{r}}}$, then $(x, x') \in R'^{\mathcal{I}_{T_G,T_\mathbf{r}}}$ for each R' s.t. $\mathcal{K} \models R \sqsubseteq R'$,
5. if $x \in B^{\mathcal{I}_{T_G,T_\mathbf{r}}}$ and $\mathcal{K} \models B \sqsubseteq \exists R$, then there exists $x' \in T_G$ such that $(x, x') \in R^{\mathcal{I}_{T_G,T_\mathbf{r}}}$.

We show item 5 holds, the rest can be shown by analogy. Assume $x \in B^{\mathcal{I}_{T_G,T_\mathbf{r}}}$ and $\mathcal{K} \models B \sqsubseteq \exists R$ for some concept B and role R. Then by definition of $\mathcal{I}_{T_G,T_\mathbf{r}}$ we have that $B, G \in V(x)$ and there exist a node $y \in T_\mathbf{r}$ with $r(y) = (x, q_B)$. Since $T_\mathbf{r}$ is a run and by definition of δ_{mod}, there exist nodes $y' = y \cdot 0_{\exists R}$ and $y'' = y' \cdot z$ in $T_\mathbf{r}$ such that $\mathbf{r}(y') = (x, q_{\exists R})$ and $\mathbf{r}(y'') = (x \cdot i, q_R)$, or $\mathbf{r}(y'') = (x, q_{R^-})$, or $\mathbf{r}(y'') = (\epsilon, q_{R_{ij}})$. In any case, it is easy to see that there is $x' \in T$ with $G \in V(x')$ (i.e., $x' \in T_G$) such that $(x, x') \in R^{\mathcal{I}_{T_G,T_\mathbf{r}}}$.

Thus, $\mathcal{I}_{T_G,T_\mathbf{r}}$ is a model of $\mathcal{K}$. $\square$

Automaton $\mathbb{A}_{\mathit{fin}}$

$\mathbb{A}_{\mathit{fin}}$ is a one-way non-deterministic automaton on infinite trees that accepts a tree if it has a finite prefix where each node is marked with the special symbol G, and no other node in the tree is marked with G. Formally, $\mathbb{A}_{\mathit{fin}} = \langle \Gamma_\mathcal{K}, Q_{\mathit{fin}}, \delta_{\mathit{fin}}, q_0, F_{\mathit{fin}} \rangle$, where $Q_{\mathit{fin}} = \{q_0, q_1\}$, $F_{\mathit{fin}} = \{q_1\}$ and transition function $\delta_{\mathit{fin}} : Q_{\mathit{fin}} \times \Gamma_\mathcal{K} \to \mathcal{B}([n] \times Q_{\mathit{fin}})$ is defined as follows:

1. For each $\sigma \in \Gamma_\mathcal{K}$:

$$\delta(q_0, \sigma) = \begin{cases} \bigwedge_{i=1}^{n} (i, q_0), & \text{if } G \in \sigma \\ \bigwedge_{i=1}^{n} (i, q_1), & \text{if } G \notin \sigma \end{cases}$$

2. For each $\sigma \in \Gamma_\mathcal{K}$:

$$\delta(q_1, \sigma) = \begin{cases} \bigwedge_{i=1}^{n} (i, q_1), & \text{if } G \notin \sigma \\ \mathit{false} & \text{if } G \in \sigma \end{cases}$$

B.3 Proof of Theorem 5.3

Proof. We prove that the non-emptiness problem for universal solutions is in NP. Assume we are given a mapping $\mathcal{M} = (\Sigma_1, \Sigma_2, \mathcal{T}_{12})$ and a source KB $\mathcal{K}_1 = \langle \mathcal{T}_1, \mathcal{A}_1 \rangle$, and we want to decide whether there exists a universal solution for $\mathcal{K}_1$ under $\mathcal{M}$ (all ABoxes are considered to be OWL 2 QL ABoxes without inequalities).

First, we check whether $\mathcal{K}_1$ and $\mathcal{M}$ are Σ_2-positive. This check can be done in polynomial time, and if it was successful, then by Lemma B.5 it remains to verify whether there exists a universal solution for $\mathcal{K}_1^{pos}$ under $\mathcal{M}^{pos}$.

Second, we construct the maximal target OWL 2 QL ABox, a candidate for universal solution. Let $\mathcal{A}_2$ be the ABox over Σ_2 containing every membership assertion α of the form $B(a)$ or $R(a,b)$ such that $\langle \mathcal{T}_1^{pos} \cup \mathcal{T}_{12}^{pos}, \mathcal{A}_1 \rangle \models \alpha$, $a, b \in \mathsf{Ind}(\mathcal{A}_1)$, B is a basic concept and R is a basic role. Then $\mathcal{A}_2$ is of polynomial size, and

Lemma B.12 *A universal solution for $\mathcal{K}_1^{pos}$ under $\mathcal{M}^{pos}$ exists iff $\mathcal{A}_2$ is a solution for $\mathcal{K}_1^{pos}$ under $\mathcal{M}^{pos}$.*

Proof. ($\Rightarrow$) Assume a universal solution for $\mathcal{K}_1^{pos}$ under $\mathcal{M}^{pos}$ exists. As it follows from Lemma B.7, there exists a universal solution $\mathcal{A}_3$ such that $\mathcal{U}_{\mathcal{A}_3} \subseteq \mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$, hence $\mathcal{A}_3 \subseteq \mathcal{A}_2$. As $\mathcal{A}_3$ is a solution, there exists $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{K}_1^{pos}$ and $(\mathcal{I}, \mathcal{U}_{\mathcal{A}_3}) \models \mathcal{T}_{12}^{pos}$. It follows that for each model $\mathcal{J}$ of $\mathcal{A}_2$, $\mathcal{J} \supseteq \mathcal{U}_{A_2} \supseteq \mathcal{U}_{\mathcal{A}_3}$, and therefore $(\mathcal{I}, \mathcal{J}) \models \mathcal{T}_{12}^{pos}$. By definition of solution, $\mathcal{A}_2$ is a solution.

($\Leftarrow$) Assume $\mathcal{A}_2$ is a solution for $\mathcal{K}_1^{pos}$ under $\mathcal{M}^{pos}$. Then $\mathcal{A}_2$ is a universal solution follows from the proof of Lemma B.3. Since $\mathcal{A}_2$ is an OWL 2 QL ABox, we conclude that a universal solution for $\mathcal{K}_1^{pos}$ under $\mathcal{M}^{pos}$ exists. $\square$

Thus, it remains only to check whether $\mathcal{A}_2$ is a solution. We need the following result to perform this check in NP.

Lemma B.13 *Let $\mathcal{A}_2$ be an (extended) ABox over Σ_2 such that it is a solution for $\mathcal{K}_1^{pos}$ under $\mathcal{M}^{pos}$. Then there exists an interpretation $\mathcal{I}$ such that $\mathcal{I}$ is of polynomial size, $\mathcal{I}$ is a model of $\mathcal{K}_1^{pos}$ and $(\mathcal{I}, \mathcal{V}_{\mathcal{A}_2}) \models \mathcal{T}_{12}^{pos}$.*

Proof. Assume $\mathcal{A}_2$ is a solution for $\mathcal{K}_1^{pos}$ under $\mathcal{M}^{pos}$, then for each model of $\mathcal{A}_2$, in particular for $\mathcal{V}_{\mathcal{A}_2}$, there exists $\mathcal{I}'$ such that $\mathcal{I}'$ is a model of $\mathcal{K}_1^{pos}$ and $(\mathcal{I}', \mathcal{V}_{\mathcal{A}_2}) \models \mathcal{T}_{12}^{pos}$. Suppose $|\mathcal{I}'|$ is more than polynomial, then since $(\mathcal{I}', \mathcal{V}_{\mathcal{A}_2}) \models \mathcal{T}_{12}^{pos}$ it follows $B^{\mathcal{I}'} \subseteq \Delta^{\mathcal{A}_2}$ and $R^{\mathcal{I}'} \subseteq \Delta^{\mathcal{A}_2} \times \Delta^{\mathcal{A}_2}$ for each basic concept B and role R that appear on the left hand side of some inclusion in $\mathcal{T}_{12}^{pos}$. Therefore, we construct an interpretation $\mathcal{I}$ of polynomial size as follows:

- $\Delta^{\mathcal{I}} = \Delta^{\mathcal{A}_2} \cup N_a \cup \{d\}$, for a fresh domain element d,
- $a^{\mathcal{I}} = a$ for $a \in N_a$,
- $A^{\mathcal{I}} = (A^{\mathcal{I}'} \cap \Delta^{\mathcal{A}_2}) \cup \{d \mid \text{ if } A^{\mathcal{I}'} \setminus \Delta^{\mathcal{A}_2} \neq \emptyset\}$ for each atomic concept A,
- $$\begin{aligned} R^{\mathcal{I}} \;=\; & (R^{\mathcal{I}'} \cap (\Delta^{\mathcal{A}_2} \times \Delta^{\mathcal{A}_2})) \;\cup \\ & \{(a,d) \mid (a,b) \in R^{\mathcal{I}'} \setminus (\Delta^{\mathcal{A}_2} \times \Delta^{\mathcal{A}_2}), a \in (\exists R)^{\mathcal{I}'} \cap \Delta^{\mathcal{A}_2}\} \;\cup \\ & \{(d,a) \mid (b,a) \in R^{\mathcal{I}'} \setminus (\Delta^{\mathcal{A}_2} \times \Delta^{\mathcal{A}_2}), a \in (\exists R^-)^{\mathcal{I}'} \cap \Delta^{\mathcal{A}_2}\} \;\cup \\ & \{(d,d) \mid (a,b) \in R^{\mathcal{I}'} \setminus (\Delta^{\mathcal{A}_2} \times \Delta^{\mathcal{A}_2}), a \notin (\exists R)^{\mathcal{I}'} \cap \Delta^{\mathcal{A}_2}, b \notin (\exists R^-)^{\mathcal{I}'} \cap \Delta^{\mathcal{A}_2}\} \end{aligned}$$
  for each atomic role R.

Note that $\mathcal{V}_{\mathcal{A}_2}$ interprets all constants as themselves, and $\mathcal{I}'$ agrees on interpretation of constants with $\mathcal{V}_{\mathcal{A}_2}$, for this reason $\Delta^{\mathcal{I}} \supseteq N_a$.

It is straightforward to verify that $\mathcal{I}$ is a model of $\mathcal{K}_1^{pos}$: clearly, $\mathcal{I}$ is a model of $\mathcal{A}_1$, we show $\mathcal{I} \models \mathcal{T}_1^{pos}$. Assume, $\mathcal{T}_1^{pos} \models B \sqsubseteq C$ for basic concepts B, C, and $b \in B^{\mathcal{I}}$. If $b \in \Delta^{\mathcal{I}'} \cap \Delta^{\mathcal{A}_2}$, then since $\mathcal{I}' \models B \sqsubseteq C$, we have that $b \in C^{\mathcal{I}'}$, which implies $b \in C^{\mathcal{I}}$. Otherwise, $b = d$ and for some $c \in \Delta^{\mathcal{I}'} \setminus \Delta^{\mathcal{A}_2}$, $c \in B^{\mathcal{I}'}$, therefore $c \in C^{\mathcal{I}'}$, and thus by definition of $\mathcal{I}$, $d \in C^{\mathcal{I}}$. Role inclusions are handled similarly. Moreover, as $\mathcal{I}$ and $\mathcal{I}'$ agree on all concepts and roles that appear on the left hand side of $\mathcal{T}_{12}^{pos}$, it follows that $(\mathcal{I}, \mathcal{V}_{\mathcal{A}_2}) \models \mathcal{T}_{12}^{pos}$. Hence, $\mathcal{I}$ is the interpretation of polynomial size we were looking for. $\square$

Finally, the NP algorithm for deciding the non-emptiness problem for universal solutions is as follows:

1. verify whether $\mathcal{K}_1$ and $\mathcal{M}$ are Σ_2-positive, if yes,
2. compute $\mathcal{A}_2$, the Σ_2-closure of $\mathcal{A}_1$ with respect to $\mathcal{T}_1^{pos} \cup \mathcal{T}_{12}^{pos}$.
3. guess a source interpretation $\mathcal{I}$ of polynomial size.
4. If $\mathcal{I} \models \mathcal{K}_1^{pos}$ and $(\mathcal{I}, \mathcal{U}_{\mathcal{A}_2}) \models \mathcal{T}_{12}^{pos}$, then a universal solution for $\mathcal{K}_1$ under $\mathcal{M}$ exists, and $\mathcal{A}_2$ is a universal solution, otherwise a universal solution does not exist.

Note that steps 1,2 and 4 can be done in polynomial time, hence this algorithm is in fact an NP algorithm. Below we prove the correctness of the algorithm.

Assume $\mathcal{I} \models \mathcal{K}_1^{pos}$ and $(\mathcal{I}, \mathcal{U}_{\mathcal{A}_2}) \models \mathcal{T}_{12}^{pos}$. Then $\mathcal{A}_2$ is a solution: for each model $\mathcal{J}$ of $\mathcal{A}_2$, it holds $\mathcal{U}_{\mathcal{A}_2} \subseteq \mathcal{J}$, therefore $(\mathcal{I}, \mathcal{J}) \models \mathcal{T}_{12}$. By Lemma B.12 we obtain that a universal solution for $\mathcal{K}_1$ under $\mathcal{M}$ exists, and from its proof it follows that $\mathcal{A}_2$ is a universal solution. Thus, the algorithm is sound.

We show the algorithm is complete. Assume $\mathcal{I} \not\models \mathcal{K}_1^{pos}$ or $(\mathcal{I}, \mathcal{U}_{\mathcal{A}_2}) \not\models \mathcal{T}_{12}^{pos}$, and to the contrary, $\mathcal{A}_2$ is a solution. The by Lemma B.13, there exists a model $\mathcal{I}'$ of $\mathcal{K}_1^{pos}$ of polynomial size such that $(\mathcal{I}', \mathcal{U}_{\mathcal{A}_2}) \models \mathcal{T}_{12}^{pos}$. Contradiction with the guessing step. Therefore, $\mathcal{A}_2$ is not a solution and there exists no universal solution. Thus, the algorithm is complete.

As a corollary we obtain an upper bound for the membership problem.

Theorem B.14 *The membership problem for universal solutions is in* NP.

□

B.4 Proof of Theorem 5.5

Proof. First we provide the PSPACE lower bound, and then present the EXPTIME automata-based algorithm for deciding the non-emptiness problem for universal solutions with extended ABoxes.

Lemma B.15 *The non-emptiness problem for universal solutions with extended ABoxes in DL-Lite$_\mathcal{R}$ is* PSPACE*-hard.*

Proof. The proof is by reduction of the satisfiability problem for quantified Boolean formulas, known to be PSPACE-complete. Suppose we are given a QBF

$$\phi = \mathsf{Q}_1 X_1 \dots \mathsf{Q}_n X_n \bigwedge_{j=1}^{m} C_j$$

where $\mathsf{Q}_i \in \{\forall, \exists\}$ and C_j, $1 \leq j \leq m$, are clauses over the variables X_i, $1 \leq i \leq n$.

Let $\Sigma_1 = \{A, Y_i^k, X_i^k, S_l, T_l, Q_i^k, P_i^k, R_j, R_j^l \mid 1 \leq j \leq m, 1 \leq i \leq n, 0 \leq l \leq n, k \in \{0,1\}\}$ where A, Y_i^k, X_i^k are concept names and the rest are role names. Let $\mathcal{T}_1$ be the following TBox over Σ_1 for $1 \leq j \leq m$, $1 \leq i \leq n$ and $k \in \{0,1\}$:

$$\begin{array}{rclrclrcl}
A & \sqsubseteq & \exists S_0^- & \exists S_{i-1}^- & \sqsubseteq & \exists Q_i^k & \multicolumn{3}{l}{\text{if } \mathsf{Q}_i = \forall} \\
 & & & \exists S_{i-1}^- & \sqsubseteq & \exists S_i & \multicolumn{3}{l}{\text{if } \mathsf{Q}_i = \exists} \\
\exists (Q_i^k)^- & \sqsubseteq & Y_i^k & Q_i^k & \sqsubseteq & S_i & \exists S_n^- & \sqsubseteq & \exists R_j \\
\exists R_j^- & \sqsubseteq & \exists R_j & & & & & & \\
 & & & & & & & & \\
A & \sqsubseteq & \exists T_0^- & \exists T_{i-1}^- & \sqsubseteq & \exists P_i^k & P_i^k & \sqsubseteq & T_i \\
\exists (P_i^k)^- & \sqsubseteq & X_i^k & X_i^0 & \sqsubseteq & \exists R_j^i & \multicolumn{3}{l}{\text{if } \neg X_i \in C_j} \\
 & & & X_i^1 & \sqsubseteq & \exists R_j^i & \multicolumn{3}{l}{\text{if } X_i \in C_j} \\
\exists (R_j^i)^- & \sqsubseteq & \exists R_j^{i-1} & & & & & &
\end{array}$$

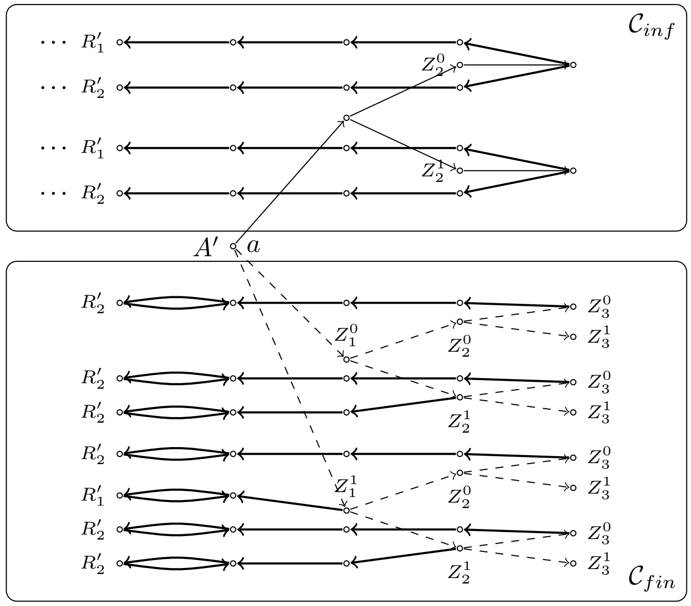
and $\mathcal{A}_1 = \{A(a)\}$.

Let $\Sigma_2 = \{A', Z_i^0, Z_i^1, S', R_j'\}$ where A', Z_i^0, Z_i^1 are concept names and S', R_j' are role names, $\mathcal{M} = (\Sigma_1, \Sigma_2, \mathcal{T}_{12})$, and $\mathcal{T}_{12}$ the following set of inclusions:

$$\begin{array}{rclrclrcl}
A & \sqsubseteq & A' & S_i & \sqsubseteq & S' & R_j & \sqsubseteq & R_j' \\
 & & & T_i & \sqsubseteq & S' & T_i & \sqsubseteq & {R_j'}^- \\
 & & & Y_i^k & \sqsubseteq & Z_i^k & R_j^i & \sqsubseteq & R_j' \\
 & & & X_i^k & \sqsubseteq & Z_i^k & R_j^0 & \sqsubseteq & {R_j'}^-
\end{array}$$

We verify that $\models \phi$ if and only if $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$ is Σ_2-homomorphically embeddable into a finite subset of itself. The latter, in turn, is equivalent to the existence of a universal solution for $\mathcal{K}_1 = \langle \mathcal{T}_1, \mathcal{A}_1 \rangle$ under $\mathcal{M}$, which is shown in Lemma B.7.

For $\phi = \exists X_1 \forall X_2 \exists X_3 (X_1 \wedge (X_2 \vee \neg X_3))$, Σ_2-reduct of $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$ can be depicted as follows:



where each edge $\longrightarrow$ is labeled with S', each edge $\dashrightarrow$ is labeled with $S', R_j'^-$ for $1 \leq j \leq m$, and the labels of edges $\longrightarrow$ are shown to the left of each infinite and finite path. The labels of the nodes (if any) are shown next to each node.

Let $\mathcal{C}_{inf}$ and $\mathcal{C}_{fin}$ be the parts of $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$ generated using the first 9 axioms and the last 9 axioms of $\mathcal{T}_1$ respectively. Note that $\mathcal{C}_{inf}$ is infinite, while $\mathcal{C}_{fin}$ is finite. One can show that $\mathcal{C}_{inf}$ is Σ_2-homomorphically embeddable into $\mathcal{C}_{fin}$ (which is equivalent to $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$ is Σ_2-homomorphically embeddable into $\mathcal{C}_{fin}$) iff ϕ is satisfiable.

The rest of the proof follows the line of the proof of Theorem 11 in [Konev *et al.*, 2011].

($\Rightarrow$) Suppose $\models \phi$. We show that the canonical model $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$ is Σ_2-homomorphically embeddable into a finite subset of itself. More precisely, let us denote with $\mathcal{T}_1^{inf}$ the subset of $\mathcal{T}_1$ consisting of the first 9 axioms, and $\mathcal{T}_1^{fin}$ the subset of $\mathcal{T}_1$ consisting of the last 9 axioms. Then $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle} = \mathcal{U}_{\langle \mathcal{T}_1^{inf} \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle} \cup \mathcal{U}_{\langle \mathcal{T}_1^{fin} \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$, and we construct a Σ_2-homomorphism $h : \Delta^{\mathcal{U}_{\langle \mathcal{T}_1^{inf} \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}} \to \Delta^{\mathcal{U}_{\langle \mathcal{T}_1^{fin} \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}}$. In the following we use $\mathcal{U}_{inf}$ to denote $\mathcal{U}_{\langle \mathcal{T}_1^{inf} \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$, and $\mathcal{U}_{fin}$ to denote $\mathcal{U}_{\langle \mathcal{T}_1^{fin} \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$.

We begin by setting $h(a^{\mathcal{U}_{inf}}) = a^{\mathcal{U}_{fin}}$. Then we define h in such a way that, for each path π in $\mathcal{U}_{inf}$ of length $i + 1 \leq n$, $h(\pi)$ is a path $a^{\mathcal{U}_{fin}} w_1 \ldots w_i$ of length $i + 1$ in $\mathcal{U}_{fin}$ and it defines an assignment $\mathfrak{a}_{h(\pi)}$ to the variables $X_1, \ldots, X_i$ by taking, for all $1 \leq i' \leq i$,

$$\mathfrak{a}_{h(\pi)}(X_{i'}) = \top \Leftrightarrow a^{\mathcal{U}_{fin}} \cdot w_1 \cdot \ldots \cdot w_{i'} \in (X_{i'}^1)^{\mathcal{U}_{fin}}$$
$$\mathfrak{a}_{h(\pi)}(X_{i'}) = \bot \Leftrightarrow a^{\mathcal{U}_{fin}} \cdot w_1 \cdot \ldots \cdot w_{i'} \in (X_{i'}^0)^{\mathcal{U}_{fin}}.$$

Such assignments $\mathfrak{a}_{h(\pi)}$ will satisfy the following:

($\mathfrak{a}$) the QBF obtained from ϕ by removing $\mathsf{Q}_1 X_1 \ldots \mathsf{Q}_i X_i$ from its prefix is true under $\mathfrak{a}_{h(\pi)}$.

For the paths of length 0 the Σ_2-homomorphism h has been defined and ($\mathfrak{a}$) trivially holds. Suppose that we have defined h for all paths in $\mathcal{U}_{inf}$ of length $i + 1 \leq n$. We extend h to all paths of length $i + 2$ in $\mathcal{U}_{inf}$ such that ($\mathfrak{a}$) holds. Let π be a path of length $i + 1$. In $\mathcal{U}_{fin}$ we have

$$\mathsf{tail}(h(\pi)) \rightsquigarrow_{\langle \mathcal{T}_1^{fin} \cup \mathcal{T}_{12}, \mathcal{A}_2 \rangle} w_{[P_i^k]}^{\mathcal{U}_{fin}}, \quad \text{and} \quad h(\pi) \cdot w_{[P_i^k]}^{\mathcal{U}_{fin}} \in (X_i^k)^{\mathcal{U}_{fin}}, \text{ for } k = 0, 1.$$

If $\mathsf{Q}_i = \forall$ then in $\mathcal{U}_{inf}$ we have

$$\mathsf{tail}(\pi) \rightsquigarrow_{\langle \mathcal{T}_1^{inf} \cup \mathcal{T}_{12}, \mathcal{A}_2 \rangle} w_{[Q_i^k]}^{\mathcal{U}_{inf}}, \quad \text{and} \quad \pi \cdot w_{[Q_i^k]}^{\mathcal{U}_{inf}} \in (X_i^k)^{\mathcal{I}}, \text{ for } k = 0, 1.$$

Thus, we set $h(\pi \cdot w_{[Q_i^k]}^{\mathcal{U}_{inf}}) = h(\pi) \cdot w_{[P_i^k]}^{\mathcal{U}_{fin}}$, for $k = 0, 1$. Clearly, ($\mathfrak{a}$) holds. Otherwise, $\mathsf{Q}_i = \exists$ and in $\mathcal{U}_{inf}$ we have

$$\mathsf{tail}(\pi) \rightsquigarrow_{\langle \mathcal{T}_1^{inf} \cup \mathcal{T}_{12}, \mathcal{A}_2 \rangle} w_{[S_i]}^{\mathcal{U}_{inf}}.$$

We know that $\models \phi$ and so, by, ($\mathfrak{a}$), the QBF obtained from π by removing $\mathsf{Q}_1 X_1 \dots \mathsf{Q}_i X_i$ is true under either $\mathfrak{a}_{h(\pi)} \cup \{X_i = \top\}$ or $\mathfrak{a}_{h(\pi)} \cup \{X_i = \bot\}$. We set $h(\pi \cdot w^{\mathcal{U}_{inf}}_{[S_i]}) = h(\pi) \cdot w^{\mathcal{U}_{fin}}_{[P^k_i]}$ with $k = 1$ in the former case, and $k = 0$ in the latter case. Either way, ($\mathfrak{a}$) holds.

Consider now in $\mathcal{U}_{inf}$ a path π of length $n+1$ from $a^{\mathcal{U}_{inf}}$ to $w^{\mathcal{U}_{inf}}_n$. By construction, we have

$$h(\pi) = a^{\mathcal{U}_{fin}} \cdot w^{\mathcal{U}_{fin}}_{[P^{k_1}_1]} \cdot \dots \cdot w^{\mathcal{U}_{fin}}_{[P^{k_n}_n]}.$$

Next, on the one hand, the path π in $\mathcal{U}_{inf}$ has m infinite extensions of the form $\pi \cdot w^{\mathcal{U}_{inf}}_{[R_j]} \cdot w^{\mathcal{U}_{inf}}_{[R_j]} \dots$, for $1 \le j \le m$. On the other hand, as $\models \phi$, by ($\mathfrak{a}$), for each clause C_j, there is some $1 \le i' \le n$ such that $h(\pi)$ contains $w^{\mathcal{U}_{fin}}_{[P^1_{i'}]}$ if $X_{i'} \in C_j$, or $w^{\mathcal{U}_{fin}}_{[P^0_{i'}]}$ if $\neg X_{i'} \in C_j$. We set for each $1 \le l \le n - i'$,

$$h(\pi \cdot \underbrace{w^{\mathcal{U}_{inf}}_{[R_j]} \cdot \dots \cdot w^{\mathcal{U}_{inf}}_{[R_j]}}_{l \text{ times}}) = a^{\mathcal{U}_{fin}} \cdot w^{\mathcal{U}_{fin}}_{[P^{k_1}_1]} \cdot \dots \cdot w^{\mathcal{U}_{fin}}_{[P^{k_{n-l}}_{n-l}]},$$

for each $n+1 \ge l > n - i'$,

$$h(\pi \cdot \underbrace{w^{\mathcal{U}_{inf}}_{[R_j]} \cdot \dots \cdot w^{\mathcal{U}_{inf}}_{[R_j]}}_{l \text{ times}}) = a^{\mathcal{U}_{fin}} \cdot w^{\mathcal{U}_{fin}}_{[P^{k_1}_1]} \cdot \dots \cdot w^{\mathcal{U}_{fin}}_{[P^{k_{i'}}_{i'}]} \cdot w^{\mathcal{U}_{fin}}_{[R^{i'}_j]} \cdot \dots \cdot w^{\mathcal{U}_{fin}}_{[R^{n-l+1}_j]},$$

and for each $l > n+1$

$$h(\pi \cdot \underbrace{w^{\mathcal{U}_{inf}}_{[R_j]} \cdot \dots \cdot w^{\mathcal{U}_{inf}}_{[R_j]}}_{l \text{ times}}) = a^{\mathcal{U}_{fin}} \cdot w^{\mathcal{U}_{fin}}_{[P^{k_1}_1]} \cdot \dots \cdot w^{\mathcal{U}_{fin}}_{[P^{k_{i'}}_{i'}]} \cdot w^{\mathcal{U}_{fin}}_{[R^{i'}_j]} \cdot w^{\mathcal{U}_{fin}}_{[R^{i'-1}_j]} \cdot \dots \cdot w^{\mathcal{U}_{fin}}_{[R^{i^\star}_j]},$$

where $i^\star = (n - l + 1) \mod 2$. It is immediate to verify that h is a Σ_2-homomorphism from $\mathcal{U}_{inf}$ to $\mathcal{U}_{fin}$.

($\Leftarrow$) Let h be a Σ_2-homomorphism from $\mathcal{U}_{inf}$ to $\mathcal{U}_{fin}$. We show that $\models \phi$.

Let π be a path of length $n+1$, $\pi = a^{\mathcal{U}_{inf}} \cdot w_1 \cdot \dots \cdot w_n$, in $\mathcal{U}_{inf}$. Then $(a^{\mathcal{U}_{inf}}, \pi_1), (\pi_i, \pi_{i+1}) \in S'^{\mathcal{U}_{inf}}$, where $\pi_i = a^{\mathcal{U}_{inf}} \cdot w_1 \cdot \dots \cdot w_i$, for $1 \le i \le n-1$. Furthermore, let $Z^{k_1}_1, Z^{k_2}_2, \dots, Z^{k_n}_n$ be the concepts containing subpaths of $h(\pi_i)$. We show that for every $1 \le j \le m$, the clause C_j contains at least one of the literals

$$\{X_i \mid k_i = 1, 1 \le i \le n\} \cup \{\neg X_i \mid k_i = 0, 1 \le i \le n\}.$$

Validity of ϕ will follow.

Consider a path of the form $\pi \cdot \underbrace{w^{\mathcal{U}_{inf}}_{[R_j]} \cdot \dots \cdot w^{\mathcal{U}_{inf}}_{[R_j]}}_{n+1 \text{ times}}$ in $\mathcal{U}_{inf}$. Then its h-image in $\mathcal{U}_{fin}$ must be of the form

$$a^{\mathcal{U}_{fin}} \cdot w^{\mathcal{U}_{fin}}_{[P^{k_1}_1]} \cdot \dots \cdot w^{\mathcal{U}_{fin}}_{[P^{k_i}_i]} \cdot w^{\mathcal{U}_{fin}}_{[R^i_j]} \cdot w^{\mathcal{U}_{fin}}_{[R^{i-1}_j]} \cdot \dots \cdot w^{\mathcal{U}_{fin}}_{[R^{i'}_j]}$$

for some $1 \le i \le n$, $i' = 0$ or $i' = 1$, and $k_i = 0$ or $k_i = 1$. If $k_i = 0$, then C_j must contain $\neg X_i$, otherwise X_i. $\square$

Lemma B.16 *The non-emptiness problem for universal solutions is in* ExpTime. *For a given DL-Lite$_\mathcal{R}$ mapping $\mathcal{M}$ and a given DL-Lite$_\mathcal{R}$ KB $\mathcal{K}_1$, if a universal solution $\mathcal{A}_2$ (an extended ABox without inequalities) exists, then it is at most exponentially large in the size of $\mathcal{K}_1 \cup \mathcal{M}$.*

Proof. First, we provide an algorithm for checking existence of a universal solution with extended ABoxes in *DL-Lite$^{pos}_\mathcal{R}$*. Given a *DL-Lite$^{pos}_\mathcal{R}$* mapping $\mathcal{M} = (\Sigma_1, \Sigma_2, \mathcal{T}_{12})$, to verify that a universal solution for $\langle \mathcal{T}_1, \mathcal{A}_1 \rangle$ under $\mathcal{M}$ exists, we check for non-emptiness of the automaton $\mathbb{B}$ defining the intersection of the automata $\pi_{\Gamma_\mathcal{K}}(\mathbb{A}^{can}_\mathcal{K})$, $\pi_{\Gamma_\mathcal{K}}(\mathbb{A}^{mod}_\mathcal{K})$, and $\mathbb{A}_{fin}$, where $\mathcal{K} = \langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle$, $\pi_{\Gamma_\mathcal{K}}(\mathbb{A}^{can}_\mathcal{K})$ is the projection of $\mathbb{A}^{can}_\mathcal{K}$ on the vocabulary $\Gamma_\mathcal{K}$, and likewise for $\pi_{\Gamma_\mathcal{K}}(\mathbb{A}^{mod}_\mathcal{K})$. If the language accepted by $\mathbb{B}$ is empty, then there is no universal solution, otherwise a universal solution exists and it is exactly the tree accepted by $\mathbb{B}$.

Proposition B.17 *Let $\mathcal{M} = (\Sigma_1, \Sigma_2, \mathcal{T}_{12})$ be a DL-Lite$^{pos}_\mathcal{R}$ mapping, and $\mathcal{K}_1 = \langle \mathcal{T}_1, \mathcal{A}_1 \rangle$ a DL-Lite$^{pos}_\mathcal{R}$ KB over Σ_1. Then, a universal solution with extended ABoxes for $\mathcal{K}_1$ under $\mathcal{M}$ exists iff the language of the automata $\mathbb{B} = \pi_{\Gamma_\mathcal{K}}(\mathbb{A}^{can}_\mathcal{K}) \cap \mathbb{A}_{fin} \cap \pi_{\Gamma_\mathcal{K}}(\mathbb{A}^{mod}_\mathcal{K})$, where $\mathcal{K} = \langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle$, is non-empty.*

Proof. ($\Leftarrow$) Assume that $\mathcal{L}(\mathbb{B}) \neq \emptyset$ and $T \in \mathcal{L}(\mathbb{B})$. Let T_G be the subtree of T defined by the G labels, and $\mathcal{I}_{T,G}$ the interpretation represented by T_G. Then from the definition of $\mathbb{B}$ it follows that

1. $\mathcal{I}_{T,G}$ is a finite interpretation of Σ_2 and $\mathcal{I}_{T,G} \subseteq \mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$,
2. there exists an interpretation $\mathcal{I}$ of Σ_1 such that $\mathcal{I} \cup \mathcal{I}_{T,G}$ is a model of $\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle$.

Since $\mathcal{I}_{T,G}$ is finite, let $\mathcal{A}_{T,G}$ be the ABox over Σ_2 such that $\mathcal{U}_{\mathcal{A}_{T,G}} = \mathcal{I}_{T,G}$. Then, $\mathcal{A}_{T,G}$ is a solution for $\mathcal{K}_1$ under $\mathcal{M}$ (by the second item). We show it is a universal solution. Let $\mathcal{J}$ be an interpretation of Σ_2 such that for some model $\mathcal{I}$ of $\mathcal{K}_1$, $(\mathcal{I}, \mathcal{J}) \models \mathcal{M}$. Then, since $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$ is the canonical model of $\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle$, there exists a homomorphism from $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$ to $\mathcal{I} \cup \mathcal{J}$ ($\mathcal{I} \cup \mathcal{J}$ is a model of $\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle$). In particular, there is a homomorphism from $\mathcal{I}_{T,G}$ to $\mathcal{I} \cup \mathcal{J}$, and as $\mathcal{I}_{T,G}$ and $\mathcal{I}$ are interpretations of disjoint signatures, there is a homomorphism h from $\mathcal{I}_{T,G}$ to $\mathcal{J}$. Hence, $\mathcal{J}$ is a model of $\mathcal{A}_{T,G}$: take h as the substitution for the labeled nulls. By definition of universal solution, $\mathcal{A}_{T,G}$ is a universal solution for $\mathcal{K}_1$ under $\mathcal{M}$.

($\Rightarrow$) Assume a universal solution for $\mathcal{K}_1$ under $\mathcal{M}$ exists. Then by Lemma B.7 there exists a universal solution $\mathcal{A}_2$ such that $\mathcal{V}_{\mathcal{A}_2} \subseteq \mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$. Therefore, the language of $\mathbb{B}$ is not empty. $\square$

As a corollary of Lemma B.5, Lemma B.6, Lemma B.7, and Proposition B.17 we obtain the exponential time upper bound of the non-emptiness problem for universal solutions with extended ABoxes in *DL-Lite*$_\mathcal{R}$. Moreover, $\mathcal{A}_{T,G}$ is at most exponentially large in the size of $\mathcal{K}_1$ and $\mathcal{M}$. $\square$

$\square$

B.5 Proof of Theorem 5.6

Proof. We show that the membership problem for universal solutions with extended ABoxes is NP-complete by first proving the lower bound, and then the upper bound.

Lemma B.18 *The membership problem for universal solutions with extended ABoxes is* NP*-hard.*

Proof. The proof is by reduction of 3-colorability of undirected graphs known to be NP-hard. Suppose we are given an undirected graph $G = (V, E)$. Let $\Sigma_1 = \{Edge\}$ and $\Sigma_2 = \{Edge'\}$. Let $r, g, b \in N_a$, $V \subseteq N_l$ and

$$\begin{array}{rcl} \mathcal{A}_1 & = & \{Edge(r,g), Edge(g,r), Edge(r,b), Edge(b,r), Edge(g,b), Edge(b,g)\}, \\ \mathcal{T}_1 & = & \{\}, \\ \mathcal{T}_{12} & = & \{Edge \sqsubseteq Edge'\}, \\ \mathcal{A}_2 & = & \{Edge'(r,g), Edge'(g,r), Edge'(r,b), Edge'(b,r), Edge'(g,b), Edge'(b,g)\} \cup \\ & & \{Edge'(x,y), Edge'(y,x) \mid (x,y) \in E\}. \end{array}$$

Note that the nodes in G become labeled nulls in $\mathcal{A}_2$.

We show that G is 3-colorable if and only if $\mathcal{A}_2$ is a universal solution for $\mathcal{K}_1 = \langle \mathcal{T}_1, \mathcal{A}_1 \rangle$ under $\mathcal{M} = (\Sigma_1, \Sigma_2, \mathcal{T}_{12})$.

($\Rightarrow$) Suppose G is 3-colorable. Then it follows that there exists a function h that assigns to each vertex from V one of the colors $\{r, g, b\}$ such that if $(x, y) \in E$, then $h(x) \neq h(y)$, hence h is a homomorphism from G to the undirected graph $(\{r, g, b\}, \{(r, g), (g, b), (b, r)\})$.

We prove that $\mathcal{A}_2$ is a universal solution for $\mathcal{K}_1$ under $\mathcal{M}$. Obviously, $\mathcal{K}_1$ and $\mathcal{M}$ are Σ_2-positive. Thus, it remains to verify that $\mathcal{V}_{\mathcal{A}_2}$ is Σ_2-homomorphically equivalent to $\mathcal{U}_{\langle \mathcal{T}_1 \langle \mathcal{T} \rangle_{12}, \mathcal{A}_1 \rangle}$. First, it is easy to see that $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$ is Σ_2-homomorphically embeddable into $\mathcal{V}_{\mathcal{A}_2}$. Second, h is also a homomorphism from $\mathcal{V}_{\mathcal{A}_2}$ to $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$, thus $\mathcal{V}_{\mathcal{A}_2}$ is homomorphically embeddable into $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$.

($\Leftarrow$) Suppose now $\mathcal{A}_2$ is a universal solution for $\mathcal{K}_1$ under $\mathcal{M}$. Then by Lemma B.3 it follows that $\mathcal{V}_{\mathcal{A}_2}$ is Σ_2-homomorphically equivalent to $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$. Let h be a homomorphism from $\mathcal{V}_{\mathcal{A}_2}$ to $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$. Then h assigns to each labeled null $x \in \Delta^{\mathcal{A}_2}$ some constant $a \in \Delta^{\mathcal{A}_1}$, and it is easy to see that h is an assignment for the vertices in V that is a 3-coloring of G. $\square$

Lemma B.19 *The membership problem for universal solutions with extended ABoxes is in* NP.

Proof. Assume we are given a mapping $\mathcal{M} = (\Sigma_1, \Sigma_2, \mathcal{T}_{12})$, a source KB $\mathcal{K}_1 = \langle \mathcal{T}_1, \mathcal{A}_1 \rangle$, and a target ABox $\mathcal{A}_2$. We want to decide whether $\mathcal{A}_2$ is a universal solution with extended ABoxes for $\mathcal{K}_1$ under $\mathcal{M}$ (ABoxes without inequalities).

We need the following proposition that provides an upper bound for checking existence of homomorphism from $\mathcal{V}_{\mathcal{A}_2}$ to $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$.

Proposition B.20 *Deciding whether $\mathcal{V}_{\mathcal{A}_2}$ is homomorphically embeddable into $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$ can be done in* NP *in the size of $\mathcal{K}_1$, $\mathcal{M}$ and $\mathcal{A}_2$.*

Proof. First, if there exists a homomorphism h from $\mathcal{V}_{\mathcal{A}_2}$ to $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$, then there exists a polynomial size witness $\mathcal{A}_3$ such that $\mathcal{V}_{\mathcal{A}_3} \subseteq \mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$ and h is a homomorphism from $\mathcal{V}_{\mathcal{A}_2}$ to $\mathcal{V}_{\mathcal{A}_3}$ (take $\mathcal{V}_{\mathcal{A}_3} = h(\mathcal{V}_{\mathcal{A}_2})$, then $|\mathcal{A}_3| \leq |\mathcal{A}_2|$). Therefore, to verify that such h exists, it is sufficient to compute $\mathcal{A}_3$ and then to check whether $\mathcal{V}_{\mathcal{A}_2}$ can be homomorphically mapped into $\mathcal{V}_{\mathcal{A}_3}$.

Second, there exists a witness $\mathcal{A}_3$ such that $\mathcal{V}_{\mathcal{A}_3} \subseteq \mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$ and every $x \in \Delta^{\mathcal{A}_3}$ is a path of polynomial length in the size of $\mathcal{T}_1 \cup \mathcal{T}_{12}$ and $\mathcal{A}_2$ (more precisely, of length smaller or equal $2m$, where m is the size of $\mathcal{T}_1 \cup \mathcal{T}_{12} \cup \mathcal{A}_2$). Proof: let h be a homomorphism from $\mathcal{V}_{\mathcal{A}_2}$ to $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$ and $\mathcal{A}_3$ an ABox such that $\mathcal{V}_{\mathcal{A}_3} = h(\mathcal{V}_{\mathcal{A}_2})$. Assume that $x \in \Delta^{\mathcal{A}_3}$ and the length of x is more than $2m$. Then x is not connected to $\mathsf{Ind}(\mathcal{A}_1)$ in $\mathcal{A}_3$, i.e., there exists no path $R_1(x_1, x_2), \dots, R_n(x_n, x_{n+1})$ with $x_1 = x$, $x_{n+1} = a \in \mathsf{Ind}(\mathcal{A}_1)$, $R_i(x_i, x_{i+1}) \in \mathcal{A}_3$ (otherwise it contradicts $\mathcal{V}_{\mathcal{A}_3} = h(\mathcal{V}_{\mathcal{A}_2})$). Let C be the maximal connected subset of $\mathcal{A}_3$ with $x \in \Delta^C$, i.e., $\Delta^C \cap \Delta^{\mathcal{A}_3 \setminus C} = \emptyset$ and for each $C' \subsetneq C$, $\Delta^{C'} \cap \Delta^{C \setminus C'} \neq \emptyset$, moreover $\Delta^C \cap \mathsf{Ind}(\mathcal{A}_1) = \emptyset$. Let y be the path (in the sense of $\mathsf{path}(\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle)$) of minimal length in C, it exists and is unique since $\mathcal{V}_{\mathcal{A}_3} \subseteq \mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$ and there are no constants in C, and for each $x \in C$, $x = y \cdot w_{[R_1]} \dots w_{[R_n]}$ for some n. Further assume $\mathsf{tail}(y) = w_{[R]}$, then let y' be a path of the minimal length in $\Delta^{\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}}$ with $\mathsf{tail}(y') = w_{[R]}$ (note that there is an infinite number of y'' with $\mathsf{tail}(y'') = w_{[R]}$). Then the length of y' is bounded by the size of $\mathcal{T}_1 \cup \mathcal{T}_{12}$ and the length of each $y' \cdot w_{[R_1]} \dots w_{[R_n]}$, where $y \cdot w_{[R_1]} \dots w_{[R_n]} \in C$, is bounded by the size of $\mathcal{T}_1 \cup \mathcal{T}_{12} \cup \mathcal{A}_2$. Now, define a new function $h' : \Delta^{\mathcal{V}_{\mathcal{A}_2}} \to \Delta^{\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}}$ such that $h'(x) = h(x)$ if $h(x) \notin C$, $h'(x) = y' \cdot w_{[R_1]} \dots w_{[R_n]}$ if $h(x) = y \cdot w_{[R_1]} \dots w_{[R_n]}$. It is easy to see that h' is a homomorphism from $\mathcal{V}_{\mathcal{A}_2}$ to $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$. We can continue this iteratively until we get that for every $x \in \Delta^{\mathcal{A}_3}$, x is a path of length bounded by $2m$, where $\mathcal{A}_3$ is an ABox such that $\mathcal{V}_{\mathcal{A}_3} = h'(\mathcal{V}_{\mathcal{A}_2})$.

Finally, our algorithm for checking existence of a homomorphism from $\mathcal{V}_{\mathcal{A}_2}$ to $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$ is as follows:

1. compute (guess) $\mathcal{A}_3$ (in NP):
   - for each $x \in \Delta^{\mathcal{A}_2}$ we guess $y \in \Delta^{\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}}$ such that there exists a $\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle$-path from some $a \in \mathsf{Ind}(\mathcal{A}_1)$ to y and y is a path of polynomial length,
   - Let W be the set of all y guessed above, then

$$\begin{array}{rcl} \mathcal{A}_3 & = & \{A(x) \mid x \in W, \mathsf{tail}(x) = w_{[R]}, \mathcal{T}_1 \cup \mathcal{T}_{12} \models \exists R^- \sqsubseteq A, A \in \Sigma_2\} \cup \\ & & \{S(x', x) \mid x, x' \in W, x = x' \cdot w_{[R]}, \mathcal{T}_1 \cup \mathcal{T}_{12} \models R \sqsubseteq S, S \in \Sigma_2\}, \end{array}$$

   $\mathcal{V}_{\mathcal{A}_3} \subseteq \mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$, $\Delta^{\mathcal{A}_3} = W$ and $\mathcal{A}_3$ is of polynomial size.
2. check whether there exists a homomorphism from $\mathcal{V}_{\mathcal{A}_2}$ to $\mathcal{V}_{\mathcal{A}_3}$ (in NP).

We prove that the above described procedure is correct.

Assume, we computed $\mathcal{A}_3$ and there exists a homomorphism h from $V_{\mathcal{A}_2}$ to $V_{\mathcal{A}_3}$. Then since $\mathcal{V}_{\mathcal{A}_3} \subseteq \mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$, it follows that h is a homomorphism from $\mathcal{V}_{\mathcal{A}_2}$ to $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$.

Now, assume that there exists no homomorphism from $V_{\mathcal{A}_2}$ to $V_{\mathcal{A}_3}$, and by contradiction there exists a homomorphism from $\mathcal{V}_{\mathcal{A}_2}$ to $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$. Then, we showed that there exists a homomorphism h' from $\mathcal{V}_{\mathcal{A}_2}$ to $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$ and an ABox $\mathcal{A}_3$ such that $\mathcal{V}_{\mathcal{A}_3} = h'(\mathcal{V}_{\mathcal{A}_2})$ and the length of every $x \Delta^{\mathcal{A}_3}$ is bounded by $2m$, where m is the size of $\mathcal{T}_1 \cup \mathcal{T}_{12} \cup \mathcal{A}_2$. Contradiction with step 1.

□

Then the membership check for universal solutions with extended ABoxes can be done as follows:

1. verify whether $\mathcal{K}_1$ and $\mathcal{M}$ are Σ_2-positive, if yes
2. check whether $\mathcal{T}_2$ is equivalent to the empty TBox, if yes
3. check whether $\mathcal{A}_2$ is a solution with extended ABoxes for $\mathcal{K}_1^{pos}$ under $\mathcal{M}^{pos}$, if yes
4. check whether $\mathcal{A}_2$ is homomorphically embeddable into $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$. If yes, then $\mathcal{K}_2$ is a universal solution for $\mathcal{K}_1$ under $\mathcal{M}$, otherwise it is not.

Steps 1 and 2 can be done in polynomial time. Step 3 can be done in NP similarly to Theorem 5.3: guess an interpretation $\mathcal{I}$ of Σ_1 of polynomial size, check whether $\mathcal{I}$ is a model of $\mathcal{K}_1^{pos}$ and $(\mathcal{I}, \mathcal{V}_{\mathcal{A}_2}) \models \mathcal{T}_{12}^{pos}$. If yes, then $\mathcal{A}_2$ is a solution: let $\mathcal{J}$ be a model of $\mathcal{A}_2$ and h a homomorphism from $\mathcal{V}_{\mathcal{A}_2}$ to $\mathcal{J}$. Then, let $\mathcal{I}^{\mathcal{J}}$ be the image of h applied to $\mathcal{I}$, $\mathcal{I}^{\mathcal{J}} = h(\mathcal{I})$. Then $\mathcal{I}^{\mathcal{J}}$ is a model of $\mathcal{K}_1^{pos}$ and $(\mathcal{I}^{\mathcal{J}}, \mathcal{J}) \models \mathcal{T}_{12}^{pos}$, hence indeed, $\mathcal{A}_2$ is a solution. Step 4 is feasible in NP, therefore in overall the membership check can be done in NP. □

□

B.6 Proof of Theorem 5.4

Proof. The proof is by reduction of the satisfiability problem for quantified Boolean formulas, known to be PSPACE-complete. Suppose we are given a QBF

$$\phi = \mathsf{Q}_1 X_1 \dots \mathsf{Q}_n X_n \bigwedge_{j=1}^{m} C_j$$

where $\mathsf{Q}_i \in \{\forall, \exists\}$ and C_j, $1 \le j \le m$, are clauses over the variables X_i, $1 \le i \le n$.

Let $\Sigma_1 = \{A, Y_i^k, X_i^k, S_l, T_l, Q_i^k, P_i^k, R_j, R_j^l \mid 1 \le j \le m, 1 \le i \le n, 0 \le l \le n, k \in \{0,1\}\}$ where A, Y_i^k, X_i^k are concept names and the rest are role names. Let $\mathcal{T}_1$ be the following TBox over Σ_1 for $1 \le j \le m$, $1 \le i \le n$ and $k \in \{0,1\}$:

$$\begin{array}{rclrcllrcl}
A & \sqsubseteq & \exists S_0^- & \exists S_{i-1}^- & \sqsubseteq & \exists Q_i^k & \text{if } \mathsf{Q}_i = \forall & & & \\
 & & & \exists S_{i-1}^- & \sqsubseteq & \exists S_i & \text{if } \mathsf{Q}_i = \exists & & & \\
\exists (Q_i^k)^- & \sqsubseteq & Y_i^k & Q_i^k & \sqsubseteq & S_i & & \exists S_n^- & \sqsubseteq & \exists R_j \\
\exists R_j^- & \sqsubseteq & \exists R_j & & & & & & & \\
 & & & & & & & & & \\
A & \sqsubseteq & \exists T_0^- & \exists T_{i-1}^- & \sqsubseteq & \exists P_i^k & & P_i^k & \sqsubseteq & T_i \\
\exists (P_i^k)^- & \sqsubseteq & X_i^k & X_i^0 & \sqsubseteq & \exists R_j^i & \text{if } \neg X_i \in C_j & & & \\
 & & & X_i^1 & \sqsubseteq & \exists R_j^i & \text{if } X_i \in C_j & & & \\
\exists (R_j^i)^- & \sqsubseteq & \exists R_j^{i-1} & & & & & & &
\end{array}$$

and $\mathcal{A}_1 = \{A(a)\}$.

Further, let $\Sigma_2 = \{A', Z_i^0, Z_i^1, S', R'_j, {P'}_i^k, T'_l, {R'}_j^l\}$ where A', Z_i^0, Z_i^1 are concept names and $S', R'_j, {P'}_i^k, T'_l, {R'}_j^l$ are role names, $\mathcal{M} = (\Sigma_1, \Sigma_2, \mathcal{T}_{12})$, and $\mathcal{T}_{12}$ the following set of inclusions:

$$\begin{array}{rclrclrcl}
A & \sqsubseteq & A' & S_i & \sqsubseteq & S' & R_j & \sqsubseteq & R'_j \\
 & & & T_i & \sqsubseteq & S' & T_i & \sqsubseteq & {R'_j}^- \\
 & & & Y_i^k & \sqsubseteq & Z_i^k & R_j^i & \sqsubseteq & R'_j \\
 & & & X_i^k & \sqsubseteq & Z_i^k & R_j^0 & \sqsubseteq & {R'_j}^- \\
 & & & & & & & & \\
P_i^k & \sqsubseteq & {P'}_i^k & T_l & \sqsubseteq & T'_l & R_j^l & \sqsubseteq & {R'}_j^l
\end{array}$$

Finally, let $\mathcal{A}_2 = \{A'(a)\}$, and $\mathcal{T}_2$ the following target TBox for $1 \le j \le m$, $1 \le i \le n$ and $k \in \{0,1\}$:

$$\begin{array}{rclrcllrcl}
A' & \sqsubseteq & \exists {T'_0}^- & \exists {T'_{i-1}}^- & \sqsubseteq & \exists {P'}_i^k & & {P'}_i^k & \sqsubseteq & T'_i \\
\exists ({P'}_i^k)^- & \sqsubseteq & Z_i^k & Z_i^0 & \sqsubseteq & \exists {R'}_j^i & \text{if } \neg X_i \in C_j & & & \\
 & & & Z_i^1 & \sqsubseteq & \exists {R'}_j^i & \text{if } X_i \in C_j & & & \\
\exists ({R'}_j^i)^- & \sqsubseteq & \exists {R'}_j^{i-1} & & & & & & & \\
 & & & T'_i & \sqsubseteq & S' & & T'_i & \sqsubseteq & {R'_j}^- \\
 & & & & & & & {R'}_j^i & \sqsubseteq & R'_j \\
 & & & & & & & {R'}_j^0 & \sqsubseteq & {R'_j}^-
\end{array}$$

We verify that $\models \phi$ if and only if $\langle \mathcal{T}_2, \mathcal{A}_2 \rangle$ is a universal UCQ-solution for $\mathcal{K}_1 = \langle \mathcal{T}_1, \mathcal{A}_1 \rangle$ under $\mathcal{M}$. From Claim A.4 it follows that $\langle \mathcal{T}_2, \mathcal{A}_2 \rangle$ is a universal UCQ-solution for $\mathcal{K}_1 = \langle \mathcal{T}_1, \mathcal{A}_1 \rangle$ under $\mathcal{M}$ iff $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$ is finitely Σ_2-homomorphically equivalent to $\mathcal{U}_{\langle \mathcal{T}_2, \mathcal{A}_2 \rangle}$. Therefore, we are going to show that $\models \phi$ if and only if $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}_1 \rangle}$ is finitely Σ_2-homomorphically equivalent to $\mathcal{U}_{\langle \mathcal{T}_2, \mathcal{A}_2 \rangle}$.

The rest of the proof is similar to Lemma B.15.

□

C Membership Problem for UCQ-representability

Note that for the ease of notation, in all proofs and statements concerning UCQ-representability we use Σ instead of Σ_1 and Ξ instead of Σ_2. At the same time, alternative syntax for the disjointness assertions is used: we write $B \sqcap B' \sqsubseteq \bot$ instead of $B \sqsubseteq \neg B'$, for basic concepts B and B'; analogiousy for roles.

We need several new definitions. For a TBox $\mathcal{T}$, a pair of basic concepts B, B' (resp., pair of roles R, R') is *$\mathcal{T}$-consistent* if $\langle \mathcal{T}, \{B(o), B'(o)\} \rangle$ (resp., $\langle \mathcal{T}, \{R(o,o'), R'(o,o')\} \rangle$) is a consistent KB. We say a concept B is *$\mathcal{T}$-consistent* if the pair B, B is $\mathcal{T}$-consistent, and we define in a similar way $\mathcal{T}$-consistency of a role R. Denote by $cons_{\mathcal{C}}(\mathcal{T})$ ($cons_{\mathcal{R}}(\mathcal{T})$) the set of all $\mathcal{T}$-consistent concepts (roles).

C.1 Basic Preliminary Results

Lemma C.1 *Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ be a KB, $a, b \in N_a$, $\sigma \in \Delta^{\mathcal{U}_\mathcal{K}}$, and $\mathsf{tail}(\sigma) \rightsquigarrow_\mathcal{K} w_{[R]}$. Then,*

- ***(i)*** $B \in \mathbf{t}^{\mathcal{U}_\mathcal{K}}(a)$ *iff* $\mathcal{A} \models B'(a)$ *and* $\mathcal{T} \vdash B' \sqsubseteq B$*;*
- ***(ii)*** $R \in \mathbf{r}^{\mathcal{U}_\mathcal{K}}(a, b)$ *iff* $\mathcal{A} \models R'(a, b)$ *and* $\mathcal{T} \vdash R' \sqsubseteq R$*;*
- ***(iii)*** $B \in \mathbf{t}^{\mathcal{U}_\mathcal{K}}(\sigma w_{[R]})$ *iff* $\mathcal{T} \vdash \exists R^- \sqsubseteq B$*;*
- ***(iv)*** $R \in \mathbf{r}^{\mathcal{U}_\mathcal{K}}(\sigma, \sigma w_{[R']})$ *iff* $\mathcal{T} \vdash R' \sqsubseteq R$*.*

Proof. For **(i)** assume, first, B is a concept name, then the proof straightforwardly follows from the definition of $\mathcal{U}_\mathcal{K}$. Let $B = \exists R$ for a role R, we show the "only if" direction. By the definition of $\mathcal{U}_\mathcal{K}$ it follows either $a \rightsquigarrow_\mathcal{K} w_{[R']}$ for some role R' such that $\mathcal{T} \vdash R' \sqsubseteq R$ or $\mathcal{K} \vdash R(a, b)$ for some $b \in N_a$. In the first case $\mathcal{K} \vdash \exists R'(a)$ and $\mathcal{T} \vdash \exists R' \sqsubseteq B$ by the definition of $\rightsquigarrow$. It is then immediate that $\mathcal{A} \models B'(a)$ and $\mathcal{T} \vdash B' \sqsubseteq B$ for some concept B'. In the second case, there is a role R'' such that $\mathcal{A} \models R''(a, b)$ and $\mathcal{T} \vdash R'' \sqsubseteq R$, so the result follows with $B' = \exists R''$. The "if" direction is similar using the definition of $\mathcal{U}_\mathcal{K}$ and $\rightsquigarrow$, which concludes the proof of **(i)**. The proof of **(ii)** is analogious.

For **(iii)** assume, first, B is a concept name, then the proof straightforwardly follows from the definition of $\mathcal{U}_\mathcal{K}$. Let $B = \exists S$ for a role S, we, first, show the "only if" direction. It follows there exists $\sigma' \in \Delta^{\mathcal{U}_\mathcal{K}}$ such that $(\sigma w_{[R]}, \sigma') \in S^{\mathcal{U}_\mathcal{K}}$. From the definition of $\mathcal{U}_\mathcal{K}$ it should be clear that either $\delta' = \delta$ and $\mathcal{T} \vdash R \sqsubseteq S^-$, or $\sigma' = \sigma w_{[R]} w_{[R']}$ for a role R' such that $w_{[R]} \rightsquigarrow_\mathcal{K} w_{[R']}$ and $\mathcal{T} \vdash R' \sqsubseteq S$. Then, from $w_{[R]} \rightsquigarrow_\mathcal{K} w_{[R']}$ we can also conclude $\mathcal{T} \vdash \exists R^- \sqsubseteq \exists R'$. One can see that in the both cases above it follows $\mathcal{T} \vdash \exists R^- \sqsubseteq \exists S$, which concludes the proof of the "only if" direction. The "if" direction is similar using the definition of $\mathcal{U}_\mathcal{K}$ and $\rightsquigarrow$. □

Lemma C.2 *Let $\langle \mathcal{T}, \mathcal{A} \rangle$ and $\langle \mathcal{T}', \mathcal{A}' \rangle$ be the KBs, such that:*

- ***(i)*** $\mathcal{T} \subseteq \mathcal{T}'$*,*
- ***(ii)*** $\mathcal{A} \models B(a)$ *implies* $\mathcal{A}' \models B(a)$ *and* $\mathcal{A} \models R(a, b)$ *implies* $\mathcal{A}' \models R(a, b)$*, for all $a, b \in N_a$, concepts B and roles R.*

Then, for each $\sigma \in \Delta^{\mathcal{U}_{\langle \mathcal{T}, \mathcal{A} \rangle}}$ there exists $\delta \in \Delta^{\mathcal{U}_{\langle \mathcal{T}', \mathcal{A}' \rangle}}$ such that

- ***(iii)*** $\mathbf{t}^{\mathcal{U}_{\langle \mathcal{T}, \mathcal{A} \rangle}}(\sigma) \subseteq \mathbf{t}^{\mathcal{U}_{\langle \mathcal{T}', \mathcal{A} \rangle}}(\delta)$*,*
- ***(iv)*** $\mathbf{r}^{\mathcal{U}_{\langle \mathcal{T}, \mathcal{A} \rangle}}(a, \sigma) \subseteq \mathbf{r}^{\mathcal{U}_{\langle \mathcal{T}', \mathcal{A} \rangle}}(a, \delta)$ *for all $a \in N_a$.*

Proof. Consider, first, the case $\sigma = b \in N_a$, then set $\delta = b$ and we show **(iii)**. Consider $B \in \mathbf{t}^{\mathcal{U}_{\langle \mathcal{T}, \mathcal{A} \rangle}}(\sigma)$, it follows by Lemma C.1 $\mathcal{A} \models B'(b)$ and $\mathcal{T} \vdash B' \sqsubseteq B$, for some concept B'. Then, by **(i)** it follows $\mathcal{T}' \vdash B' \sqsubseteq B$ and by **(ii)** it follows $\mathcal{A}' \models B'(b)$, therefore, by Lemma C.1 we obtain $B \in \mathbf{t}^{\mathcal{U}_{\langle \mathcal{T}', \mathcal{A} \rangle}}(\delta)$. The proof for **(iv)** is analogious.

Now, assume the lemma holds for $\sigma' \in \Delta^{\mathcal{U}_{\langle \mathcal{T}, \mathcal{A} \rangle}}$; we show it also holds for $\sigma = \sigma' w_{[R]} \in \Delta^{\mathcal{U}_{\langle \mathcal{T}, \mathcal{A} \rangle}}$ for a role R. By the definition of $\mathcal{U}_{\langle \mathcal{T}, \mathcal{A} \rangle}$ it follows $\mathsf{tail}(\sigma') \rightsquigarrow_{\langle \mathcal{T}, \mathcal{A} \rangle} w_{[R]}$ and so $\exists R \in \mathbf{t}^{\mathcal{U}_{\langle \mathcal{T}, \mathcal{A} \rangle}}(\sigma')$. By Lemma C.1 it follows

$$\mathcal{T} \vdash \exists R^- \sqsubseteq B \text{ for each } B \in \mathbf{t}^{\mathcal{U}_{\langle \mathcal{T}, \mathcal{A} \rangle}}(\sigma) \tag{1}$$

$$\mathcal{T} \vdash R \sqsubseteq Q \text{ for each } Q \in \mathbf{t}^{\mathcal{U}_{\langle \mathcal{T}, \mathcal{A} \rangle}}(a, \sigma) \tag{2}$$

On the other hand, observe by our induction hypothesis that there exists $\delta' \in \Delta^{\langle \mathcal{T}', \mathcal{A}' \rangle}$ such that $\mathbf{t}^{\mathcal{U}_{\langle \mathcal{T}, \mathcal{A} \rangle}}(\sigma') \subseteq \mathbf{t}^{\mathcal{U}_{\langle \mathcal{T}', \mathcal{A}' \rangle}}(\delta')$; therefore, $\exists R \in \mathbf{t}^{\mathcal{U}_{\langle \mathcal{T}', \mathcal{A}' \rangle}}(\delta')$. It follows there exists $\delta'' \in \Delta^{\mathcal{U}_{\langle \mathcal{T} \cup \mathcal{T}', \mathcal{A} \rangle}}$ such that $(\delta', \delta'') \in R^{\mathcal{U}_{\langle \mathcal{T}', \mathcal{A}' \rangle}}$. We select δ (for σ) equal to δ''; using (1), **(i)** and Lemma C.1 one can easily show **(iii)**, and using (2) and **(i)** one can show **(iv)**. □

Lemma C.3 *Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ and assume $a \rightsquigarrow_\mathcal{K} w_{[R]}$ for some basic role R. Then there exists a basic concept B, such that $\mathcal{A} \models B(a)$, and:*

- ***(i)*** $o \rightsquigarrow_{\langle \mathcal{T}, B(o) \rangle} w_{[R]}$*;*
- ***(ii)*** $\mathbf{t}^{\mathcal{U}_\mathcal{K}}(a w_{[R]}) \subseteq \mathbf{t}^{\mathcal{U}_{\langle \mathcal{T}, B(o) \rangle}}(o w_{[R]})$*;*
- ***(iii)*** $\mathbf{r}^{\mathcal{U}_\mathcal{K}}(a, a w_{[R]}) \subseteq \mathbf{r}^{\mathcal{U}_{\langle \mathcal{T}, B(o) \rangle}}(o, o w_{[R]})$*.*

Proof. Consequence of Lemma C.1. □

Lemma C.4 *Let $\mathcal{A}$ be an ABox, $\mathbf{B}$ a set of basic concepts, and $\mathcal{T}, \mathcal{T}'$ TBoxes. Let $\mathcal{B} = \langle \mathcal{T}, \{B(o) \mid B \in \mathbf{B}\}\rangle$, and assume $y \in \Delta^{\mathcal{U}_{\mathcal{B}}}$. If $\sigma \in \Delta^{\mathcal{U}_{\langle \mathcal{T}\cup\mathcal{T}',\mathcal{A}\rangle}}$ and $\mathbf{B} \subseteq \mathbf{t}^{\mathcal{U}_{\langle \mathcal{T}\cup\mathcal{T}',\mathcal{A}\rangle}}(\sigma)$, then there exists $\delta \in \Delta^{\mathcal{U}_{\langle \mathcal{T}\cup\mathcal{T}',\mathcal{A}\rangle}}$ such that*

(i) $\mathbf{t}^{\mathcal{U}_{\mathcal{B}}}(y) \subseteq \mathbf{t}^{\mathcal{U}_{\langle \mathcal{T}\cup\mathcal{T}',\mathcal{A}\rangle}}(\delta)$

(ii) $\mathbf{r}^{\mathcal{U}_{\mathcal{B}}}(o, y) \subseteq \mathbf{r}^{\mathcal{U}_{\langle \mathcal{T}\cup\mathcal{T}',\mathcal{A}\rangle}}(\sigma, \delta)$

Proof. Straightforward consequence of Lemma C.2. □

Lemma C.5 *For each $\sigma \in \Delta^{\mathcal{U}_{\langle \mathcal{T}_1\cup\mathcal{T}_{12},\mathcal{A}\rangle}}$ and $B' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\langle \mathcal{T}_1\cup\mathcal{T}_{12},\mathcal{A}\rangle}}(\sigma)$ one of the following holds:*

(i) *there exists a concept B over Σ such that $B \in \mathbf{t}^{\mathcal{U}_{\langle \mathcal{T}_1\cup\mathcal{T}_{12},\mathcal{A}\rangle}}(\sigma)$ and $\mathcal{T}_{12} \vdash B \sqsubseteq B'$;*

(ii) $\mathbf{t}^{\mathcal{U}_{\langle \mathcal{T}_1\cup\mathcal{T}_{12},\mathcal{A}\rangle}}(\sigma) = \{B'\}$.

Proof. Using Lemma C.1, and considering the structure of $\mathcal{T}_1 \cup \mathcal{T}_{12}$ with $\Sigma \cap \Xi = \emptyset$. The case **(ii)** occurs, when $\mathsf{tail}(\sigma) = w_{[Q]}$ for a role Q is over Ξ. □

Lemma C.6 *A DL-Lite$_{\mathcal{R}}$ KB $\langle \mathcal{T}, \mathcal{A}\rangle$ is consistent iff*

(i) *B, B' is $\mathcal{T}$-consistent for each pair of basic concepts B, B' and each $a \in \mathsf{Ind}(\mathcal{A})$ such that $\mathcal{A} \models B(a)$ and $\mathcal{A} \models B'(a)$;*

(ii) *R, R' is $\mathcal{T}$-consistent for each pair of roles R, R' and each $a, b \in \mathsf{Ind}(\mathcal{A})$ such that $\mathcal{A} \models R(a, b)$ and $\mathcal{A} \models R'(a, b)$*

Proof. ($\Rightarrow$) Assume **(i)** is violated, so there exist B_1, B_2 and $a \in \mathsf{Ind}(\mathcal{A})$ such that $\mathcal{A} \models B_1(a)$, $\mathcal{A} \models B_2(a)$, and $\langle \mathcal{T}, \{B_1(o), B_2(o)\}\rangle$ is inconsistent. It follows that $\mathcal{U}_{\langle \mathcal{T}, \{B_1(o), B_2(o)\}\rangle}$ is not a model of $\langle \mathcal{T}, \{B_1(o), B_2(o)\}\rangle$, so there is $\delta \in \Delta^{\mathcal{U}_{\langle \mathcal{T}, \{B_1(o), B_2(o)\}\rangle}}$ and a disjointness assertion $B \sqcap C \sqsubseteq \bot \in \mathcal{T}$ (note that inclusion assertions $B \sqsubseteq C \in \mathcal{T}$ cannot cause inconsistency) such that $B, C \in \mathbf{t}^{\mathcal{U}_{\langle \mathcal{T}, \{B_1(o), B_2(o)\}\rangle}}(\delta)$. Obviously, $\{B_1, B_2\} \subseteq \mathbf{t}^{\mathcal{U}_{\langle \mathcal{T},\mathcal{A}\rangle}}(a)$, then by Lemma C.4 we obtain $\delta \in \Delta^{\mathcal{U}_{\langle \mathcal{T},\mathcal{A}\rangle}}$ such that $B, C \in \mathbf{t}^{\mathcal{U}_{\langle \mathcal{T},\mathcal{A}\rangle}}(\delta)$. Hence, $\mathcal{U}_{\langle \mathcal{T},\mathcal{A}\rangle}$ is not a model of $\langle \mathcal{T}, \mathcal{A}\rangle$, which contradicts Claim A.1 since $\langle \mathcal{T}, \mathcal{A}\rangle$ is consistent.

It can be also the case that $\mathcal{U}_{\langle \mathcal{T}, \{B_1(o), B_2(o)\}\rangle}$ is inconsistent due to the disjointness assertion $R \sqcap Q \sqsubseteq \bot \in \mathcal{T}$. Then the proof is similar using Lemmas C.4 and Claim A.1.

Assume now **(ii)** is violated, the proof is a straightforward modification of the proof above.

($\Leftarrow$) The proof is analogous to ($\Rightarrow$). □

Lemma C.7 *If a KB $\langle \mathcal{T}, \mathcal{A}\rangle$ is consistent, then for all $\delta, \sigma \in \Delta^{\mathcal{U}_{\langle \mathcal{T},\mathcal{A}\rangle}}$,*

(i) *B is $\mathcal{T}$-consistent for each $B \in \mathbf{t}^{\mathcal{U}_{\langle \mathcal{T},\mathcal{A}\rangle}}(\delta)$;*

(ii) *R is $\mathcal{T}$-consistent for each $R \in \mathbf{r}^{\mathcal{U}_{\langle \mathcal{T},\mathcal{A}\rangle}}(\delta, \sigma)$.*

Proof. Similar to Lemma C.6. □

C.2 Homomorphism Lemmas

Here we present a series of important lemmas used in the proof the main results in the following sections.

Lemma C.8 *Assume a mapping $\mathcal{M} = (\Sigma, \Xi, \mathcal{T}_{12})$, ABoxes $\mathcal{A}$ and $\mathcal{A}'$ over, respectively, Σ and Ξ, and a Ξ-TBox over $\mathcal{T}_2$. If $\mathcal{U}_{\langle \mathcal{T}_{12},\mathcal{A}\rangle}$ is Ξ homomorphically embeddable into $\mathcal{U}_{\mathcal{A}'}$, then $\mathcal{U}_{\langle \mathcal{T}_2\cup\mathcal{T}_{12},\mathcal{A}\rangle}$ is Ξ homomorphically embeddable into $\mathcal{U}_{\langle \mathcal{T}_2,\mathcal{A}'\rangle}$.*

Proof. Consider the Ξ homomorphism $h : \Delta^{\mathcal{U}_{\langle \mathcal{T}_{12},\mathcal{A}\rangle}} \mapsto \Delta^{\mathcal{U}_{\mathcal{A}'}}$ from $\mathcal{U}_{\langle \mathcal{T}_{12},\mathcal{A}\rangle}$ to $\mathcal{U}_{\mathcal{A}'}$, we are going to construct the Σ homomorphism $h' : \mathcal{U}_{\langle \mathcal{T}_2\cup\mathcal{T}_{12},\mathcal{A}\rangle} \mapsto \mathcal{U}_{\langle \mathcal{T}_2,\mathcal{A}'\rangle}$ from $\mathcal{U}_{\langle \mathcal{T}_2\cup\mathcal{T}_{12},\mathcal{A}\rangle}$ to $\mathcal{U}_{\langle \mathcal{T}_2,\mathcal{A}'\rangle}$. Initially, we define $h'(a) = a$, let us immediately verify that $\mathbf{t}_{\Xi}^{\mathcal{U}_{\langle \mathcal{T}_2\cup\mathcal{T}_{12},\mathcal{A}\rangle}}(a) \subseteq \mathbf{t}_{\Xi}^{\mathcal{U}_{\langle \mathcal{T}_2,\mathcal{A}'\rangle}}(h'(a))$. Notice that by the definition of h we have:

$$\mathbf{t}_{\Xi}^{\mathcal{U}_{\langle \mathcal{T}_{12},\mathcal{A}\rangle}}(a) \subseteq \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{A}'}}(h(a)), \tag{3}$$

$$h(a) = h'(a). \tag{4}$$

Let $C \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\langle \mathcal{T}_2\cup\mathcal{T}_{12},\mathcal{A}\rangle}}(a)$, it follows by Lemma C.1 **(i)** there exists B over Σ, such that $\mathcal{A} \models B(a)$ and $\mathcal{T} \cup \mathcal{T}' \vdash B \sqsubseteq C$. Taking into account the shape of $\mathcal{T}_2$ and $\mathcal{T}_{12}$, t follows also there exists D over Ξ such that $\mathcal{T}_{12} \vdash B \sqsubseteq D$ and $\mathcal{T}_2 \vdash D \sqsubseteq C$. Observe that $B \in \mathbf{t}^{\mathcal{U}_{\langle \mathcal{T}_{12},\mathcal{A}\rangle}}(a)$, then by Lemma C.1 **(i)** and **(iii)** it follows $D \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\langle \mathcal{T}_{12},\mathcal{A}\rangle}}(a)$ and taking into account (3) and (4) we conclude $D \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{A}'}}(h'(a))$. Finally, using again Lemma C.1 **(i)** and **(iii)** we obtain $C \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\langle \mathcal{T}_2,\mathcal{A}'\rangle}}(h'(a))$. The proof that $\mathbf{r}_{\Xi}^{\mathcal{U}_{\langle \mathcal{T}_2\cup\mathcal{T}_{12},\mathcal{A}\rangle}}(a,b) \subseteq \mathbf{r}_{\Xi}^{\mathcal{U}_{\langle \mathcal{T}_2,\mathcal{A}'\rangle}}(h'(a),h'(b))$ for all constants a and b is analogious.

Now we show how to define h' for $\sigma = aw_{[R]} \in \mathsf{path}(\langle \mathcal{T}_2 \cup \mathcal{T}_{12}, \mathcal{A}\rangle)$. It follows $a \rightsquigarrow_{\langle \mathcal{T}_2\cup\mathcal{T}_{12},\mathcal{A}\rangle} w_{[R]}$, then two cases are possible:

(I) R is over Σ;

(II) R is over Ξ.

In case **(I)** it follows $a \rightsquigarrow_{\langle \mathcal{T}_{12},\mathcal{A}\rangle} w_{[R]}$ and by the condition of the current lemma it follows there is $\delta \in \Delta^{\mathcal{U}_{\mathcal{A}'}}$ such that:

$$\mathbf{t}_{\Xi}^{\mathcal{U}_{\langle \mathcal{T}_{12},\mathcal{A}\rangle}}(aw_{[R]}) \subseteq \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{A}'}}(\delta), \tag{5}$$

$$\mathbf{r}_{\Xi}^{\mathcal{U}_{\langle \mathcal{T}_{12},\mathcal{A}\rangle}}(a, aw_{[R]}) \subseteq \mathbf{r}_{\Xi}^{\mathcal{U}_{\mathcal{A}'}}(a,\delta). \tag{6}$$

Then, using Lemma C.2 (with $\mathcal{A}' = \mathcal{A}$, $\mathcal{T} = \emptyset$, $\mathcal{T}' = \mathcal{T}_2$) we obtain $\gamma \in \Delta^{\mathcal{U}_{\langle \mathcal{T}_2,\mathcal{A}'\rangle}}$ such that

$$\mathbf{t}^{\mathcal{U}_{\mathcal{A}'}}(\delta) \subseteq \mathbf{t}^{\mathcal{U}_{\langle \mathcal{T}_2,\mathcal{A}'\rangle}}(\gamma), \tag{7}$$

$$\mathbf{r}^{\mathcal{U}_{\mathcal{A}'}}(a,\delta) \subseteq \mathbf{r}^{\mathcal{U}_{\langle \mathcal{T}_2,\mathcal{A}'\rangle}}(a,\gamma). \tag{8}$$

Now define $h'(\sigma) = \gamma$; we need to show

$$\mathbf{t}_{\Xi}^{\mathcal{U}_{\langle \mathcal{T}_2\cup\mathcal{T}_{12},\mathcal{A}\rangle}}(\sigma) \subseteq \mathbf{t}_{\Xi}^{\mathcal{U}_{\langle \mathcal{T}_2,\mathcal{A}'\rangle}}(\gamma), \tag{9}$$

$$\mathbf{r}_{\Xi}^{\mathcal{U}_{\langle \mathcal{T}_2\cup\mathcal{T}_{12},\mathcal{A}\rangle}}(a,\sigma) \subseteq \mathbf{r}_{\Xi}^{\mathcal{U}_{\langle \mathcal{T}_2,\mathcal{A}'\rangle}}(a,\gamma). \tag{10}$$

For (9) consider the set $\vec{B} = \{B \text{ over } \Xi \mid \mathcal{T}_{12} \vdash \exists R^- \sqsubseteq B\}$ and observe that $\vec{B} \subseteq \mathbf{t}_{\Xi}^{\mathcal{U}_{\langle \mathcal{T}_{12},\mathcal{A}\rangle}}(aw_{[R]})$ and also by Lemma C.1 and the structure of $\mathcal{T}_2 \cup \mathcal{T}_{12}$, for each $B' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\langle \mathcal{T}_{12},\mathcal{A}\rangle}}(aw_{[R]})$ there exists $B \in \vec{B}$ such that $\mathcal{T}_2 \vdash B \sqsubseteq B'$. By (5) and (7) we obtain $\vec{B} \subseteq \mathbf{t}_{\Xi}^{\mathcal{U}_{\langle \mathcal{T}_2,\mathcal{A}'\rangle}}(\gamma)$; then using Lemma C.1 it can be easily verified $B' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\langle \mathcal{T}_2,\mathcal{A}'\rangle}}(\gamma)$ for all B' as above, which concludes the proof of (9). The (10) is analogious using (6), (8), and the set $\vec{S} = \{S \text{ over } \Xi \mid \mathcal{T}_{12} \vdash R \sqsubseteq S\}$.

Consider the case **(II)**, using Lemma C.1 and the structure of $\mathcal{T}_2 \cup \mathcal{T}_{12}$ and $\mathcal{A}$, one can show:

$$\exists R \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\langle \mathcal{T}_2\cup\mathcal{T}_{12},\mathcal{A}\rangle}}(a), \tag{11}$$

$$\mathcal{T}_2 \vdash \exists R^- \sqsubseteq B \text{ for all } B \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\langle \mathcal{T}_2\cup\mathcal{T}_{12},\mathcal{A}\rangle}}(\sigma), \tag{12}$$

$$\mathcal{T}_2 \vdash R \sqsubseteq S \text{ for all } S \in \mathbf{r}_{\Xi}^{\mathcal{U}_{\langle \mathcal{T}_2\cup\mathcal{T}_{12},\mathcal{A}\rangle}}(a,\sigma). \tag{13}$$

Provided that the homomorphism h' is defined for a, it follows $\exists R \in \mathbf{t}^{\mathcal{U}_{\langle \mathcal{T}_2,\mathcal{A}'\rangle}}(h'(a))$, therefore, there exists $\gamma \in \Delta^{\mathcal{U}_{\langle \mathcal{T}_2,\mathcal{A}'\rangle}}$ such that $R \in \mathbf{r}^{\mathcal{U}_{\langle \mathcal{T}_2,\mathcal{A}'\rangle}}(h'(a),\gamma)$. Now define $h'(\sigma) = \gamma$; we need to show

$$\mathbf{t}_{\Xi}^{\mathcal{U}_{\langle \mathcal{T}_2\cup\mathcal{T}_{12},\mathcal{A}\rangle}}(\sigma) \subseteq \mathbf{t}^{\mathcal{U}_{\langle \mathcal{T}_2,\mathcal{A}'\rangle}}(\gamma), \tag{14}$$

$$\mathbf{r}_{\Xi}^{\mathcal{U}_{\langle \mathcal{T}_2\cup\mathcal{T}_{12},\mathcal{A}\rangle}}(a,\sigma) \subseteq \mathbf{r}^{\mathcal{U}_{\langle \mathcal{T}_2,\mathcal{A}'\rangle}}(a,\gamma). \tag{15}$$

For (14) consider (12) and Lemma C.1; similarly, for (15) consider (13).

Assume now $\sigma = \sigma' w_{[R]}$ and the homomorphism from $\mathcal{U}_{\langle \mathcal{T}_2\cup\mathcal{T}_{12},\mathcal{A}\rangle}$ to $\mathcal{U}_{\langle \mathcal{T}_2,\mathcal{A}'\rangle}$ is defined for σ'. The proof is done in the same way as for the case **(II)**, all the statements are valid if one substitutes a by σ'. $\square$

Let $\mathcal{M} = (\Sigma, \Xi, \mathcal{T}_{12})$ be a mapping, and, $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, Σ- and Ξ-TBoxes. Define KBs $\mathcal{S}_B = \langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \{B(o)\}\rangle$ and $\mathcal{X}_B = \langle \mathcal{T}_2 \cup \mathcal{T}_{12}, \{B(o)\}\rangle$ for a basic concept B over Σ. We slightly abuse the notation and write $\mathcal{S}_{\mathcal{A}}$ to denote the KB $\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}\rangle$ for a given ABox $\mathcal{A}$, analogously we use $\mathcal{X}_{\mathcal{A}}$ to denote $\langle \mathcal{T}_2 \cup \mathcal{T}_{12}, \mathcal{A}\rangle$. We show

Lemma C.9 *Let $\mathcal{A}$ be an ABox over Σ and assume for each concept B, role R, and all $\sigma, \delta \in \Delta^{\mathcal{U}_{\mathcal{S}_{\mathcal{A}}}}$ such that*

(i) $B \in \mathbf{t}_{\Sigma}^{\mathcal{S}_{\mathcal{A}}}(\sigma)$,

(ii) $R \in \mathbf{r}_{\Sigma}^{\mathcal{S}_{\mathcal{A}}}(\sigma,\delta)$, *the following conditions hold*

(iii) $\mathbf{t}_{\Xi}^{\mathcal{X}_B}(o) \subseteq \mathbf{t}^{\mathcal{S}_B}(o)$*;*

(iv) $\mathcal{T}_2 \cup \mathcal{T}_{12} \vdash R \sqsubseteq R'$ *implies* $\mathcal{T}_1 \cup \mathcal{T}_{12} \vdash R \sqsubseteq R'$ *for all roles* R' *over* Ξ*;*

(v) *for each role* R *such that* $o \rightsquigarrow_{\mathcal{X}_B} w_{[R]}$ *there exists* $y \in \Delta^{\mathcal{U}_{\mathcal{S}_B}}$ *such that*

- ***(a)*** $\mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{X}_B}}(ow_{[R]}) \subseteq \mathbf{t}^{\mathcal{U}_{\mathcal{S}_B}}(y)$*,*
- ***(b)*** $\mathbf{r}_{\Xi}^{\mathcal{U}_{\mathcal{X}_B}}(o, ow_{[R]}) \subseteq \mathbf{r}^{\mathcal{U}_{\mathcal{S}_B}}(o, y)$*.*

Then $\mathcal{U}_{\mathcal{X}_\mathcal{A}}$ *is finitely homomorphically embeddable into* $\mathcal{U}_{\mathcal{S}_\mathcal{A}}$*.*

Proof. Let $\mathcal{A}$ as above and assume the condition of the lemma are satisfied. We build a mapping h from $\mathsf{path}(\mathcal{X}_\mathcal{A})$ to $\mathsf{path}(\mathcal{S}_\mathcal{A})$ such that for any finite subinterpretation of $\mathcal{U}_{\mathcal{X}_\mathcal{A}}$ the restriction of h to it is a homomorphism to $\mathcal{U}_{\mathcal{S}_\mathcal{A}}$. Initially, we define $h(a) = a$, let us immediately verify that $\mathbf{t}^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}(a) \subseteq \mathbf{t}^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(a)$. Let $C \in \mathbf{t}^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}(a)$, it follows by Lemma C.1 **(i)** there exists B over Σ such that $\mathcal{A} \models B(a)$ and $\mathcal{T}_2 \cup \mathcal{T}_{12} \vdash B \sqsubseteq C$. Observe that $B \in \mathbf{t}^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(a)$; now if C is over Σ it follows $C = B$, so $C \in \mathbf{t}^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(a)$ and the proof is done. Otherwise, $C \in \mathbf{s}_{\Xi}^{\mathcal{X}_B}(o)$, then by **(iii)** $C \in \mathbf{s}^{\mathcal{S}_B}(o)$, so $\mathcal{T}_1 \cup \mathcal{T}_{12} \vdash B \sqsubseteq C$. Finally, using Lemma C.1 **(i)** obtain $C \in \mathbf{t}^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(a)$. The proof of $\mathbf{r}^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}(a, b) \subseteq \mathbf{r}^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(a, b)$ is analogous using Lemma C.1 **(ii)** and current **(iv)**.

Now we show how to define h for $\sigma = aw_{[R]} \in \mathsf{path}(\mathcal{X}_\mathcal{A})$. It follows $a \rightsquigarrow_{\mathcal{X}_\mathcal{A}} w_{[R]}$, then by Lemma C.3 (with $\mathcal{K} = \mathcal{X}_\mathcal{A}$) there exists B over Σ such that $\mathcal{A} \models B(a)$, $o \rightsquigarrow_{\mathcal{X}_B} w_{[R]}$, and

$$\mathbf{t}^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}(aw_{[R]}) \subseteq \mathbf{t}^{\mathcal{U}_{\mathcal{X}_B}}(ow_{[R]}) \tag{16}$$

$$\mathbf{r}^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}(a, aw_{[R]}) \subseteq \mathbf{r}^{\mathcal{U}_{\mathcal{X}_B}}(o, ow_{[R]}). \tag{17}$$

We are going to show now there exists $y \in \Delta^{\mathcal{U}_{\mathcal{S}_B}}$ such that

$$\mathbf{t}^{\mathcal{U}_{\mathcal{X}_B}}(ow_{[R]}) \subseteq \mathbf{t}^{\mathcal{U}_{\mathcal{S}_B}}(y) \text{ and} \tag{18}$$

$$\mathbf{r}^{\mathcal{U}_{\mathcal{X}_B}}(o, ow_{[R]}) \subseteq \mathbf{r}^{\mathcal{U}_{\mathcal{S}_B}}(o, y). \tag{19}$$

Assume, first, $\mathbf{t}_{\Sigma}^{\mathcal{U}_{\mathcal{X}_B}}(ow_{[R]}) = \emptyset$, then also $\mathbf{r}_{\Sigma}^{\mathcal{U}_{\mathcal{X}_B}}(o, ow_{[R]}) = \emptyset$; it remains to observe that from $\mathcal{A} \models B(a)$ it follows **(i)** is satisfied with $\sigma = a$, then by **(v)** we obtain y satisfying (18) and (19).

Assume now $\mathbf{t}_{\Sigma}^{\mathcal{U}_{\mathcal{X}_B}}(ow_{[R]}) \neq \emptyset$, it follows $B = \exists R$, $\mathbf{t}_{\Sigma}^{\mathcal{U}_{\mathcal{X}_B}}(ow_{[R]}) = \{\exists R^-\}$, and $\mathbf{r}_{\Sigma}^{\mathcal{U}_{\mathcal{X}_B}}(o, ow_{[R]}) = \{R\}$. Since $B = \exists R$, there must exists a role Q such that $o \rightsquigarrow_{\mathcal{S}_B} w_{[Q]}$ and $\mathcal{T}_1 \cup \mathcal{T}_{12} \vdash Q \sqsubseteq R$, we choose $w_{[Q]}$ to be the required y; it is immediate to see $\mathbf{t}_{\Sigma}^{\mathcal{U}_{\mathcal{X}_B}}(ow_{[R]}) \subseteq \mathbf{t}^{\mathcal{U}_{\mathcal{S}_B}}(y)$, and $\mathbf{r}_{\Sigma}^{\mathcal{U}_{\mathcal{X}_B}}(o, ow_{[R]}) \subseteq \mathbf{r}^{\mathcal{U}_{\mathcal{S}_B}}(o, y)$. To prove also $\mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{X}_B}}(ow_{[R]}) \subseteq \mathbf{t}^{\mathcal{U}_{\mathcal{S}_B}}(y)$ and $\mathbf{r}_{\Xi}^{\mathcal{U}_{\mathcal{X}_B}}(o, ow_{[R]}) \subseteq \mathbf{r}^{\mathcal{U}_{\mathcal{X}_B}}(o, y)$ we are going to use **(iii)** and **(iv)**, but we need $\exists R^- \in \mathbf{t}_{\Sigma}^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(\sigma)$ and $R \in \mathbf{r}_{\Sigma}^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(\sigma, \delta)$ for some $\sigma, \delta \in \Delta^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}$. To get the latter two facts it is sufficient to notice $\exists R \in \mathbf{t}^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}(a)$ (since $a \rightsquigarrow_{\mathcal{X}_B} w_{[R]}$) and $\mathbf{t}^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}(a) \subseteq \mathbf{t}^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(a)$ proven above.

The proof of $\mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{X}_B}}(ow_{[R]}) \subseteq \mathbf{t}^{\mathcal{U}_{\mathcal{S}_B}}(y)$ is as follows: assume $B' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{X}_B}}(ow_{[R]})$, then since R is over Σ it follows $B' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{X}_{\exists R^-}}}(o)$. By $\exists R^- \in \mathbf{t}_{\Sigma}^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(\sigma)$ and **(iii)** obtain $B' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_{\exists R^-}}}(o)$, then since $\mathcal{T}_1 \cup \mathcal{T}_{12} \vdash Q \sqsubseteq R$ it follows $\mathbf{t}^{\mathcal{U}_{\mathcal{S}_{\exists R^-}}}(o) \subseteq \mathbf{t}^{\mathcal{U}_{\mathcal{S}_B}}(ow_{[Q]})$ and we obtain $B' \in \mathbf{t}^{\mathcal{U}_{\mathcal{S}_B}}(y)$. The proof of $\mathbf{r}_{\Xi}^{\mathcal{U}_{\mathcal{X}_B}}(o, ow_{[R]}) \subseteq \mathbf{r}^{\mathcal{U}_{\mathcal{X}_B}}(o, y)$ is analogous using $R \in \mathbf{r}_{\Sigma}^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(\sigma, \delta)$, **(iv)**, and $\mathcal{T}_1 \cup \mathcal{T}_{12} \vdash Q \sqsubseteq R$. We finished showing there exists $y \in \Delta^{\mathcal{U}_{\mathcal{S}_B}}$, such that (18) and (19).

To continue the proof consider $\{B\} \subseteq \mathbf{t}^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(a)$ and Lemma C.4 (with $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_{12}$ and $\mathcal{T}' = \emptyset$) there exists $\delta \in \Delta^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}$ such that $\mathbf{t}^{\mathcal{U}_{\mathcal{S}_B}}(y) \subseteq \mathbf{t}^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(\delta)$ and $\mathbf{r}^{\mathcal{U}_{\mathcal{S}_B}}(o, y) \subseteq \mathbf{r}^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(a, \delta)$. It follows now using (16) and (18) that $\mathbf{t}^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}(aw_{[R]}) \subseteq \mathbf{t}^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(\delta)$. Analogously using (17) and (19) one obtains $\mathbf{r}^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}(a, aw_{[R]}) \subseteq \mathbf{r}^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(a, \delta)$.

We show how to define the homomorphism for $\sigma w_{[R]} \in \mathsf{path}(\mathcal{X}_\mathcal{A})$ with $\mathsf{tail}(\sigma) = w_{[R']}$ given that the homomorphism for $h(\sigma)$ is defined. It follows $w_{[R']} \rightsquigarrow_{\mathcal{X}_\mathcal{A}} w_{[R]}$ and by definition of $\rightsquigarrow$ and the structure of $\mathcal{T}_2 \cup \mathcal{T}_{12}$ we obtain $\mathcal{T}_2 \cup \mathcal{T}_{12} \vdash \exists R'^- \sqsubseteq \exists R$ and R is a Ξ role *different from* R^-. By Lemma C.1 it also follows $\{\exists R'^-, \exists R\} \subseteq \mathbf{t}^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}(\sigma)$. Since h is a homomorphism, $\{\exists R'^-, \exists R\} \subseteq \mathbf{t}^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(\delta)$ for $\delta = h(\sigma) \in \Delta^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}$. We use Lemma C.5 to obtain B over Σ such that $B \in \mathbf{t}^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(\delta)$ and $\mathcal{T}_{12} \vdash B \sqsubseteq \exists R$. Notice that such B exists: since $\exists R'^-$ and $\exists R$ are different concepts, **(ii)** of Lemma C.5 is excluded, so **(i)** holds.

Then in $\mathcal{X}_B$ we have that $o \rightsquigarrow_{\mathcal{X}_B} w_{[R]}$ for a Ξ role R, and the proof continues analogously to the proof for the case $\sigma = aw_{[R]}$ above using the conditions **(ii)**, **(iii)** and Lemmas C.4 to obtain δ' in $\Delta^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}$ such that $\mathbf{t}^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}(\sigma w_{[R]}) \subseteq \mathbf{t}^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(\delta')$ and $\mathbf{r}^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}(\sigma, \sigma w_{[R]}) \subseteq \mathbf{r}^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(\delta, \delta')$. We assign $h(\sigma w_{[R]}) = \delta'$.

Thus, we defined the mapping h that is clearly a Ξ-homomorphism from each finite subinterpretation of $\mathcal{U}_{\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}\rangle}$ into $\mathcal{U}_{\langle \mathcal{T}_2 \cup \mathcal{T}_{12}, \mathcal{A}\rangle}$. □

Lemma C.10 *Let $\mathcal{A}$ be an ABox over Σ and assume for each concept B, role R, and all $\sigma, \delta \in \Delta^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}$ such that*

- ***(i)*** $B \in \mathbf{t}_\Sigma^{\mathcal{S}_\mathcal{A}}(\sigma)$,
- ***(ii)*** $R \in \mathbf{r}_\Sigma^{\mathcal{S}_\mathcal{A}}(\sigma, \delta)$

the following conditions hold

- ***(iii)*** $\mathbf{t}_\Xi^{\mathcal{U}_{\mathcal{S}_B}}(o) \subseteq \mathbf{t}^{\mathcal{U}_{\mathcal{X}_B}}(o)$;
- ***(iv)*** $\mathcal{T}_1 \cup \mathcal{T}_{12} \vdash R \sqsubseteq R'$ *implies* $\mathcal{T}_2 \cup \mathcal{T}_{12} \vdash R \sqsubseteq R'$ *for all roles R' over Ξ;*
- ***(v)*** *for each role R such that $o \rightsquigarrow_{\mathcal{U}_{\mathcal{S}_B}} w_{[R]}$ there exists $y \in \Delta^{\mathcal{U}_{\mathcal{X}_B}}$ such that*
  - ***(a)*** $\mathbf{t}_\Xi^{\mathcal{U}_{\mathcal{S}_B}}(ow_{[R]}) \subseteq \mathbf{t}^{\mathcal{U}_{\mathcal{X}_B}}(y)$,
  - ***(b)*** $\mathbf{r}_\Xi^{\mathcal{S}_B}(o, ow_{[R]}) \subseteq \mathbf{r}^{\mathcal{U}_{\mathcal{X}_B}}(o, y)$.

Then $\mathcal{U}_{\mathcal{S}_\mathcal{A}}$ is finitely Ξ-homomorphically embeddable into $\mathcal{U}_{\mathcal{X}_\mathcal{A}}$.

Proof. Assume the condition of the lemma is satisfied, and let $\mathcal{A}$ be an ABox over Σ. We build a mapping h from $\mathsf{path}(\mathcal{S}_\mathcal{A})$ to $\mathsf{path}(\mathcal{X}_\mathcal{A})$ such that for any finite subinterpretation of $\mathcal{U}_{\mathcal{S}_\mathcal{A}}$ the restriction of h to it is a Ξ-homomorphism to $\mathcal{U}_{\mathcal{T}_\mathcal{A}}$. Initially, we define $h(a) = a$, let us immediately verify that $\mathbf{t}_\Xi^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(a) \subseteq \mathbf{t}_\Xi^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}(a)$. Let $B' \in \mathbf{t}_\Xi^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(a)$, it follows by Lemma C.1 **(i)** there exists B over Σ such that $\mathcal{A} \models B(a)$ and $\mathcal{T}_1 \cup \mathcal{T}_{12} \vdash B \sqsubseteq B'$. Observe that $B \in \mathbf{t}^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(a)$, then by **(iii)** $B' \in \mathbf{t}^{\mathcal{U}_{\mathcal{X}_B}}(o)$, so $\mathcal{T}_2 \cup \mathcal{T}_{12} \vdash B \sqsubseteq B'$. Finally, using Lemma C.1 **(i)** obtain $B' \in \mathbf{t}^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(a)$. The proof of $\mathbf{r}_\Xi^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(a, b) \subseteq \mathbf{r}_\Xi^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}(a, b)$ is analogious using Lemma C.1 **(ii)** and current **(iv)**.

Now we show how to define h for $\sigma = aw_{[R]} \in \mathsf{path}(\mathcal{S}_\mathcal{A})$. It follows $a \rightsquigarrow_{\mathcal{S}_\mathcal{A}} w_{[R]}$ and by Lemma C.3 (with $\mathcal{K} = \mathcal{S}_\mathcal{A}$) we obtain B over Σ such that $\mathcal{A} \models B(a)$, $o \rightsquigarrow_{\mathcal{S}_B} w_{[R]}$, and

$$\mathbf{t}^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(aw_{[R]}) \subseteq \mathbf{t}^{\mathcal{U}_{\mathcal{S}_B}}(ow_{[R]}) \tag{20}$$
$$\mathbf{r}^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(a, aw_{[R]}) \subseteq \mathbf{r}^{\mathcal{U}_{\mathcal{S}_B}}(o, ow_{[R]}). \tag{21}$$

Notice that $B \in \mathbf{s}_\Sigma^{\mathcal{S}_B}(a)$ (that is, **(i)**), then by **(v)** there exists $y \in \Delta^{\mathcal{X}_B}$ such that

$$\mathbf{t}_\Xi^{\mathcal{U}_{\mathcal{S}_B}}(w_{[R]}) \subseteq \mathbf{t}^{\mathcal{U}_{\mathcal{X}_B}}(y), \tag{22}$$
$$\mathbf{r}_\Xi^{\mathcal{U}_{\mathcal{S}_B}}(o, w_{[R]}) \subseteq \mathbf{r}^{\mathcal{U}_{\mathcal{X}_B}}(o, y). \tag{23}$$

Since $\{B\} \subseteq \mathbf{t}^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}(a)$, by Lemma C.4 (with $\mathcal{T} = \mathcal{T}_2 \cup \mathcal{T}_{12}$ and $\mathcal{T}' = \emptyset$) there exists $\delta \in \Delta^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}$ such that $\mathbf{t}^{\mathcal{U}_{\mathcal{X}_B}}(y) \subseteq \mathbf{t}^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(\delta)$ and $\mathbf{r}^{\mathcal{U}_{\mathcal{X}_B}}(o, y) \subseteq \mathbf{r}^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}(a, \delta)$. It follows now using (20) and (22) that $\mathbf{t}_\Xi^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(aw_{[R]}) \subseteq \mathbf{t}_\Xi^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}(\delta)$. Analogously using (21) and (23) one obtains $\mathbf{r}_\Xi^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(a, aw_{[R]}) \subseteq \mathbf{r}_\Xi^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}(a, \delta)$. We assign $h(\sigma) = \delta$.

We show how to define the homomorphism for $\sigma w_{[R]} \in \mathsf{path}(\mathcal{S}_\mathcal{A})$ with $\sigma = \sigma' w_{[R']}$ given that the homomorphism $h(\sigma)$ and $h(\sigma')$ is defined. It follows $w_{[R']} \rightsquigarrow_{\mathcal{S}_\mathcal{A}} w_{[R]}$ and it that case R' is over Σ by the structure of $\mathcal{T}_1 \cup \mathcal{T}_{12}$. Analogously to the proof of Lemma C.3 it can be verified $o \rightsquigarrow_{\mathcal{S}_{(\exists R'^-)}} w_{[R]}$ and

$$\mathbf{t}^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(\sigma w_{[R]}) \subseteq \mathbf{t}^{\mathcal{U}_{\mathcal{S}_{(\exists R'^-)}}}(ow_{[R]}) \text{ and} \tag{24}$$
$$\mathbf{r}^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(\sigma, \sigma w_{[R]}) \subseteq \mathbf{r}^{\mathcal{U}_{\mathcal{S}_{(\exists R'^-)}}}(o, ow_{[R]}). \tag{25}$$

Observe that $\exists R'^- \in \mathbf{t}_\Sigma^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(\sigma)$ (that is, **(i)**), then by **(v)** there is $y \in \Delta^{\mathcal{U}_{\mathcal{X}_{(\exists R'^-)}}}$ satisfying **(a)** and **(b)**. Given the structure of $\mathcal{T}_2 \cup \mathcal{T}_{12}$ two cases are possible:

(III) $y \in \Delta^{\mathcal{U}_{\langle \mathcal{T}_2, \{B(o) \mid B \in \mathbf{B}\}\rangle}}$ for the set $\mathbf{B}$ of all concepts B over Ξ such that $\mathcal{T}_{12} \vdash \exists R'^- \sqsubseteq B$,

$$\mathbf{t}_\Xi^{\mathcal{U}_{\mathcal{S}_{(\exists R'^-)}}}(ow_{[R]}) \subseteq \mathbf{t}^{\mathcal{U}_{\langle \mathcal{T}_2, \{B(o) \mid B \in \mathbf{B}\}\rangle}}(y), \text{ and} \tag{26}$$

$$\mathbf{r}_\Xi^{\mathcal{U}_{\mathcal{S}_{(\exists R'^-)}}}(o, ow_{[R]}) \subseteq \mathbf{r}^{\mathcal{U}_{\langle \mathcal{T}_2, \{B(o) \mid B \in \mathbf{B}\}\rangle}}(o, y). \tag{27}$$

(IV) $o \leadsto_{\mathcal{X}_{(\exists R'^-)}} w_{[R'^-]}$,

$$y \in \Delta^{\mathcal{U}_{\langle \mathcal{T}_2, \{B(o) \mid B \in \mathbf{B}\}\rangle}}, \text{ for the set } \mathbf{B} \text{ of all concepts } B \text{ over } \Xi \text{ such that } \mathcal{T}_{12} \vdash \exists R' \sqsubseteq B, \tag{28}$$

$$\mathbf{t}_\Xi^{\mathcal{U}_{\mathcal{S}_{(\exists R'^-)}}}(ow_{[R]}) \subseteq \mathbf{t}^{\mathcal{U}_{\langle \mathcal{T}_2, \{B(o) \mid B \in \mathbf{B}\}\rangle}}(y), \text{ and} \tag{29}$$

$$\mathbf{r}_\Xi^{\mathcal{U}_{\mathcal{S}_{(\exists R'^-)}}}(o, ow_{[R]}) \subseteq \mathbf{r}^{\mathcal{U}_{\mathcal{X}_{(\exists R'^-)}}}(o, ow_{[R'^-]}). \tag{30}$$

Consider **(III)**; then, $\mathbf{B} \subseteq \mathbf{t}_\Xi^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}(h(\sigma))$, since obviously $\mathbf{B} \subseteq \mathbf{t}_\Xi^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(\sigma)$ and h is a homomorphism on σ. By Lemma C.4 (with $\mathcal{T} = \mathcal{T}_2$ and $\mathcal{T}' = \mathcal{T}_{12}$) we obtain $\delta \in \Delta^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}$ such that $\mathbf{t}^{\mathcal{U}_{\langle \mathcal{T}_2, \{B(o) \mid B \in \mathbf{B}\}\rangle}}(y) \subseteq \mathbf{t}^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}(\delta)$ and $\mathbf{r}^{\mathcal{U}_{\langle \mathcal{T}_2, \{B(o) \mid B \in \mathbf{B}\}\rangle}}(o, y) \subseteq \mathbf{r}^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}(h(\sigma), \delta)$. Note that using (24) and (26) we obtain $\mathbf{t}^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(\sigma w_{[R]}) \subseteq \mathbf{t}^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}(\delta)$; also using (25) and (27) we obtain $\mathbf{r}_\Xi^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(\sigma, \sigma w_{[R]}) \subseteq \mathbf{r}_\Xi^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}(h(\sigma), \delta)$. We assign $h(\sigma w_{[R]}) = \delta$ which concludes the proof.

Consider **(IV)**; at this point we need

$$\mathbf{B} \subseteq \mathbf{t}^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(\sigma') \text{ and} \tag{31}$$

$$\mathbf{R} \subseteq \mathbf{r}^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(\sigma, \sigma'), \tag{32}$$

for $\mathbf{R} = \{R'' \mid \mathcal{T}_{12} \vdash R'^- \sqsubseteq R''\}$. Indeed, (31) follows since $\exists R' \in \mathbf{t}^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(\sigma')$, by the definition of $\mathbf{B}$, and Lemma C.1 **(i)** and **(iii)**. For (32) let $R'' \in \mathbf{R}$ it follows $[R'^-] \leq_{\mathcal{T}_1 \cup \mathcal{T}_{12}} [R'']$ and so $[R'] \leq_{\mathcal{T}_1 \cup \mathcal{T}_{12}} [R''^-]$. Then by the definition of $\mathcal{U}_{\mathcal{S}_\mathcal{A}}$ obtain $R''^- \in \mathbf{r}^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(\sigma', \sigma)$, so obviously $R'' \in \mathbf{r}^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(\sigma, \sigma')$.

Observe that, since h is a Ξ-homomorphism on σ' and (31), it follows

$$\mathbf{B} \subseteq \mathbf{t}_\Xi^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}(h(\sigma')) \tag{33}$$

and distinguish two subcases:

(V) $\mathbf{r}_\Xi^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(\sigma, \sigma w_{[R]}) = \emptyset$;

(VI) $\mathbf{r}_\Xi^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(\sigma, \sigma w_{[R]}) \neq \emptyset$.

In case **(V)** consider (28), (33) and Lemma C.4 to obtain $\delta \in \Delta^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}$ such that $\mathbf{t}^{\mathcal{U}_{\langle \mathcal{T}_2, \{B(o) \mid B \in \mathbf{B}\}\rangle}}(y) \subseteq \mathbf{t}^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}(\delta)$. Then using (24) and (29) one obtains $\mathbf{t}_\Xi^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(\sigma w_{[R]}) \subseteq \mathbf{t}_\Xi^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}(\delta)$. Taking $\delta = h(\sigma w_{[R]})$ completes the proof of the first subcase.

In the alternative case **(VI)**, it follows by (25) that $\mathbf{r}_\Xi^{\mathcal{U}_{\mathcal{S}_{(\exists R'^-)}}}(o, w_{[R]}) \neq \emptyset$ therefore $y = o$ (c.f. (28)). We assign $h(\delta w_{[R]}) = h(\sigma')$ and we prove $\mathbf{t}_\Xi^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(\sigma w_{[R]}) \subseteq \mathbf{t}^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}(h(\sigma'))$, and $\mathbf{r}_\Xi^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(\sigma, \sigma w_{[R]}) \subseteq \mathbf{r}^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}(h(\sigma), h(\sigma'))$.

Indeed, let $B \in \mathbf{t}_\Xi^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(\sigma w_{[R]})$, by (24) $B \in \mathbf{s}_\Xi^{\mathcal{S}_{(\exists R'^-)}}(ow_{[R]})$, then by (29) there exists $B' \in \mathbf{B}$ such that $\mathcal{T}_2 \vdash B' \sqsubseteq B$. Using (33) and Lemma C.1 **(iii)** obtain $B \in \mathbf{t}^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}(h(\sigma'))$.

Let now $Q \in \mathbf{r}_\Xi^{\mathcal{U}_{\mathcal{S}_\mathcal{A}}}(\sigma, \sigma w_{[R]})$, by (25) it follows $Q \in \mathbf{r}_\Xi^{\mathcal{U}_{\mathcal{S}_{(\exists R'^-)}}}(o, ow_{[R]})$, then by (30) there exists $R'' \in \mathbf{R}$ such that $\mathcal{T}_2 \vdash R'' \sqsubseteq Q$. Since h is a homomorphism on σ, σ' and (32) obtain $R'' \in \mathbf{t}_\Xi^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}(h(\sigma), h(\sigma'))$. By the definition of $\mathcal{U}_{\mathcal{X}_\mathcal{A}}$ we conclude also $Q \in \mathbf{r}_\Xi^{\mathcal{U}_{\mathcal{X}_\mathcal{A}}}(h(\sigma), h(\sigma'))$. This concludes the proof of the second subcase and the whole case **(IV)**. We have shown how to define h for $\sigma w_{[R]} \in \mathsf{path}(\mathcal{S}_\mathcal{A})$ so that h is Ξ-homomorphism. $\square$

C.3 Proof of Proposition 6.1

This proof can be obtained as an easy consequence of the following

Lemma C.11 *Let $\mathcal{M} = (\Sigma, \Xi, \mathcal{T}_{12})$ be a mapping, and $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, Σ- and Ξ-TBoxes, $q(\vec{x})$ a Ξ-query, and $\mathcal{A}$ a Σ ABox. Then*

$$\bigcap_{\substack{\mathcal{A}' - \text{ABox, s.t. it is} \\ \text{UCQ-solution for } \mathcal{A} \\ \text{under } \mathcal{T}_{12}}} cert(q, \langle \mathcal{T}_2, \mathcal{A}' \rangle) \subseteq \bigcap_{\substack{\mathcal{A}' - \text{extended ABox, s.t.} \\ \text{it is UCQ-solution for } \mathcal{A} \\ \text{under } \mathcal{T}_{12}}} cert(q, \langle \mathcal{T}_2, \mathcal{A}' \rangle).$$

Proof. Consider a tuple of constants $\vec{a}$ such that $\langle\mathcal{T}_2, \mathcal{A}'\rangle \models q[\vec{a}]$ for all Ξ-ABoxes $\mathcal{A}'$, such that $\mathcal{A}'$ is a UCQ-solution for $\mathcal{A}$ under $\mathcal{T}_{12}$. Assume an extended ABox $\mathcal{A}'$, such that it is a UCQ-solution for $\mathcal{A}$; we are going to show $\langle\mathcal{T}_2, \mathcal{A}'\rangle \models q[\vec{a}]$. If $\langle\mathcal{T}_2, \mathcal{A}'\rangle$ is inconsistent, the proof is done; otherwise, take an interpretation $\mathcal{I} \models \langle\mathcal{T}_2, \mathcal{A}'\rangle$. It follows there exists a substitution over $\mathcal{I}$, such that $h(u) \in B^{\mathcal{I}}$ for every $B(u) \in \mathcal{A}$, and $(h(u), h(v)) \in R^{\mathcal{I}}$ for all $R(u, v) \in \mathcal{A}$. We associate with every null n in $\mathcal{A}'$ a *fresh* (w.r.t. constants in $\mathcal{A}'$, $\vec{a}$, and $q(\vec{x})$) constant $a_n \in N_a$; then take $\mathcal{A}^*$ the result of the substitution of each n by a_n in $\mathcal{A}'$. Consider an interpretation $\mathcal{I}^*$, such that it is equal to $\mathcal{I}$, except for a_n, such that n is a null in $\mathcal{A}'$, we set $a_n^{\mathcal{I}^*} = h(n)$. It should be clear that $\mathcal{I}^* \models \langle\mathcal{T}_2, \mathcal{A}^*\rangle$, then we obtain $\mathcal{I}^* \models q[\vec{a}]$. It remains to show $\mathcal{I} \models q[\vec{a}]$; for that assume $\vec{x} = (x_1, \dots x_n)$, $\vec{a} = (a_1, \dots, a_n)$, and

$$q(\vec{x}) = \exists y_1, \dots, y_m \varphi(\vec{x}, y_1, \dots, y_m, b_1, \dots, b_k),$$

where b_i are constants and φ a quantifier-free formula. It follows, there exist $d_1, \dots, d_n, e_1, \dots e_m, f_1, \dots, f_k \in \Delta^{\mathcal{I}^*}$, such that $d_i = a_i^{\mathcal{I}^*}$, $f_i = b_i^{\mathcal{I}^*}$, and

$$\mathcal{I}^* \models \varphi(d_1, \dots d_n, e_1, \dots, e_m, f_1, \dots, f_k).$$

It remains to observe that all of d_i, e_i, f_i belong to the interpretation of the same concepts/roles in $\mathcal{I}$ as in $\mathcal{I}'$, and $a_i^{\mathcal{I}} = d_i$, $b_i^{\mathcal{I}} = f_i$. Therefore, $\mathcal{I} \models \varphi(d_1, \dots d_n, e_1, \dots, e_m, f_1, \dots, f_k)$, and, finally, $\mathcal{I} \models q[\vec{a}]$. $\square$

C.4 Proof of Proposition 6.2

The result is proved in Theorem C.16, which is based the series of lemmas.

Lemma C.12 *Let $\mathcal{M} = (\Sigma, \Xi, \mathcal{T}_{12})$ be a mapping, and $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, Σ- and Ξ-TBoxes. Then $\mathcal{T}_2$ is a* UCQ-representation of $\mathcal{T}_1$ under $\mathcal{T}_{12}$ *if and only if $\langle\mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A}\rangle$ is Ξ-query equivalent to $\langle\mathcal{T}_2 \cup \mathcal{T}_{12}, \mathcal{A}\rangle$ for every ABox $\mathcal{A}$ over Σ such that $\langle\mathcal{T}_1, \mathcal{A}\rangle$ is consistent.*

Proof. We first prove the following:

Proposition C.13 *Let $\mathcal{M} = (\Sigma, \Xi, \mathcal{T}_{12})$ be a mapping, and $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, Σ- and Ξ-TBoxes, $\mathcal{A}$ a Σ-ABox, such that $\langle\mathcal{T}_1, \mathcal{A}\rangle$ is consistent, $q(\vec{x})$ a Ξ query and $\vec{a}$ a tuple of constants. Then $\langle\mathcal{T}_2 \cup \mathcal{T}_{12}, \mathcal{A}\rangle \models q[\vec{a}]$ iff $\langle\mathcal{T}_2, \mathcal{A}'\rangle \models q[\vec{a}]$ for all Ξ-ABoxes $\mathcal{A}'$ such that $\mathcal{A}'$ is a UCQ-solution for $\mathcal{A}$ under $\mathcal{M}$.*

Proof. ($\Rightarrow$) Let $\mathcal{A}$ and $\mathcal{A}'$ as above; we show $\langle\mathcal{T}_2, \mathcal{A}'\rangle$ Ξ-query entails $\langle\mathcal{T}_2 \cup \mathcal{T}_{12}, \mathcal{A}\rangle$. Notice that since $\mathcal{A}'$ is a UCQ-solution, it follows $\langle\mathcal{T}_{12}, \mathcal{A}\rangle$ Ξ-query entails $\mathcal{A}'$; and since $\mathcal{A}$ is consistent, $\langle\mathcal{T}_{12}, \mathcal{A}\rangle$ is consistent as well. Using Claim A.4, we obtain that $\mathcal{U}_{\langle\mathcal{T}_{12}, \mathcal{A}\rangle}$ is Ξ homomorphically embeddable into $\mathcal{U}_{\mathcal{A}'}$. By Lemma C.8 it follows $\mathcal{U}_{\langle\mathcal{T}_2 \cup \mathcal{T}_{12}, \mathcal{A}\rangle}$ is Ξ homomorphically embeddable into $\mathcal{U}_{\langle\mathcal{T}_2, \mathcal{A}'\rangle}$. Now, if $\langle\mathcal{T}_2 \cup \mathcal{T}_{12}, \mathcal{A}\rangle$ is inconsistent, it can be shown in the way similar to the proof of Lemma C.6 that $\langle\mathcal{T}_2, \mathcal{A}'\rangle$ is inconsistent, then the proof is done. Otherwise, we use Claim A.4 to conclude $\langle\mathcal{T}_2, \mathcal{A}'\rangle$ Ξ-query entails $\langle\mathcal{T}_2 \cup \mathcal{T}_{12}, \mathcal{A}\rangle$.

($\Leftarrow$) Let $\mathcal{A}$, $q(\vec{x})$, and $\vec{a}$ as above; assume $\langle\mathcal{T}_2, \mathcal{A}'\rangle \models q[\vec{a}]$ for all solutions $\mathcal{A}'$ for $\mathcal{A}$ under $\mathcal{T}_{12}$. We are going to show $\langle\mathcal{T}_2 \cup \mathcal{T}_{12}, \mathcal{A}\rangle \models q[\vec{a}]$. If $\langle\mathcal{T}_2 \cup \mathcal{T}_{12}, \mathcal{A}\rangle$ is inconsistent, the proof is done; assume the opposite, then we will show $\mathcal{U}_{\langle\mathcal{T}_2 \cup \mathcal{T}_{12}, \mathcal{A}\rangle} \models q[\vec{a}]$, using Theorem A.3 the proof will be done. Consider $\mathcal{U}_{\langle\mathcal{T}_{12}, \mathcal{A}\rangle}$ and define the set

$$\Theta_{\mathcal{T}_{12}, \mathcal{A}} = \mathsf{Ind}(\mathcal{A}) \cup \{a w_{[R]} \in \Delta^{\mathcal{U}_{\langle\mathcal{T}_{12}, \mathcal{A}\rangle}} \mid a \in \mathsf{Ind}(\mathcal{A}) \text{ and } R \text{ over } \Sigma\}.$$

For each $\sigma \in \Theta_{\mathcal{T}_{12}, \mathcal{A}}$ define $t_\sigma = a$ if $\sigma = a \in \mathsf{Ind}(\mathcal{A})$, and $t_\sigma = a_\sigma$ for a *fresh w.r.t.* $\mathcal{A}$, $\vec{a}$, and $q(\vec{x})$ constant a_σ, otherwise. Now, define

$$\begin{aligned} \mathcal{A}' = &\{B(t_\sigma) \mid B \text{ basic conc. over } \Xi, \sigma \in B^{\mathcal{U}_{\langle\mathcal{T}_{12}, \mathcal{A}\rangle}} \cap \Theta_{\mathcal{T}_{12}, \mathcal{A}}\} \cup \\ &\{P(t_\sigma, t_{\sigma'}) \mid P \text{ role name over } \Xi, (\sigma, \sigma') \in P^{\mathcal{U}_{\langle\mathcal{T}_{12}, \mathcal{A}\rangle}} \cap (\Theta_{\mathcal{T}_{12}, \mathcal{A}} \times \Theta_{\mathcal{T}_{12}, \mathcal{A}})\}. \end{aligned}$$

It is straightforward to build a Ξ-homomorphism from $\langle\mathcal{T}_{12}, \mathcal{A}\rangle$ to $\mathcal{A}'$ and use Claim A.4 to show $\mathcal{A}'$ is a UCQ-solution for $\mathcal{A}$ under $\mathcal{T}_{12}$. Consider now $\mathcal{U}_{\langle\mathcal{T}_2, \mathcal{A}'\rangle}$ and a mapping $g : \Delta^{\mathcal{U}_{\langle\mathcal{T}_2, \mathcal{A}'\rangle}} \mapsto \Delta^{\mathcal{U}_{\langle\mathcal{T}_2 \cup \mathcal{T}_{12}, \mathcal{A}\rangle}}$ defined in the following way:

$$g(a w_{[R_1]} \dots w_{[R_n]}) = \begin{cases} \sigma w_{[R_1]} \dots w_{[R_n]}, & \text{if } a = t_\sigma \text{ and } \sigma \in \Theta_{\mathcal{T}_{12}, \mathcal{A}}, \\ a, & \text{otherwise,} \end{cases}$$

where $n \geq 0$. Notice that g is not a homomorphism, however, using the definitions of $\mathcal{U}_{\langle\mathcal{T}_2, \mathcal{A}'\rangle}$ and $\mathcal{U}_{\langle\mathcal{T}_2 \cup \mathcal{T}_{12}, \mathcal{A}\rangle}$ one can straightforwardly verify

$$\mathbf{t}^{\langle\mathcal{T}_2, \mathcal{A}'\rangle}(\delta) \subseteq \mathbf{t}^{\langle\mathcal{T}_2 \cup \mathcal{T}_{12}, \mathcal{A}\rangle}(g(\delta)), \tag{34}$$

$$\mathbf{r}^{\langle\mathcal{T}_2, \mathcal{A}'\rangle}(\delta, \delta') \subseteq \mathbf{r}^{\langle\mathcal{T}_2 \cup \mathcal{T}_{12}, \mathcal{A}\rangle}(g(\delta), g(\delta')), \tag{35}$$

for all $\delta, \delta' \in \Delta^{\mathcal{U}_{\langle \mathcal{T}_2, \mathcal{A}' \rangle}}$. This is sufficint to prove in the way analogious to the proof of Lemma C.6, that $\mathcal{U}_{\langle \mathcal{T}_2, \mathcal{A}' \rangle}$ is consistent. Using Claim A.3 one can obtain $\mathcal{U}_{\langle \mathcal{T}_2, \mathcal{A}' \rangle} \models q[\vec{a}]$. Finally, observe

$$g(a) = a, \tag{36}$$

for all a in $\mathsf{Ind}(\mathcal{A})$, $\vec{a}$, or $q(\vec{x})$; then using (34), (35), (36) in the same way as the proof of Claim A.4 one can show $\mathcal{U}_{\langle \mathcal{T}_2 \cup \mathcal{T}_{12}, \mathcal{A} \rangle} \models q[\vec{a}]$, which concludes the proof.

□

Now, given a Σ ABox $\mathcal{A}$ such that $\langle \mathcal{T}_1, \mathcal{A} \rangle$ is consistent, we show that $\langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A} \rangle$ is Ξ-query equivalent to $\langle \mathcal{T}_2 \cup \mathcal{T}_{12}, \mathcal{A} \rangle$ *if and only if* for every Ξ query $q(\vec{x})$ it holds

$$cert(q, \langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A} \rangle) = \bigcap_{\mathcal{A}' - \text{ solution for } \mathcal{A} \text{ under } \mathcal{T}_{12}} cert(q, \langle \mathcal{T}_2, \mathcal{A}' \rangle). \tag{37}$$

($\Rightarrow$) Let $q(\vec{x})$ be a Ξ query, it follows $cert(q, \langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A} \rangle) = cert(q, \langle \mathcal{T}_2 \cup \mathcal{T}_{12}, \mathcal{A} \rangle)$, and we easily obtain (37) using Proposition C.13. ($\Leftarrow$) Let $q(\vec{x})$ be a Ξ query, we need to show $cert(q, \langle \mathcal{T}_1 \cup \mathcal{T}_{12}, \mathcal{A} \rangle) = cert(q, \langle \mathcal{T}_2 \cup \mathcal{T}_{12}, \mathcal{A} \rangle)$, which is easily concluded using Proposition C.13 and (37). □

Lemma C.14 *The Ξ-TBox $\mathcal{T}_2$ is a* UCQ*-representation of Σ-TBox $\mathcal{T}_1$ under the mapping* $\mathcal{M} = (\Sigma, \Xi, \mathcal{T}_{12})$ if and only if *following conditions hold:*

(i) *for each pair of $\mathcal{T}_1$-consistent concepts B, B' over Σ, B, B' is $\mathcal{T}_1 \cup \mathcal{T}_{12}$-consistent iff B, B' is $\mathcal{T}_2 \cup \mathcal{T}_{12}$-consistent;*

(ii) *for each pair of $\mathcal{T}_1$-consistent roles R, R' over Σ, R, R' is $\mathcal{T}_1 \cup \mathcal{T}_{12}$-consistent iff R, R' is $\mathcal{T}_2 \cup \mathcal{T}_{12}$-consistent;*

(iii) *for each $\mathcal{T}_1 \cup \mathcal{T}_{12}$-consistent concept B over Σ and each B' over Σ_2, $\mathcal{T}_1 \cup \mathcal{T}_{12} \vdash B \sqsubseteq B'$ iff $\mathcal{T}_2 \cup \mathcal{T}_{12} \vdash B \sqsubseteq B'$;*

(iv) *for each $\mathcal{T}_1 \cup \mathcal{T}_{12}$-consistent role R over Σ and each R' over Σ_2, $\mathcal{T}_1 \cup \mathcal{T}_{12} \vdash R \sqsubseteq R'$ iff $\mathcal{T}_2 \cup \mathcal{T}_{12} \vdash R \sqsubseteq R'$;*

(v) *for each $B \in cons_{\mathcal{C}}(\mathcal{T}_1 \cup \mathcal{T}_{12})$ over Σ and each role R such that $o \rightsquigarrow_{\mathcal{S}_B} w_{[R]}$ there exists $y \in \Delta^{\mathcal{U}_{\mathcal{X}_B}}$ such that*

  (a) $\mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(ow_{[R]}) \subseteq \mathbf{t}^{\mathcal{U}_{\mathcal{X}_B}}(y)$*,*

  (b) $\mathbf{r}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o, ow_{[R]}) \subseteq \mathbf{r}^{\mathcal{U}_{\mathcal{X}_B}}(o, y)$*;*

(vi) *for each $B \in cons_{\mathcal{C}}(\mathcal{T}_1 \cup \mathcal{T}_{12})$ over Σ and each role R such that $o \rightsquigarrow_{\mathcal{X}_B} w_{[R]}$ there exists y such that $y \in \Delta^{\mathcal{U}_{\mathcal{S}_B}}$ and*

  (a) $\mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{X}_B}}(ow_{[R]}) \subseteq \mathbf{t}^{\mathcal{U}_{\mathcal{S}_B}}(y)$*,*

  (b) $\mathbf{r}_{\Xi}^{\mathcal{U}_{\mathcal{X}_B}}(o, ow_{[R]}) \subseteq \mathbf{r}^{\mathcal{U}_{\mathcal{S}_B}}(o, y)$

Proof. ($\Leftarrow$) Let the conditions above hold for $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_{12}$. Let $\mathcal{A}$ be an ABox over Σ such that $\langle \mathcal{T}_1, \mathcal{A} \rangle$ is consistent, we show $\mathcal{S}_{\mathcal{A}}$ is Ξ-query equivalent to $\mathcal{X}_{\mathcal{A}}$.

Observe that $\mathcal{S}_{\mathcal{A}}$ is consistent iff $\mathcal{X}_{\mathcal{A}}$ is consistent. Indeed, if $\mathcal{S}_{\mathcal{A}}$ is inconsistent then by Lemma C.6 one of the following holds:

(VII) B_1, B_2 is $\mathcal{T}_1 \cup \mathcal{T}_{12}$-inconsistent for some basic concepts B_1, B_2 and $a \in \mathsf{Ind}(\mathcal{A})$ such that $\mathcal{A} \models B_1(a)$, $\mathcal{A} \models B_2(a)$;

(VIII) R_1, R_2 is $\mathcal{T}_1 \cup \mathcal{T}_{12}$-inconsistent for some roles R_1, R_2 and $a, b \in \mathsf{Ind}(\mathcal{A})$ such that $\mathcal{A} \models R_1(a, b)$, $\mathcal{A} \models R_2(a, b)$

Consider **(VII)** and observe that by Lemma C.6 B_1, B_2 are $\mathcal{T}_1$ consistent. Then by **(i)** B_1, B_2 are $\mathcal{T}_2 \cup \mathcal{T}_{12}$-inconsistent and again by Lemma C.6 $\mathcal{X}_{\mathcal{A}}$ is inconsistent. The proof for the case of **(VIII)** is similar using **(ii)**. The proof can be inverted to show $\mathcal{X}_{\mathcal{A}}$ is inconsistent implies $\mathcal{S}_{\mathcal{A}}$ is inconsistent.

First, assume $\mathcal{S}_{\mathcal{A}}$ is inconsistent, it follows $\mathcal{S}_{\mathcal{A}} \models q[\vec{a}]$ for all $\vec{a} \subseteq \mathsf{Ind}(\mathcal{A})$ and Ξ-queries q. By the paragraph above, $\mathcal{X}_{\mathcal{A}}$ is inconsistent, so $\mathcal{X}_{\mathcal{A}} \models q[\vec{a}]$ for all $\vec{a} \subseteq \mathsf{Ind}(\mathcal{A})$ and Ξ-queries q, and so $\mathcal{S}_{\mathcal{A}}$ is Ξ-query equivalent to $\mathcal{X}_{\mathcal{A}}$.

Now assume $\mathcal{S}_{\mathcal{A}}$ is consistent, by Lemma C.7 each B is $\mathcal{T}_1 \cup \mathcal{T}_{12}$-consistent for all $\delta, \sigma \in \Delta^{\mathcal{U}_{\mathcal{S}_{\mathcal{A}}}}$, each B such that $B \in \mathbf{t}^{\mathcal{U}_{\mathcal{S}_{\mathcal{A}}}}(\delta)$, and each R such that $R \in \mathbf{r}^{\mathcal{U}_{\mathcal{S}_{\mathcal{A}}}}(\delta, \sigma)$. It follows from **(iii)**, **(iv)** and **(v)**, that all the conditions of Lemma C.10 are satisfied, therefore we conclude $\mathcal{U}_{\mathcal{S}_{\mathcal{A}}}$ is finitely Ξ-homomorphically embeddable into $\mathcal{U}_{\mathcal{X}_{\mathcal{A}}}$. Since $\mathcal{X}_{\mathcal{A}}$ is consistent, then we can apply Theorem A.4 to obtain $\mathcal{X}_{\mathcal{A}}$ Ξ-query entails $\mathcal{S}_\mathcal{A}$. On the other hand, **(iii)**, **(iv)** and **(vi)** imply that all the conditions of Lemma C.9 are satisfied, therefore we conclude $\mathcal{U}_{\mathcal{X}_\mathcal{A}}$ is finitely Ξ-homomorphically embeddable into $\mathcal{U}_{\mathcal{S}_\mathcal{A}}$ and $\mathcal{S}_\mathcal{A}$ Ξ-query entails $\mathcal{X}_\mathcal{A}$ by Theorem A.4. We again obtain $\mathcal{S}_\mathcal{A}$ is Ξ-query equivalent to $\mathcal{X}_\mathcal{A}$.

($\Rightarrow$) Assume, by contraction, one of the conditions **(i)** – **(vi)** is not satisfied. We produce a $\mathcal{T}_1$-consistent ABox $\mathcal{A}$ over Σ and a *instance* Ξ-query $q[]$ such that it is not the case that $\mathcal{S}_\mathcal{A} \models q$ iff $\mathcal{X}_\mathcal{A} \models q$.

Assume, first, the condition **(i)** is violated, then we take $\mathcal{A} = \{B_1(o), B_2(o)\}$ violating it and $q = B_1(a)$ for some constant $a \neq o$. If B_1, B_2 are $\mathcal{T}_1 \cup \mathcal{T}_{12}$-consistent, but $\mathcal{T}_2 \cup \mathcal{T}_{12}$ inconsistent, it follows $\mathcal{S}_\mathcal{A} \not\models q$ and $\mathcal{X}_\mathcal{A} \models q$, and the opposite holds if B_1, B_2 are $\mathcal{T}_2 \cup \mathcal{T}_{12}$-consistent, but $\mathcal{T}_1 \cup \mathcal{T}_{12}$-inconsistent. If **(ii)** is violated, the proof is analogous.

Let now the condition **(iii)** be violated for $B \in cons_\mathcal{C}(\mathcal{T}_1 \cup \mathcal{T}_{12})$ over Σ. Assume, first, there is $B' \in \mathbf{t}_\Xi^{\mathcal{U}_{\mathcal{S}_B}}(o) \setminus \mathbf{t}_\Xi^{\mathcal{U}_{\mathcal{X}_B}}(o)$, then we take $q = B'(o)$. By definition of $\mathcal{U}_{\mathcal{S}_B}, \mathcal{U}_{\mathcal{X}_B}$ and Lemma C.1 it follows $\mathcal{U}_{\mathcal{S}_B} \models q$ and $\mathcal{U}_{\mathcal{X}_B} \not\models q$; then by Claim A.3 it follows $\mathcal{S}_B \models q$ and $\mathcal{X}_B \not\models q$. The opposite follows if there exists $B' \in \mathbf{t}_\Xi^{\mathcal{U}_{\mathcal{X}_B}}(o) \setminus \mathbf{t}_\Xi^{\mathcal{U}_{\mathcal{S}_B}}(o)$, which completes the proof for this case. If **(iv)** is violated, the proof is analogios.

To prove the case when **(v)** is violated, we need an additional lemma below. Before we present it, notice that, w.l.o.g., one can consider UCQ's with atoms over *basic* concepts $B(t)$; one can convert such a UCQ into the one over the standard syntax by using fresh existentially quantified variables.

Lemma C.15 *Let $\mathcal{T}$ TBox, B a concept, $\vec{B}$ and $\vec{R}$ the sets of concepts and roles, respectively, and the instance query*

$$q_{\vec{B},\vec{R}} = \exists x \big(\bigwedge_{B' \in \vec{B}} B'(x) \wedge \bigwedge_{R' \in \vec{R}} R'(o,x) \big).$$

Then $\mathcal{U}_{\langle \mathcal{T}, \{B(o)\}\rangle} \models q_{\vec{B},\vec{R}}$ iff there exists $y \in \Delta^{\mathcal{U}_{\langle \mathcal{T}, \{B(o)\}\rangle}}$, such that

(i) $\vec{B} \subseteq \mathbf{t}^{\mathcal{U}_{\langle \mathcal{T}, \{B(o)\}\rangle}}(y)$,

(ii) $\vec{R} \subseteq \mathbf{r}^{\mathcal{U}_{\langle \mathcal{T}, \{B(o)\}\rangle}}(o, y)$.

Proof. Straightforward using Lemma C.4 and the definition of $\mathcal{U}_{\langle \mathcal{T}, \{B(o)\}\rangle}$. □

Now, assume **(v)** is violated, so there exists $B \in cons_\mathcal{C}(\mathcal{T}_1 \cup \mathcal{T}_{12})$ over Σ and a role R such that $o \rightsquigarrow_{\mathcal{S}_B} w_{[R]}$ and for all $y \in \Delta^{\mathcal{U}_{\mathcal{X}_B}}$ either $\mathbf{t}_\Xi^{\mathcal{U}_{\mathcal{S}_B}}(ow_{[R]}) \not\subseteq \mathbf{t}^{\mathcal{U}_{\mathcal{X}_B}}(y)$ or $\mathbf{r}_\Xi^{\mathcal{U}_{\mathcal{S}_B}}(o, w_{[R]}) \not\subseteq \mathbf{r}^{\mathcal{U}_{\mathcal{X}_B}}(o, y)$. Then, by Lemma C.15 with $\vec{B} = \mathbf{t}_\Xi^{\mathcal{U}_{\mathcal{S}_B}}(ow_{[R]})$, $\vec{R} = \mathbf{r}_\Xi^{\mathcal{U}_{\mathcal{S}_B}}(o, ow_{[R]})$ and $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_{12}$ it follows $\mathcal{U}_{\mathcal{S}_B} \models q_{\vec{B},\vec{R}}$. On the other hand, by Lemma C.15 with $\mathcal{T} = \mathcal{T}_2 \cup \mathcal{T}_{12}$ it follows $\mathcal{U}_{\mathcal{X}_B} \not\models q_{\vec{B},\vec{R}}$. Using Claim A.3 we then obtain $\mathcal{S}_B \models q_{\vec{B},\vec{R}}$ and $\mathcal{X}_B \not\models q_{\vec{B},\vec{R}}$.

The case when **(vi)** is violated is analogous to the case above. The proof is complete.

□

Theorem C.16 *The membership problem for* UCQ*-representability is* NLOGSPACE*-complete.*

Proof. The lower bound can be obtained by the reduction from the directed graph reachability problem, which is known to be NLogSpace-hard: given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and a pair of vertices $v_k, v_m \in \mathcal{V}$, decide if there is a directed path from v_k to v_m. To encode the problem, we need a set of Σ concept names $\{V_i \mid v_i \in \mathcal{V}\}$ and a set of Ξ concept names $\{V_i' \mid v_i \in \mathcal{V}\}$. Consider $\mathcal{T}_1 = \{V_k \sqsubseteq V_m\} \cup \{V_i \sqsubseteq V_j \mid (v_i, v_j) \in \mathcal{E}\}$, $\mathcal{T}_{12} = \{V_i \sqsubseteq V_i' \mid v_i \in \mathcal{V}\}$, and $\mathcal{T}_2 = \{V_i' \sqsubseteq V_j' \mid (v_i, v_j) \in \mathcal{E}\}$. One can easily verify that the condition **(iii)** of Lemma C.14 is satisfied iff there is a directed path from v_k to v_m in $\mathcal{G}$, whereas the other conditions of Lemma C.14 are satisfied trivially. Therefore,

Proposition C.17 *There is a directed path from v_k to v_m in $\mathcal{G}$ iff $\mathcal{T}_2$ is a representation for $\mathcal{T}_1$ under $\mathcal{M} = (\Sigma, \Xi, \mathcal{T}_{12})$.*

This concludes the proof of the lower bound. For the upper bound, we show that the conditions **(i)**–**(vi)** of Lemma C.14 can be verified in NLOGSPACE. It is well known (see, e.g., [Artale *et al.*, 2009]), that given a pair of *DL-Lite*$_\mathcal{R}$ concepts B, B', and a TBox $\mathcal{T}$, it can be verified in NLOGSPACE, if B, B' is $\mathcal{T}$ consistent (using an algorithm, based on directed graph reachability solving procedure); the same holds for a pair of *DL-Lite*$_\mathcal{R}$ roles R, R'. The same algorithm can be straightforwardly adopted to check, if $\mathcal{T} \vdash B \sqsubseteq B'$ or $\mathcal{T} \vdash R \sqsubseteq R'$. Therefore, clearly, the conditions **(i)**–**(iv)** can be verified in NLOGSPACE.

The conditions **(v)** and **(vi)** are slightly more involved; first of all, observe that, given a concept B and a role R, it can be checked in NLOGSPACE, whether $o \rightsquigarrow_{\langle \mathcal{T}, \{B(o)\}\rangle} w_{[R]}$, using an algorithm based on the directed graph reachability solving procedure. At the same time, given $z \in \{o\} \cup \{w_{[R]} \mid R - \text{ role}\}$, we can verify, if there exists $y \in \Delta^{\mathcal{U}_{\langle \mathcal{T}, \{B(o)\}\rangle}}$ with $z = \mathsf{tail}(y)$: we "follow" the sequence of roles $R_1, \dots, R_n = R$ (with $n \geq 0$) in the way that when we "guess" R_{i+1}, we check $w_{[R_i]} \rightsquigarrow_{\langle \mathcal{T}, \{B(o)\}\rangle} w_{[R_{i+1}]}$ (by the algorithm, similar to the one for checking $o \rightsquigarrow_{\langle \mathcal{T}, \{B(o)\}\rangle} w_{[R]}$), and "forget" R_i.

Furthermore, in a similar way, as testing $\mathcal{T} \vdash B \sqsubseteq B'$, one can, check for a concept B', if $B' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\langle \mathcal{T}, \{B(o)\}\rangle}}(ow_{[R]})$ in NLogSpace; the same holds for checking if a role $R' \in \mathbf{r}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o, ow_{[R]})$, and, then, for checking $B' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\langle \mathcal{T}, \{B(o)\}\rangle}}(y)$, for y as above. By combining the algorithms outlined above, one can produce a procedure that checks the conditions **(v)** and **(vi)** in NLogSpace. $\square$

D Non-emptyness Problem for UCQ-representability

The definitions that follow are needed for the non-emptyness problem of UCQ-representability. Let the mapping $\mathcal{M} = (\Sigma, \Xi, \mathcal{T}_{12})$, $\mathcal{T}_1$ and $\mathcal{T}_2$ TBoxes over, respectively, Σ and Ξ. For a pair of concepts B', C' be over Ξ, we say that $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is *closed under inclusion between B' and C'* if the following is satisfied for each $\mathcal{T}_1$-consistent concept B over Σ:

(IX) $\mathcal{T}_1 \cup \mathcal{T}_{12} \vdash B \sqsubseteq B'$ implies $\mathcal{T}_1 \cup \mathcal{T}_{12} \vdash B \sqsubseteq C'$;

(X) if $B' = \exists Q'$, then $\exists Q'^- \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o)$ implies $o \rightsquigarrow_{\mathcal{S}_B} w_{[Q]}$ for some role Q such that $Q'^- \in \mathbf{r}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o, ow_{[Q]})$ and $C' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(ow_{[Q]})$.

Then, for a pair R', Q' of roles over Ξ, we say $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is *closed under inclusion between R' and Q'* if the following is satisfied:

(XI) $\mathcal{T}_1 \cup \mathcal{T}_{12} \vdash R \sqsubseteq R'$ implies $\mathcal{T}_1 \cup \mathcal{T}_{12} \vdash R \sqsubseteq Q'$ for each $\mathcal{T}_1$-consistent role R over Σ;

(XII) $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under inclusion between $\exists R'$ and $\exists Q'$;

(XIII) $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under inclusion between $\exists R'^-$ and $\exists Q'^-$.

Next, we say $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is *closed under disjointness between* B' and C' if the following is satisfied:

(XIV) for each $\mathcal{T}_1 \cup \mathcal{T}_{12}$-consistent pair of concepts B, C over Σ it is *not the case* $\mathcal{T}_1 \cup \mathcal{T}_{12} \vdash B \sqsubseteq B'$ and $\mathcal{T}_1 \cup \mathcal{T}_{12} \vdash C \sqsubseteq C'$;

(XV) for each $\mathcal{T}_1 \cup \mathcal{T}_{12}$-consistent concept B over Σ_1 and each role R such that $o \rightsquigarrow_{\mathcal{S}_B} w_{[R]}$ it is *not the case* $B', C' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(ow_{[R]})$.

Then, $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is *closed under disjointness between R' and Q'* if the following is satisfied:

(XVI) for each $\mathcal{T}_1 \cup \mathcal{T}_{12}$-consistent pair of roles R, Q over Σ it is *not the case* $\mathcal{T}_1 \cup \mathcal{T}_{12} \vdash R \sqsubseteq R'$ and $\mathcal{T}_1 \cup \mathcal{T}_{12} \vdash Q \sqsubseteq Q'$;

(XVII) for each $\mathcal{T}_1 \cup \mathcal{T}_{12}$-consistent concept B over Σ_1 and each role R such that $o \rightsquigarrow_{\mathcal{S}_B} w_{[R]}$ it is *neither the case $R', Q' \in \mathbf{r}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o, ow_{[R]})$ nor $R'^-, Q'^- \in \mathbf{r}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o, ow_{[R]})$*

Define a *generating pass for a concept* B over Σ as a pair $\pi = (\langle C_0, C_1, \dots C_n\rangle, L)$, where $\langle C_0, C_1, \dots C_n\rangle$ a is tuple of concepts of the length greater or equal 1, $C_0 = B$, and for each $1 \leq i \leq n$ it holds $C_i = \exists Q_i^-$ for some role Q_i; then L is a labeling function

$$L : C_i \cup C_i \times C_j \mapsto 2^{\Xi\text{ - concepts}} \cup 2^{\Xi\text{ -roles}}$$

such that $L(C_i, C_j) = \emptyset$ for $j \neq i + 1$. It is said that a generating pass π for B is *conform with* $\mathcal{T}_1 \cup \mathcal{T}_{12}$ if the following is satisfied:

(XVIII) $\exists Q \in L(C_i)$ or $\exists Q = C_i$ for all $0 \leq i < n$ and roles Q such that $C_{i+1} = \exists Q^-$;

(XIX) For each $0 \leq i \leq n$ and $B' \in L(C_i)$ there exists C' over Ξ such that $\mathcal{T}_{12} \vdash C_i \sqsubseteq C'$ and $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under inclusion between C' and B'.

(XX) For each $0 \leq i < n$, role Q such that $C_{i+1} = \exists Q^-$ and $R' \in L(C_i, C_{i+1})$ there exists Q' over Ξ such that $\mathcal{T}_{12} \vdash Q \sqsubseteq Q'$ and $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under inclusion between Q' and R'.

D.1 Basic Preliminary Results

Lemma D.1 *Let $\mathcal{M} = (\Sigma, \Xi, \mathcal{T}_{12})$ be a mapping, and a Σ-TBox $\mathcal{T}_2$ be is a representation for a Ξ-TBox $\mathcal{T}_1$ under $\mathcal{T}_{12}$. Then $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under:*

(i) *inclusion between concepts B' and C' (roles R' and Q') for all B', C' over Ξ (R', Q' over Ξ) such that $\mathcal{T}_2 \vdash B' \sqsubseteq C'$ ($\mathcal{T}_2 \vdash R' \sqsubseteq Q'$);*

(ii) *disjointness between concepts B' and C' (roles R' and Q') for all B', C' over Ξ (R', Q' over Ξ) such that $\mathcal{T}_2 \vdash B' \sqsubseteq D'$, $\mathcal{T}_2 \vdash C' \sqsubseteq E'$, and $(D' \sqcap E' \sqsubseteq \bot) \in \mathcal{T}_2$ for some concepts D', E' over Ξ ($\mathcal{T}_2 \vdash R' \sqsubseteq S'$, $\mathcal{T}_2 \vdash Q' \sqsubseteq T'$, and $(S' \sqcap T' \sqsubseteq \bot) \in \mathcal{T}_2$ for some roles S', T' over Ξ);*

(iii) *disjointness between B' and B' (R' and R') for all $\mathcal{T}_2$ inconsistent concepts B' (roles R').*

Proof. We assume that $\mathcal{T}_2$ is a representation, but **(i)**, **(ii)** or **(iii)** is violated, and derive a contradiction. Let, first, **(i)** be violated for concepts, i.e., there are B', C' over Ξ such that $\mathcal{T}_2 \vdash B' \sqsubseteq C'$ and $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is *not* closed under inclusion between B' and C'. Then, **(IX)** or **(X)** must be violated for some $B \in cons_{\mathcal{C}}(\mathcal{T}_1 \cup \mathcal{T}_{12})$ over Σ. Assume **(IX)** is violated, i.e., $B' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o)$ and $C' \notin \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o)$. By Lemma C.14 **(iii)** we get the contradiction. If **(X)** is violated, i.e., $B' = \exists Q'$, $\exists Q'^- \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o)$, and for all roles Q such that $o \rightsquigarrow_{\mathcal{S}_B} w_{[Q]}$ and $Q'^- \in \mathbf{r}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o, ow_{[Q]})$ it holds $C' \notin \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o)$. By Lemma C.14 **(iii)** obtain $\exists Q'^- \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{X}_B}}(o)$ and since $\mathcal{T}_2 \vdash B' \sqsubseteq C'$ it follows there exists a role Q such that $o \rightsquigarrow_{\mathcal{X}_B} w_{[Q]}$, $Q'^- \in \mathbf{r}_{\Xi}^{\mathcal{U}_{\mathcal{X}_B}}(o, ow_{[Q]})$, and $C' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{X}_B}}(ow_{[Q]})$. Using **(vi)** we obtain a contraction.

Suppose there are roles R', Q' over Ξ such that $\mathcal{T}_2 \vdash R' \sqsubseteq Q'$ and $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is *not* closed under inclusion between R' and Q', then one of **(XI)**, **(XII)**, **(XIII)** is violated. Assume it is **(XI)**, then $\mathcal{T}_1 \cup \mathcal{T}_{12} \vdash R \sqsubseteq R'$ and it is not the case $\mathcal{T}_1 \cup \mathcal{T}_{12} \vdash R \sqsubseteq Q'$ for some $R \in cons_{\mathcal{R}}(\mathcal{T}_1 \cup \mathcal{T}_{12})$ over Σ. Using Lemma C.14 **(iv)** we get the contradiction. Assume **(XII)** is violated, then there is $B \in cons_{\mathcal{C}}(\mathcal{T}_1 \cup \mathcal{T}_{12})$ over Σ such that $\exists R'^- \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o)$ and for all roles Q such that $o \rightsquigarrow_{\mathcal{S}_B} w_{[Q]}$ and $R'^- \in \mathbf{r}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o, ow_{[Q]})$ it holds $\exists Q' \notin \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o)$. By Lemma C.14 **(iii)** obtain $\exists R'^- \in \mathbf{s}_{\Xi}^{\mathcal{U}_{\mathcal{X}_B}}(o)$ and since $\mathcal{T}_2 \vdash R' \sqsubseteq Q'$ it follows there exists a role Q such that $o \rightsquigarrow_{\mathcal{X}_B} w_{[Q]}$, $R'^- \in \mathbf{r}_{\Xi}^{\mathcal{U}_{\mathcal{X}_B}}(o, ow_{[Q]})$, and $\exists Q' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{X}_B}}(ow_{[Q]})$. Using **(vi)** we obtain a contraction. The proof when **(XIII)** is violated is analogous.

Let now **(ii)** be violated for concepts, i.e., there are B', C', D', E' over Ξ such that $\mathcal{T}_2 \vdash B' \sqsubseteq D'$, $\mathcal{T}_2 \vdash C' \sqsubseteq E'$, $(D' \sqcap E' \sqsubseteq \bot) \in \mathcal{T}_2$ and $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is *not* closed under disjointness between B' and C'. Then, **(XIV)** or **(XV)** must be violated. If it is **(XIV)** there is a $\mathcal{T}_1 \cup \mathcal{T}_{12}$-consistent pair of concepts B, C such that $B' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o)$ and $C' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_C}}(o)$. By Lemma C.14 **(iii)** obtain $B' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{X}_B}}(o)$ and $C' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{X}_C}}(o)$, then by $\mathcal{T}_2 \vdash B' \sqsubseteq D'$, $\mathcal{T}_2 \vdash C' \sqsubseteq E'$ and $(D' \sqcap E' \sqsubseteq \bot) \in \mathcal{T}_2$ it follows the pair B, C is $\mathcal{T}_2 \cup \mathcal{T}_{12}$ inconsistent. We obtained a contradiction to Lemma C.14 **(i)**. If **(XV)** is violated there is $B \in cons_{\mathcal{C}}(\mathcal{T}_1 \cup \mathcal{T}_{12})$ over Σ and a role R such that $o \rightsquigarrow_{\mathcal{S}_B} w_{[R]}$ and $B', C' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(ow_{[R]})$. By Lemma C.14 **(v)** there is $y \in \Delta^{\mathcal{U}_{\mathcal{X}_B}}$ such that $B', C' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{X}_B}}(y)$. Using Lemmas C.4, C.7, $\mathcal{T}_2 \vdash B' \sqsubseteq D'$, $\mathcal{T}_2 \vdash C' \sqsubseteq E'$ and $(D' \sqcap E' \sqsubseteq \bot) \in \mathcal{T}_2$ it follows B is $\mathcal{T}_2 \cup \mathcal{T}_{12}$ inconsistent, which is a contradiction to $B \in cons_{\mathcal{C}}(\mathcal{T}_1 \cup \mathcal{T}_{12})$ over Σ by Lemma C.14 **(i)**. The proof of the case when **(ii)** is violated for roles is analogous to the case of concepts above.

Finally, assume **(iii)** is violated for concepts, i.e., there is $\mathcal{T}_2$ inconsistent B' such that $\mathcal{T}_2 \cup \mathcal{T}_{12}$ is not closed under inclusion between B' and B'. It follows **(XIV)** or **(XV)** must be violated. If it is **(XIV)** there is a $\mathcal{T}_1 \cup \mathcal{T}_{12}$-consistent pair of concepts B, C such that $B' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o)$ and $B' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_C}}(o)$. By Lemma C.14 **(iii)** obtain $B' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{X}_B}}(o)$ and $B' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{X}_C}}(o)$, then by Lemmas C.4 and C.7 we obtain that the pair B, C is $\mathcal{T}_2 \cup \mathcal{T}_{12}$-inconsistent. We obtained a contradiction to Lemma C.14 **(i)**. If **(XV)** is violated there is $B \in cons_{\mathcal{C}}(\mathcal{T}_1 \cup \mathcal{T}_{12})$ over Σ and a role R such that $o \rightsquigarrow_{\mathcal{S}_B} w_{[R]}$ and $B' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(ow_{[R]})$. By Lemma C.14 **(v)** there is $y \in \Delta^{\mathcal{U}_{\mathcal{X}_B}}$ such that $B' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{X}_B}}(y)$. Using Lemmas C.4 and C.7 it follows B is $\mathcal{T}_2 \cup \mathcal{T}_{12}$-inconsistent, which is a contradiction to $B \in cons_{\mathcal{C}}(\mathcal{T}_1 \cup \mathcal{T}_{12})$ over Σ by Lemma C.14 **(i)**. The proof of the case when **(iii)** is violated for roles is analogous to the case of concepts above. □

D.2 Proof of Proposition 6.3

The result is shown in Theorem D.4; we need a series of lemmas before we present the proof.

Lemma D.2 *Given a mapping $\mathcal{M} = (\Sigma, \Xi, \mathcal{T}_{12})$ and a Σ-TBox $\mathcal{T}_1$, there exists Ξ-TBox $\mathcal{T}_2$, such that it is a* UCQ*-representation of $\mathcal{T}_1$ under $\mathcal{M}$, if and only if the following conditions are satisfied:*

(i) *For each $\mathcal{T}_1 \cup \mathcal{T}_{12}$-consistent concept B over Σ and each B' over Ξ such that $\mathcal{T}_1 \cup \mathcal{T}_{12} \vdash B \sqsubseteq B'$ there exists C' over Ξ such that $\mathcal{T}_{12} \vdash B \sqsubseteq C'$ and $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under the inclusion between C' and B'.*

(ii) *For each $\mathcal{T}_1 \cup \mathcal{T}_{12}$-consistent role R over Σ and each R' over Ξ such that $\mathcal{T}_1 \cup \mathcal{T}_{12} \vdash R \sqsubseteq R'$ there exists Q' over Ξ such that $\mathcal{T}_{12} \vdash R \sqsubseteq Q'$ and $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under inclusion between Q' and R'.*

(iii) *For each $\mathcal{T}_1 \cup \mathcal{T}_{12}$-consistent concept B over Σ and each role R such that $o \rightsquigarrow_{\mathcal{S}_B} w_{[R]}$ there exists a generating pass $\pi = (\langle C_0, \dots C_n \rangle, L)$ for B conform with $\mathcal{T}_1 \cup \mathcal{T}_{12}$, such that:*

- *(a)* $\mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(ow_{[R]}) \subseteq L(C_n)$.
- *(b)* $\mathbf{r}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o, ow_{[R]}) \subseteq L(C_0, C_n)$;

(iv) *For each $\mathcal{T}_1$-consistent pair of concepts B_1, B_2 over Σ, such that B_1, B_2 is $\mathcal{T}_1 \cup \mathcal{T}_{12}$-inconsistent, there are concepts B, C such that one of the following holds:*

- *(a)* *$B, C \in \{B_1, B_2\}$ and one of the following holds:*
  - *(1) $\mathcal{T}_{12} \vdash B \sqsubseteq B'$, $\mathcal{T}_{12} \vdash C \sqsubseteq C'$, and $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under the disjointness between B' and C';*
  - *(2) $\mathcal{T}_{12} \vdash B \sqsubseteq B'$, $\mathcal{T}_{12} \ni (C \sqcap C' \sqsubseteq \bot)$, and $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under inclusion between B' and C'.*
- *(b)* *$\exists R \in \{B_1, B_2\}$ and one of the following holds:*
  - *(1) $\mathcal{T}_{12} \vdash \exists R^- \sqsubseteq B'$, $\mathcal{T}_{12} \vdash \exists R^- \sqsubseteq C'$, and $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under disjointness between B' and C';*
  - *(2) $\mathcal{T}_{12} \vdash \exists R^- \sqsubseteq B'$, $\mathcal{T}_{12} \ni (\exists R^- \sqcap C' \sqsubseteq \bot)$, and $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under inclusion between B' and C';*
  - *(3) $\mathcal{T}_{12} \vdash R \sqsubseteq R'$, $\mathcal{T}_{12} \vdash R \sqsubseteq Q'$, and $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under the disjointness between R' and Q';*
  - *(4) $\mathcal{T}_{12} \vdash R \sqsubseteq R'$, $\mathcal{T}_{12} \ni (R \sqcap Q' \sqsubseteq \bot)$, and $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under inclusion between R' and Q'.*

(v) *For all $\mathcal{T}_1$-consistent pairs of roles R_1, R_2, such that R_1, R_2 is $\mathcal{T}_1 \cup \mathcal{T}_{12}$-inconsistent one of the following holds:*

- *(a)* *there are roles $R, Q \in \{R_1, R_2\}$ and R', Q' over Ξ such that one of the following holds:*
  - *(1) $\mathcal{T}_{12} \vdash R \sqsubseteq R'$, $\mathcal{T}_{12} \vdash Q \sqsubseteq Q'$, and $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under disjointness between R' and Q';*
  - *(2) $\mathcal{T}_{12} \vdash R \sqsubseteq R'$, $\mathcal{T}_{12} \ni (Q \sqcap Q' \sqsubseteq \bot)$, and $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under inclusion between R' and Q';*
- *(b)* *there exist $B, C \in \{\exists R_1, \exists R_2\}$ or $B, C \in \{\exists R_1^-, \exists R_2^-\}$ such that one of the following holds:*
  - *(1) $\mathcal{T}_{12} \vdash B \sqsubseteq B'$, $\mathcal{T}_{12} \vdash C \sqsubseteq C'$, and $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under disjointness between B' and C';*
  - *(2) $\mathcal{T}_{12} \vdash B \sqsubseteq B'$, $\mathcal{T}_{12} \ni (C \sqcap C' \sqsubseteq \bot)$, and $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under inclusion between B' and C'.*

Proof. ($\Leftarrow$) Assume the conditions **(i)** – **(v)** are satisfied, we construct a TBox $\mathcal{T}_2$ and prove it is a UCQ-representation for $\mathcal{T}_1$ under $\mathcal{M}$. The required $\mathcal{T}_2$ will be given as the union of the fives sets of axioms presented below. First, take $B \in cons_{\mathcal{C}}(\mathcal{T}_1 \cup \mathcal{T}_{12})$ over Σ, $B' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o)$, then let $ax_1(B, B') = \{C' \sqsubseteq B'\}$ for C' given by the condition **(i)**. For $R \in cons_{\mathcal{R}}(\mathcal{T}_1 \cup \mathcal{T}_{12})$ over Σ and R' over Ξ, such that $\mathcal{T}_1 \cup \mathcal{T}_{12} \vdash R \sqsubseteq R'$, define $ax_2(R, R') = \{Q' \sqsubseteq R'\}$ for Q' given by the condition **(ii)**. For each $B \in cons_{\mathcal{C}}(\mathcal{T}_1 \cup \mathcal{T}_{12})$ over Σ and each role R such that $o \rightsquigarrow_{\mathcal{S}_B} w_{[R]}$ define the set $ax_3(B, R)$ from the generating pass $\langle C_0, \dots, C_n \rangle$ for B conform with $\mathcal{T}_1 \cup \mathcal{T}_{12}$ that satisfies **(iii)**. Take $ax_3(B, R)$ equal to the set of all axioms $C' \sqsubseteq B'$ satisfying **(XIX)** and all axioms $Q' \sqsubseteq R'$ satisfying **(XX)**. Now let $B_, B_2$ be a $\mathcal{T}_1$-consistent and $\mathcal{T}_1 \cup \mathcal{T}_{12}$-inconsistent pair of Σ concepts, then define a set $ax_4(B_1, B_2)$ to be equal to $\{B' \sqcap C' \sqsubseteq \bot\}$ for the corresponding B' and C', if **(iv)(a)**(1) or **(iv)(b)**(1) is satisfied; and $\{R' \sqcap Q' \sqsubseteq \bot\}$ for the corresponding R' and Q', if **(iv)(b)**(3) is satisfied. On the other hand, define $ax_4(B_1, B_2)$ to be equal to $\{B' \sqsubseteq C'\}$ for the corresponding B' and C', if **(iv)(a)**(2) or **(iv)(b)**(2) is satisfied; and $\{R' \sqsubseteq Q'\}$ for the corresponding R' and Q', if **(iv)(b)**(4) is satisfied. Finally, we define $ax_5(R_1, R_2)$ for $\mathcal{T}_1$-consistent and $\mathcal{T}_1 \cup \mathcal{T}_{12}$-inconsistent pair of Σ roles R_1, R_2 analogously to $ax_4(B_1, B_2)$ using the conditions **(v)(b)** and **(v)(b)**. Finally we have:

$$\mathcal{T}_2 = \bigcup_{\substack{B \in cons_{\mathcal{C}}(\mathcal{T}_1 \cup \mathcal{T}_{12}) \text{ over } \Sigma, \\ B' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o)}} ax_3(B, B') \cup \bigcup_{\substack{R \in cons_{\mathcal{R}}(\mathcal{T}_1 \cup \mathcal{T}_{12}) \text{ over } \Sigma, \\ R' \text{ over } \Xi,\ \mathcal{T}_1 \cup \mathcal{T}_{12} \vdash R \sqsubseteq R'}} ax_4(R, R') \cup$$
$$\bigcup_{\substack{B \in cons_{\mathcal{C}}(\mathcal{T}_1 \cup \mathcal{T}_{12}) \text{ over } \Sigma, \\ o \rightsquigarrow_{\mathcal{S}_B} w_{[R]}}} ax_5(B, R) \cup \bigcup_{\substack{B_0, B_1 \text{ conc. over } \Sigma, \\ \mathcal{T}_1 - \text{consist. and} \\ \mathcal{T}_1 \cup \mathcal{T}_{12} - \text{inconsist.}}} ax_1(B_0, B_1) \cup \bigcup_{\substack{R_0, R_1 \text{ roles over } \Sigma, \\ \mathcal{T}_1 - \text{consist. and} \\ \mathcal{T}_1 \cup \mathcal{T}_{12} - \text{inconsist.}}} ax_2(R_0, R_1)$$

We need the following intermediate result:

Lemma D.3 *For all concepts $B', C' \in \Xi$ (roles R', Q' over Ξ), if $\mathcal{T}_2 \vdash B' \sqsubseteq C'$ ($\mathcal{T}_2 \vdash R' \sqsubseteq Q'$) then $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under inclusion between B' and C' (R' and Q').*

Proof. Notice that for all concepts B' and C' (roles R' and Q') such that $(B' \sqsubseteq C') \in \mathcal{T}_2$ ($(R' \sqsubseteq Q') \in \mathcal{T}_2$) it holds $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under inclusion between B' and C' (R' and Q'). First we prove the statement of the lemma for roles, if $\mathcal{T}_2 \vdash R' \sqsubseteq Q'$ there is a sequence of roles $Q_1, \ldots, Q_n$ such that $Q_1 = R'$, $Q_n = Q'$, and for each $1 \leq i < n$ one of the following holds:

(XXI) $(Q_i \sqsubseteq Q_{i+1}) \in \mathcal{T}_2$

(XXII) $(Q_i^- \sqsubseteq Q_{i+1}^-) \in \mathcal{T}_2$

We show $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under inclusion between R' and Q_i by induction on i. For $i = 1$ the proof is trivial, assume $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under inclusion between R' and Q_i, we show now its closure under inclusion between R' and Q_{i+1}. Let, first, **(XXI)**, we show **(XI)**. Assume $\mathcal{T}_1 \cup \mathcal{T}_{12} \vdash R \sqsubseteq R'$ for some $R \in cons_{\mathcal{R}}(\mathcal{T}_1 \cup \mathcal{T}_{12})$ over Σ, since $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under inclusion between R' and Q_i, it follows by **(XI)** $\mathcal{T}_1 \cup \mathcal{T}_{12} \vdash R \sqsubseteq Q_i$, then, again by closure under inclusion between Q_i and Q_{i+1} obtain $\mathcal{T}_1 \cup \mathcal{T}_{12} \vdash R \sqsubseteq Q_{i+1}$.

To show **(XII)** we need to prove **(IX)** and **(X)** for $B' = \exists R'$ and $C' = \exists Q_{i+1}$. For **(IX)** assume $\exists R' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o)$ for some $B \in cons_{\mathcal{C}}(\mathcal{T}_1 \cup \mathcal{T}_{12})$ over Σ; since $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under inclusion between R' and Q_i, it follows by **(XII)** that $\exists Q_i \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o)$, and again by closure under inclusion between Q_i and Q_{i+1} obtain $\exists Q_{i+1} \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o)$. For **(X)** assume $\exists R'^- \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o)$ for some $B \in cons_{\mathcal{C}}(\mathcal{T}_1 \cup \mathcal{T}_{12})$ over Σ and consider two cases: $R' = Q_i$ and $R' \neq Q_i$. In the first case $o \rightsquigarrow_{\mathcal{S}_B} w_{[Q]}$ for some role Q such that $R'^- \in \mathbf{r}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o, ow_{[Q]})$ and $\exists Q_{i+1} \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(ow_{[Q]})$ immediately follows, since $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under inclusion between Q_i and Q_{i+1}.

Assume $R' \neq Q_i$, since $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under inclusion between R' and Q_i, it follows by **(XII)** and the structure of $\mathcal{S}_B$ that $o \rightsquigarrow_{\mathcal{S}_B} w_{[Q]}$ for some role Q over Σ such that $R'^- \in \mathbf{r}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o, ow_{[Q]})$ and $\exists Q_i \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(ow_{[Q]})$. Since $\mathcal{S}_B$ is consistent, it can be easily shown that $\exists Q^- \in cons_{\mathcal{C}}(\mathcal{T}_1 \cup \mathcal{T}_{12})$. Observe now that $\exists Q_i \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_{\exists Q^-}}}(o)$; then since $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closured under inclusion between Q_i and Q_{i+1} and **(XIII)**, obtain $\exists Q_{i+1} \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_{\exists Q^-}}}(o)$. Finally, it follows $\exists Q_{i+1} \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(ow_{[Q]})$, which completes the proof for the case **(XXI)**. The proof for the case **(XXII)** is analogous.

To prove the lemma for concepts we exploit that $\mathcal{T}_2 \vdash B' \sqsubseteq C'$ implies there exists a sequence of Ξ concepts $B_1, \ldots, B_n$ such that $B_1 = B'$, $B_n = C'$, and for each $1 \leq i < n$ one of the following holds:

(XXIII) $(B_i \sqsubseteq B_{i+1}) \in \mathcal{T}_2$

(XXIV) $B_i = \exists R'$, $B_{i+1} = \exists Q'$, $(R' \sqsubseteq Q') \in \mathcal{T}_2$

(XXV) $B_i = \exists R'^-$, $B_{i+1} = \exists Q'^-$, $(R' \sqsubseteq Q') \in \mathcal{T}_2$

We show $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under inclusion between B' and B_i by induction on i. For $i = 1$ the proof is trivial, assume $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under inclusion between B' and B_i, we show now its closure under inclusion between B' and B_{i+1}. First we consider the case of B_i and B_{i+1} are as in **(XXIII)**. To show **(IX)** assume $B \in cons_{\mathcal{C}}(\mathcal{T}_1 \cup \mathcal{T}_{12})$ over Σ and $B' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o)$; by closure under inclusion between B' and B_i and **(IX)** it follows $B_i \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o)$, then by closure under inclusion between B_i and B_{i+1} and **(IX)** obtain $B_{i+1} \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o)$.

To show **(X)** assume $B' = \exists Q'$ and $\exists Q'^- \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o)$. If $B' = B_i$, then $o \rightsquigarrow_{\mathcal{S}_B} w_{[Q]}$ for some role Q such that $Q' \in \mathbf{r}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o, ow_{[Q]})$ and $B_{i+1} \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(ow_{[Q]})$ by closure under inclusion between B_i and B_{i+1}, and **(X)**. On the other hand, if $B' \neq B_i$, it follows by closure under inclusion between B' and B_i, and **(X)** $o \rightsquigarrow_{\mathcal{S}_B} w_{[Q]}$ for a role Q over Σ such that $Q' \in \mathbf{r}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o, ow_{[Q]})$ and $B_i \in \mathbf{t}_{\Xi}^{\mathcal{S}_B}(ow_{[Q]})$. Since $\mathcal{S}_B$ is consistent, it follows $\exists R^- \in cons_{\mathcal{C}}(\mathcal{T}_1 \cup \mathcal{T}_{12})$, then by closure under inclusion between B_i and B_{i+1} and **(IX)** we conclude $B_{i+1} \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(ow_{[R]})$, which concludes the proof. The proof for the cases **(XXIV)** and **(XXV)** is analogios. $\square$

We return to the proof of ($\Leftarrow$) of Lemma D.2; we prove $\mathcal{T}_2$ above is a representation of $\mathcal{T}_1$ under $\mathcal{T}_{12}$ by showing the conditions **(i)** – **(vi)** of Lemma C.14 are satisfied. We start from **(iii)** (consistency conditions will be shown in the end.) Let $B \in cons_{\mathcal{C}}(\mathcal{T}_1 \cup \mathcal{T}_{12})$ over Σ, $B' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o)$, then $B' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{X}_B}}(o)$ follows straightforwardly from current **(i)**. Assume now some $B' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{X}_B}}(o)$, it follows $\mathcal{T}_{12} \vdash B \sqsubseteq C'$

and $\mathcal{T}_2 \vdash C' \sqsubseteq B'$ for some concept C' over Ξ; by Lemma D.3 it follows $C' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o)$ implies $B' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o)$, and since $\mathcal{T}_{12} \vdash B \sqsubseteq C'$ conclude $B' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o)$. The proof that **(iv)** of Lemma D.2 is satisfied is analogous to the proof that **(iii)** is satisfied above, using current **(ii)** and Lemma D.3.

The **(v)** of Lemma D.2 follows straightforwardly from current **(iii)**, the definition of a Ξ pass conform with $\mathcal{T}_1 \cup \mathcal{T}_{12}$, and the structure of $\mathcal{T}_2$. To show **(vi)** of Lemma D.2 assume $B \in cons_{\mathcal{C}}(\mathcal{T}_1 \cup \mathcal{T}_{12})$ over Σ and a role Q such that $o \rightsquigarrow_{\mathcal{X}_B} w_{[Q]}$. We first consider the case Q over Ξ, by the structure of $\mathcal{T}_2$ (see the proof that **(iii)** of Lemma C.14 is satisfied) it follows $\exists Q \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o)$. If $\mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{X}_B}}(w_{[Q]}) = \{\exists Q^-\}$, the proof is done; otherwise, $\mathcal{T}_2 \vdash \exists Q^- \sqsubseteq C'$ for some $C' \neq \exists Q^-$, then by Lemma D.3 and **(X)** it follows there exists R such that $o \rightsquigarrow_{\mathcal{S}_B} w_{[R]}$, $Q \in \mathbf{r}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o, ow_{[R]})$ and $C' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o)$; also by $C' \neq \exists Q^-$ and the structure of $\mathcal{S}_B$ it follows R is over Σ. Notice that $\exists R^- \in cons_{\mathcal{C}}(\mathcal{T}_1 \cup \mathcal{T}_{12})$ since $\mathcal{S}_B$ is consistent. For each (other) $C' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{X}_B}}(ow_{[Q]})$ we show $C' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(w_{[R]})$. Indeed, it follows $\mathcal{T}_2 \vdash \exists Q^- \sqsubseteq C'$; then using Lemma D.3, **(IX)**, $\exists Q^- \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_{\exists R^-}}}(o)$, we can conclude $C' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_{\exists R^-}}}(o)$, and so $C' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(ow_{[R]})$. To show $\mathbf{r}_{\Xi}^{\mathcal{U}_{\mathcal{X}_B}}(o, ow_{[Q]}) \subseteq \mathbf{r}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o, ow_{[R]})$ consider that $R \in cons_{\mathcal{R}}(\mathcal{T}_1 \cup \mathcal{T}_{12})$ and assume $Q' \in \mathbf{r}_{\Xi}^{\mathcal{U}_{\mathcal{X}_B}}(o, ow_{[Q]})$; it follows $\mathcal{T}_2 \vdash Q \sqsubseteq Q'$, then by Lemma D.3, **(XI)** and $\mathcal{T}_1 \cup \mathcal{T}_{12} \vdash R \sqsubseteq Q$ it follows $Q' \in \mathbf{r}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o, ow_{[R]})$, which concludes the proof.

Consider now the case Q over Σ, then, clearly, $o \rightsquigarrow_{\mathcal{S}_B} w_{[R]}$ and $Q \in \mathbf{r}^{\mathcal{U}_{\mathcal{S}_B}}(o, ow_{[R]})$ for some role R over Σ. We show now $\mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{X}_B}}(ow_{[Q]}) \subseteq \mathbf{t}^{\mathcal{U}_{\mathcal{S}_B}}(ow_{[R]})$: let $C' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{X}_B}}(ow_{[Q]})$, then $\mathcal{T}_{12} \vdash \exists Q^- \sqsubseteq B'$ and $\mathcal{T}_2 \vdash B' \sqsubseteq C'$ for some B' over Ξ. It follows $B' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(ow_{[R]})$, then by Lemma D.3 and **(IX)** obtain $C' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(ow_{[R]})$. The proof of $\mathbf{r}_{\Xi}^{\mathcal{U}_{\mathcal{X}_B}}(o, ow_{[Q]}) \subseteq \mathbf{r}^{\mathcal{U}_{\mathcal{S}_B}}(o, ow_{[R]})$ is analogous.

Now we show that the consistency conditions of Lemma C.14 are satisfied. For **(i)** assume a pair B_1, B_2 of $\mathcal{T}_1$ consistent and $\mathcal{T}_1 \cup \mathcal{T}_{12}$-inconsistent concepts; then B_1, B_2 is $\mathcal{T}_2 \cup \mathcal{T}_{12}$-inconsistent follows easily from current **(iv)** and definition of $\mathcal{T}_2$. Assume B_1, B_2 are $\mathcal{T}_1$ consistent and $\mathcal{T}_2 \cup \mathcal{T}_{12}$-inconsistent; it follows there exists $\delta, \sigma \in \Delta^{\mathcal{X}_{\{B_1(o), B_2(o)\}}}$ such that one of the following holds:

(XXVI) There are concepts $C, C' \in \mathbf{t}^{\mathcal{U}_{\mathcal{X}_{\{B_1(o), B_2(o)\}}}}(\delta)$ such that $(C \sqcap C' \sqsubseteq \bot) \in \mathcal{T}_2 \cup \mathcal{T}_{12}$;

(XXVII) There are roles $Q, Q' \in \mathbf{r}^{\mathcal{U}_{\mathcal{X}_{\{B_1(o), B_2(o)\}}}}(\delta, \sigma)$ such that $(Q \sqcap Q' \sqsubseteq \bot) \in \mathcal{T}_2 \cup \mathcal{T}_{12}$.

Assume for the sake of contradiction that B_1, B_2 is $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is consistent. By Lemma C.7 it follows for each $\delta, \sigma \in \Delta^{\mathcal{U}_{\mathcal{S}_{\{B_1(o), B_2(o)\}}}}$, every $B \in \mathbf{t}_{\Sigma}^{\mathcal{U}_{\mathcal{S}_{\{B_1(o), B_2(o)\}}}}(\delta)$ and $R \in \mathbf{r}_{\Sigma}^{\mathcal{U}_{\mathcal{S}_{\{B_1(o), B_2(o)\}}}}(\delta, \sigma)$ are $\mathcal{T}_1 \cup \mathcal{T}_{12}$-consistent. By the structure of $\mathcal{T}_2$ for all such B and R the conditions **(iii)** – **(v)** of Lemma C.10 are satisfied (see the proof that **(iii)**, **(iv)** and **(vi)** of Lemma C.14 are satisfied above). Then, by Lemma C.10 we have that there exist $\delta, \sigma \in \Delta^{\mathcal{S}_{\{B_1(o), B_2(o)\}}}$ such that one of the following holds:

(XXVIII) There are concepts $C, C' \in \mathbf{t}^{\mathcal{U}_{\mathcal{S}_{\{B_1(o), B_2(o)\}}}}(\delta)$ such that $(C \sqcap C' \sqsubseteq \bot) \in \mathcal{T}_2 \cup \mathcal{T}_{12}$;

(XXIX) There are roles $Q, Q' \in \mathbf{r}^{\mathcal{U}_{\mathcal{S}_{\{B_1(o), B_2(o)\}}}}(\delta, \sigma)$ such that $(Q \sqcap Q' \sqsubseteq \bot) \in \mathcal{T}_2 \cup \mathcal{T}_{12}$.

Assume **(XXVIII)** and observe that w.l.o.g. C' is over Ξ, whereas $C \in \Sigma \cup \Xi$. If $C \in \Sigma$ it follows $(C \sqcap C') \in \mathcal{T}_{12}$ and we immediately have the contradiction to the fact that B_1, B_2 is $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is consistent. So let C be over Ξ, it follows $(C \sqcap C') \in \mathcal{T}_2$, and $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under disjointness between C and C' by the definition of $\mathcal{T}_2$. Consider, first, the case $\delta \neq o$: by Lemma C.1 $C, C' \in \mathbf{s}_{\Xi}^{\mathcal{U}_{\mathcal{S}_{\exists Q^-}}}(o)$ for the role Q such that $\mathsf{tail}(\delta) = w_{[Q]}$. If Q is over Σ we derive the contradiction because $\exists Q^-$ is $\mathcal{T}_1 \cup \mathcal{T}_{12}$-consistent and **(XIV)**. On the other hand, if Q is over Ξ, it can be seen by the structure of $\mathcal{U}_{\mathcal{S}_{\{B_1(o), B_2(o)\}}}$ that $o \rightsquigarrow_{\mathcal{S}_B} w_{[Q]}$ for some $B \in cons_{\mathcal{C}}(\mathcal{T}_1 \cup \mathcal{T}_{12})$ over Σ; then $C, C' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(ow_{[Q]})$ and we derive the contradiction because of **(XV)**. Finally, consider the case $\delta = o$, then by the structure of $\mathcal{U}_{\mathcal{S}_{\{B_1(o), B_2(o)\}}}$ there are concepts $B, D \in \{B_1, B_2\}$ such that $C \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o)$ and $C' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_D}}(o)$. By **(XIV)** we again have a contradiction.

Assume **(XXIX)**, then again, assuming Q is over Σ produces an immediate contradiction; if, however, Q is over Ξ, we obtain by the definition of $\mathcal{T}_2$, that $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under disjointness between Q and Q'. By the structure of $\mathcal{U}_{\mathcal{S}_{\{B_1(o), B_2(o)\}}}$ we need to consider two cases: $\sigma = \delta w_{[R]}$, $Q, Q' \in \mathbf{r}^{\mathcal{U}_{\mathcal{S}_{\{B_1(o), B_2(o)\}}}}(\delta, \sigma)$ and $\delta = \sigma w_{[R]}$, $Q^-, Q'^- \in \mathbf{r}^{\mathcal{U}_{\mathcal{S}_{\{B_1(o), B_2(o)\}}}}(\sigma, \delta)$. In the first case, $o \rightsquigarrow_{\mathcal{S}_B} w_{[R]}$ for some $B \in cons_{\mathcal{C}}(\mathcal{T}_1 \cup \mathcal{T}_{12})$ over Σ and $Q, Q' \in \mathbf{r}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o, ow_{[R]})$; using **(XVII)** we derive the contradiction. The second case is proved analogously using **(XVII)**.

Thus, assuming the pair B_1, B_2 is $\mathcal{T}_1 \cup \mathcal{T}_{12}$-consistent produces a contradiction, therefore B_1, B_2 is $\mathcal{T}_1 \cup \mathcal{T}_{12}$ inconsistent. This concludes the proof that **(i)** of Lemma D.2 is satisfied. Analogously, using **(v)**, Lemma C.10, **(XIV)**, **(XV)**, **(XVI)**, **(XVII)**, it can be shown that **(ii)** of Lemma D.2 is satisfied, which concludes the proof ($\Leftarrow$) of Lemma D.2.

($\Rightarrow$) Assume $\mathcal{T}_2$ is a representation for $\mathcal{T}_1$ under $\mathcal{T}_{12}$, we show that **(iv)** – **(iii)** are satisfied. For **(iv)** assume a $\mathcal{T}_1$-consistent pair of concepts B_1, B_2, such that B_1, B_2 is $\mathcal{T}_1 \cup \mathcal{T}_{12}$-inconsistent; it follows Lemma by C.14 **(i)** that $\mathcal{X}_{\{B_1(o), B_2(o)\}}$ is inconsistent. Then, one of the following holds:

(XXX) There are concepts $C, C' \in \mathbf{t}^{\mathcal{U}_{\mathcal{X}_{\{B_1(o),B_2(o)\}}}}(\delta)$ such that $(C \sqcap C' \sqsubseteq \bot) \in \mathcal{T}_2 \cup \mathcal{T}_{12}$;

(XXXI) There are roles $Q, Q' \in \mathbf{r}^{\mathcal{U}_{\mathcal{X}_{\{B_1(o),B_2(o)\}}}}(\delta, \sigma)$ such that $(Q \sqcap Q' \sqsubseteq \bot) \in \mathcal{T}_2 \cup \mathcal{T}_{12}$.

Assume **(XXX)** is the case and notice that w.l.o.g. we can assume C' is over Ξ and C is over $\Sigma \cup \Xi$. Let, first, $\delta = o$, by the structure of $\mathcal{X}_{\{B_1(o), B_2(o)\}}$ it follows there are $B \in \{B_1, B_2\}$ and B' is over Ξ such that $\mathcal{T}_{12} \vdash B \sqsubseteq B'$ and $\mathcal{T}_2 \vdash B' \sqsubseteq C'$. Suppose C is over Ξ, then it follows $(C \sqcap C' \sqsubseteq \bot) \in \mathcal{T}_2$ and, again, there are $D \in \{B_1, B_2\}$ and D' is over Ξ such that $\mathcal{T}_{12} \vdash D \sqsubseteq D'$ and $\mathcal{T}_2 \vdash D' \sqsubseteq C$. By Lemma D.1 **(ii)** it follows $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under disjointness between B' and D', so **(iv)(a)**(1) is satisfied. Suppose C is over Σ, then $(C \sqcap C' \sqsubseteq \bot) \in \mathcal{T}_{12}$, and by the structure of $\mathcal{X}_{\{B_1(o), B_2(o)\}}$ it follows $C \in \{B_1, B_2\}$. By Lemma D.1 **(ii)** it follows $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under inclusion between B' and C', so **(iv)(a)**(2) is satisfied. Consider now the case $\mathsf{tail}(\delta) = w_{[R]}$ for $R \in \Sigma$; by the structure of $\mathcal{X}_{\{B_1(o), B_2(o)\}}$ it follows $\exists R \in \{B_1, B_2\}$ and by Lemma C.1 $\mathcal{T}_2 \cup \mathcal{T}_{12} \vdash \exists R^- \sqsubseteq C$, $\mathcal{T}_2 \cup \mathcal{T}_{12} \vdash \exists R^- \sqsubseteq C'$. Now we can repeat the argument above with $B = D = \exists R^-$ to conclude either that either **(iv)(b)**(1) or (2) is satisfied.

Finally, consider the case $\mathsf{tail}(\delta) = w_{[R']}$ with R' over Ξ. By Lemma C.1 it is the case $\mathcal{T}_2 \vdash \exists R'^- \sqsubseteq C$, $\mathcal{T}_2 \vdash \exists R'^- \sqsubseteq C'$. If $o \rightsquigarrow_{\mathcal{X}_{\{B_1(o),B_2(o)\}}} w_{[R']}$, then by the structure of $\mathcal{X}_{\{B_1(o), B_2(o)\}}$ it follows $\mathcal{T}_{12} \vdash B \sqsubseteq B'$ and $o \rightsquigarrow_{\langle \mathcal{T}_2, \{B'(o)\}\rangle} w_{[R']}$ for some $B \in \{B_1, B_2\}$, B' over Ξ; also by Lemmas C.4 and C.7 it follows the concept B' is $\mathcal{T}_2$ inconsistent. Since by Lemma D.1 **(iii)** $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under the disjointness between B' and B', it follows **(iv)(a)**(1) is satisfied. If it is not the case $o \rightsquigarrow_{\mathcal{X}_{\{B_1(o),B_2(o)\}}} w_{[R']}$, it follows $\exists R \in \{B_1, B_2\}$ and $\mathcal{T}_{12} \vdash \exists R^- \sqsubseteq B'$ for some B' over Ξ, such that there is $\sigma \in \Delta^{\mathcal{U}_{\langle \mathcal{T}_2, \{B'(o)\}\rangle}}$ with $\mathsf{tail}(\sigma) = w_{[R']}$. Again, by Lemmas C.4 and C.7 B' is $\mathcal{T}_2$ inconsistent, then by Lemma D.1 **(iii)** $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under disjointness between B' and B', so **(iv)(b)**(1) is satisfied.

Assume **(XXXI)** is the case and notice that w.l.o.g. we can assume Q' is over Ξ and Q is over $\Sigma \cup \Xi$. By the structure of $\mathcal{U}_{\mathcal{X}_{\{B_1(o),B_2(o)\}}}$ we need to consider two cases: $\sigma = \delta w_{[R]}$, $Q, Q' \in \mathbf{r}^{\mathcal{U}_{\mathcal{X}_{\{B_1(o),B_2(o)\}}}}(\delta, \sigma)$ and $\delta = \sigma w_{[R]}$, $Q^-, Q'^- \in \mathbf{r}^{\mathcal{U}_{\mathcal{X}_{\{B_1(o),B_2(o)\}}}}(\sigma, \delta)$. We show only the first case, the second case is analogous. Assume $\sigma = o$, $o \rightsquigarrow_{\mathcal{X}_{\{B_1(o),B_2(o)\}}} w_{[R]}$ for R over Σ, and Q over Ξ. It follows $\mathcal{T}_{12} \vdash R \sqsubseteq R'$ and $\mathcal{T}_2 \vdash R' \sqsubseteq Q'$, and also $\mathcal{T}_{12} \vdash R \sqsubseteq S$ and $\mathcal{T}_2 \vdash S \sqsubseteq Q'$ for some R', S over Ξ. Since $(Q \sqcap Q' \sqsubseteq \bot) \in \mathcal{T}_2$ by Lemma D.1**(ii)** we get $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under inclusion between R' and S, so **(iv)(b)**(3) is satisfied. Let $Q \in \Sigma$, it follows $o \rightsquigarrow_{\mathcal{X}_{\{B_1(o),B_2(o)\}}} w_{[R]}$ and $R = Q$. It follows also $\mathcal{T}_{12} \vdash Q \sqsubseteq R'$ and $\mathcal{T}_2 \vdash R' \sqsubseteq Q'$, then by Lemma D.1**(i)** we get $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under inclusion between R' and Q', so, since $(Q \sqcap Q' \sqsubseteq \bot) \in \mathcal{T}_{12}$, we conclude **(iv)(b)**(4) is satisfied. Consider now the case $o \rightsquigarrow_{\mathcal{X}_{\{B_1(o),B_2(o)\}}} w_{[R]}$ for R over Ξ, which implies Q is over Ξ and $(Q \sqcap Q' \sqsubseteq \bot) \in \mathcal{T}_2$; then $\mathcal{T}_{12} \vdash B \sqsubseteq B'$ and $\mathcal{T}_2 \vdash B' \sqsubseteq \exists R$ for some concepts $B \in \{B_1, B_2\}$ and B' over Ξ. It follows $o \rightsquigarrow_{\langle \mathcal{T}_2, \{B'(o)\}\rangle} w_{[R]}$, then by Lemmas C.4 and C.7 B' is $\mathcal{T}_2$ inconsistent, then by Lemma D.1 **(iii)** $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under disjointness between B' and B', so **(iv)(b)**(1) is satisfied. This concludes the proof for the case $\sigma = o$.

Assume $\mathsf{tail}(\sigma) = w_{[R']}$ for R' over Σ, this implies $o \rightsquigarrow_{\mathcal{X}_B\{B_1(o),B_2(o)\}} w_{[R']}$, and we lead the proof analogously to the case above to show there is a $\mathcal{T}_2$ inconsistent B' such that $\mathcal{T}_{12} \vdash \exists R^- \sqsubseteq B'$ and **(iv)(b)**(3) is satisfied. If $\mathsf{tail}(\sigma) = w_{[R']}$ for R' over Ξ it can be easily verified **(iv)(b)**(1) or **(iv)(b)**(3) is satisfied. This concludes the proof that **(iv)** is satisfied; then **(v)** can be shown analogously.

To show **(i)** is satisfied assume $B \in cons_{\mathcal{C}}(\mathcal{T}_1 \cup \mathcal{T}_{12})$ over Σ and $B' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o)$. By Lemma C.14 **(iii)** it follows $B' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{X}_B}}(o)$, so there exists C' over Ξ such that $\mathcal{T}_{12} \vdash B \sqsubseteq C'$ and $\mathcal{T}_2 \vdash C' \sqsubseteq B'$. By Lemma D.1 **(i)** it follows $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under inclusion between C' and B'; then **(ii)** can be shown analogously.

Finally, we show **(iii)** is satisfied; assume $B \in cons_{\mathcal{C}}(\mathcal{T}_1 \cup \mathcal{T}_{12})$ over Σ and $o \rightsquigarrow_{\mathcal{S}_B} w_{[R]}$ for some role R, by Lemma C.14 **(v)** it follows there exists $y \in \Delta^{\mathcal{U}_{\mathcal{X}_B}}$ such that $\mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(ow_{[R]}) \subseteq \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{X}_B}}(y)$, and $\mathbf{r}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o, ow_{[R]}) \subseteq \mathbf{r}_{\Xi}^{\mathcal{U}_{\mathcal{X}_B}}(o, y)$. By the structure of $\mathcal{X}_B$ it follows there exists a sequence of concepts $\langle C_0, \ldots, C_n \rangle = \langle B, \exists Q_1^-, \ldots, \exists Q_n^- \rangle$ such that $\mathcal{T}_2 \cup \mathcal{T}_{12} \vdash C_i \sqsubseteq \exists Q$ for all $0 \leq i < n$ and roles Q such that $C_{i+1} = \exists Q^-$, $\mathcal{T}_2 \cup \mathcal{T}_{12} \vdash C_n \sqsubseteq B'$ for all $B' \in \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(ow_{[R]})$, and $\mathbf{r}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o, ow_{[R]}) \neq \emptyset$ implies $n = 1$ and $\mathcal{T}_2 \cup \mathcal{T}_{12} \vdash Q \sqsubseteq R'$ for all $R' \in \mathbf{r}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o, ow_{[R]})$ and Q such that $C_1 = \exists Q^-$. We define a generating pass for B conform with $\mathcal{T}_1 \cup \mathcal{T}_{12}$ as follows: $L(C_n) = \mathbf{s}_{\Xi}^{\mathcal{S}_B}(w_{[R]})$, $L(C_1, C_n) = \mathbf{q}_{\Xi}^{\mathcal{S}_B}(o, w_{[R]})$, $L(C_i) = \{\exists Q \mid C_{i+1} = \exists Q^-, B \neq \exists Q\}$ for all $0 \leq i < n$, and $L(C_i, C_j) = \emptyset$ for $j \neq i+1$. It can be straightforwardly verified that **(XVIII)** holds, then also **(XIX)** and **(XX)** follow using Lemma D.1. We have shown **(iii)** is satisfied, which concludes the proof ($\Rightarrow$) of Lemma D.2. $\square$

Theorem D.4 *The non-emptyness problem for* UCQ*-representability is* NLOGSPACE*-complete.*

Proof. As in the case of Theorem C.16, the lower bound is shown by the reduction from the directed graph reachability problem, however, we need a slightly more involved encoding.

Lemma D.5 *The non-emptyness problem for* UCQ*-representability is* NLOGSPACE*-hard.*

Proof. To encode the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, we need a set of Σ-concept names $\{V_i \mid v_i \in \mathcal{V}\} \cup \{S, F, X, Y\}$ and a set of Ξ-concept names $\{V_i' \mid v_i \in \mathcal{V}\} \cup \{S', X', Y'\}$. Consider the TBox

$$\mathcal{T}_1 = \{V_i \sqsubseteq V_j \mid (v_i, v_j) \in \mathcal{E}\} \cup \{S \sqsubseteq V_k, V_m \sqsubseteq F, X \sqsubseteq Y\},$$

where v_k and v_m are, respectively, the initial and final vertices. Then, let

$$\mathcal{T}_{12} = \{V_i \sqsubseteq V_i' \mid v_i \in \mathcal{V}\} \cup \{S \sqsubseteq S', S \sqsubseteq X', F \sqsubseteq Y', X \sqsubseteq X', Y \sqsubseteq Y'\};$$

we will show:

Proposition D.6 *There is a directed path from v_k to v_m in $\mathcal{G}$ iff there exists a representation for $\mathcal{T}_1$ under $\mathcal{M} = (\Sigma, \Xi, \mathcal{T}_{12})$.*

Indeed, using Lemma D.2, there exists a representation iff the condition **(i)** is satisfied. By the structure of $\mathcal{T}_1 \cup \mathcal{T}_{12}$ one can see that it is the case iff $\mathcal{T}_1 \cup \mathcal{T}_{12}$ is closed under the inclusion between X' and Y'. The latter is the case iff $\mathcal{T}_1 \cup \mathcal{T}_{12} \vdash S \sqsubseteq X'$ implies $\mathcal{T}_1 \cup \mathcal{T}_{12} \vdash S \sqsubseteq Y'$, and that holds iff $\mathcal{T}_1 \vdash S \sqsubseteq F$, which is the case iff there exists a path from v_k to v_m in $\mathcal{G}$. This completes the proof of Lemma D.4. $\square$

To show the upper bound, we prove that the conditions **(i)**–**(v)** of Lemma D.2 can be checked in NLOGSPACE. In fact, these conditions can be checked using the algorithm, based on directed graph reachability solving procedure, similar to the proof of Theorem C.16. The only new case is the condition **(iii)**; to verify that there exists a generating pass $\pi = (\langle C_0, \dots C_n \rangle, L)$ for a concept B conform with $\mathcal{T}_1 \cup \mathcal{T}_{12}$, we can use the following procedure, running in NLOGSPACE. First, we take $C_0 = B$ and decide, if the pass ends here (i.e., $n = 1$). If we decided so, it only remains to take $L(C_0) = \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(ow_{[R]})$, for $\mathcal{S}_B$ and R as in the condition **(iii)**, and verify **(XIX)**. This verification can be performed in NLOGSPACE, similarly to the method described in the proof of Theorem C.16. If, on the other hand, we decide, that the pass continues, we "guess" $C_1 = \exists Q^-$ for some role Q, and verify that for some $L(C_0) \subseteq \{\exists Q\}$ the **(XVIII)** and **(XIX)** are satisfied. Now, if we decide that the pass stops, it remains to take $L(C_1) = \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(ow_{[R]})$ and $L(C_0, C_1) = \mathbf{r}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(o, ow_{[R]})$, for $\mathcal{S}_B$ and R as in the condition **(iii)**, and verify **(XIX)** and **(XX)**. If, on the contrary, we decide that the pass continues, we can "forget" C_0, "guess" C_2, and proceed with it in the same way, as we did with C_1. Finally, when we reach the concept C_n, such that the algorithm decides to stop, it remains to verify **(XIX)** for $L(C_n) = \mathbf{t}_{\Xi}^{\mathcal{U}_{\mathcal{S}_B}}(ow_{[R]})$. It should be clear that whenever the generating pass $\pi = (\langle C_0, \dots C_n \rangle, L)$ for a concept B conform with $\mathcal{T}_1 \cup \mathcal{T}_{12}$ exists, we can find it by the above non-determinictic procedure. $\square$