Logic

Foundations of Propositional Logic

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Knowledge bases

Inference engine	$\longleftarrow$	domain-independent algorithms
Knowledge base	$\longleftarrow$	domain-specific content

- Knowledge base = set of *sentences* in a *formal* language = logical *theory*
- *Declarative* approach to building an intelligent agent:

  TELL it what it needs to know
- Then it can ASK itself what to do—answers should follow from the KB
- Agents can be viewed at the *knowledge level*

  i.e., what they know, regardless of how implemented
- Or at the *implementation level*

  i.e., data structures in KB and algorithms that manipulate them

Logic in general

- *Logics* are formal languages for representing information such that conclusions can be drawn
- *Syntax* defines the sentences in the language
- *Semantics* define the "meaning" of sentences; i.e., define *truth* of a sentence in a world
- E.g., the language of arithmetic

  $x + 2 \geq y$ is a sentence; $x2 + y >$ is not a sentence

  $x + 2 \geq y$ is true iff the number $x + 2$ is no less than the number y

  $x + 2 \geq y$ is true in a world where $x = 7,\ y = 1$

  $x + 2 \geq y$ is false in a world where $x = 0,\ y = 6$

  $x + 2 \geq x + 1$ is true in every world

The *one and only* Logic?

- Logics of higher order
- Modal logics
  - epistemic
  - temporal and spatial
  - ...
- Description logic
- Non-monotonic logic
- Intuitionistic logic
- ...

But: There are “standard approaches”

$\leadsto$ propositional and predicate logic

Types of logic

- Logics are characterized by what they commit to as "primitives"
- Ontological commitment: what exists—facts? objects? time? beliefs?
- Epistemological commitment: what states of knowledge?

Language	Ontological Commitment (What exists in the world)	Epistemological Commitment (What an agent believes about facts)
Propositional logic	facts	true/false/unknown
First-order logic	facts, objects, relations	true/false/unknown
Temporal logic	facts, objects, relations, times	true/false/unknown
Probability theory	facts	degree of belief 0…1
Fuzzy logic	degree of truth	degree of belief 0…1

Classical logics are based on the notion of TRUTH

Entailment – Logical Implication

$$\boxed{KB \models \alpha}$$

- Knowledge base KB *entails* sentence α

  if and only if

  α is true in all worlds where KB is true

- E.g., the KB containing "Manchester United won" and "Manchester City won" entails "Either Manchester United won or Manchester City won"

Models

- Logicians typically think in terms of *models*, which are formally *structured worlds* with respect to which truth can be evaluated
- We say m is a *model* of a sentence α if α is true in m
- $M(\alpha)$ is the set of all models of α
- Then $KB \models \alpha$ if and only if $M(KB) \subseteq M(\alpha)$
- E.g. KB = United won and City won

  α = City won

  or

  α = Manchester won

  or

  α = either City or Manchester won

Inference – Deduction – Reasoning

$$\boxed{KB \vdash_i \alpha}$$

- $KB \vdash_i \alpha$ = sentence α can be derived from KB by **procedure** i
- *Soundness*: i is sound if

  whenever $KB \vdash_i \alpha$, it is also true that $KB \models \alpha$
- *Completeness*: i is complete if

  whenever $KB \models \alpha$, it is also true that $KB \vdash_i \alpha$
- We will define a logic (first-order logic) which is expressive enough to say almost anything of interest, and for which there exists a sound and complete inference procedure.

Propositional Logics: Basic Ideas

Statements:

The elementary building blocks of propositional logic are *atomic statements* that cannot be decomposed any further: *propositions*. E.g.,

- “The block is red”
- “The proof of the pudding is in the eating”
- “It is raining”

and logical connectives “and”, “or”, “not”, by which we can build **propositional formulas**.

Propositional Logics: Reasoning

We are interested in the questions:

- when is a statement **logically implied** by a set of statements, in symbols: $\Theta \models \phi$
- can we define **deduction** in such a way that deduction and entailment coincide?

Syntax of Propositional Logic

Countable alphabet Σ of **atomic propositions**: $a, b, c, \ldots$.

Propositional formulas:

$$
\begin{array}{rcll}
\phi, \psi & \longrightarrow & a & \textit{atomic formula} \\
 & \mid & \bot & \textit{false} \\
 & \mid & \top & \textit{true} \\
 & \mid & \neg\phi & \textit{negation} \\
 & \mid & \phi \wedge \psi & \textit{conjunction} \\
 & \mid & \phi \vee \psi & \textit{disjunction} \\
 & \mid & \phi \rightarrow \psi & \textit{implication} \\
 & \mid & \phi \leftrightarrow \psi & \textit{equivalence}
\end{array}
$$

- **Atom:** atomic formula
- **Literal:** (negated) atomic formula
- **Clause:** disjunction of literals

Semantics: Intuition

- Atomic statements can be *true* $\mathtt{T}$ or *false* $\mathtt{F}$.
- The truth value of formulas is determined by the truth values of the atoms (*truth value assignment* or *interpretation*).

Example: $(a \vee b) \wedge c$

- If a and b are wrong and c is true, then the formula is not true.
- Then *logical entailment* could be defined as follows:
- ϕ is implied by Θ, if ϕ is true in all “states of the world”, in which Θ is true.

Semantics: Formally

A **truth value assignment** (or **interpretation**) of the atoms in Σ is a function $\mathcal{I}$:

$$\mathcal{I}\colon \Sigma \to \{\mathtt{T}, \mathtt{F}\}.$$

Instead of $\mathcal{I}(a)$ we also write $a^{\mathcal{I}}$.

A formula ϕ is *satisfied* by an interpretation $\mathcal{I}$ ($\mathcal{I} \models \phi$) or is *true* under $\mathcal{I}$:

$$
\begin{array}{rcl}
\mathcal{I} \models \top & & \\
\mathcal{I} \not\models \bot & & \\
\mathcal{I} \models a & \text{iff} & a^{\mathcal{I}} = \mathtt{T} \\
\mathcal{I} \models \neg\phi & \text{iff} & \mathcal{I} \not\models \phi \\
\mathcal{I} \models \phi \wedge \psi & \text{iff} & \mathcal{I} \models \phi \text{ and } \mathcal{I} \models \psi \\
\mathcal{I} \models \phi \vee \psi & \text{iff} & \mathcal{I} \models \phi \text{ or } \mathcal{I} \models \psi
\end{array}
$$

$$
\begin{array}{rcl}
\mathcal{I} \models \phi \to \psi & \text{iff} & \text{if } \mathcal{I} \models \phi, \text{ then } \mathcal{I} \models \psi \\
\mathcal{I} \models \phi \leftrightarrow \psi & \text{iff} & \mathcal{I} \models \phi, \text{ if and only if } \mathcal{I} \models \psi
\end{array}
$$

Example

$$\mathcal{I}\colon \begin{cases} a & \mapsto & \mathtt{T} \\ b & \mapsto & \mathtt{F} \\ c & \mapsto & \mathtt{F} \\ d & \mapsto & \mathtt{T} \\ & \vdots & \end{cases}$$

$$((a \vee b) \leftrightarrow (c \vee d)) \wedge (\neg(a \wedge b) \vee (c \wedge \neg d)).$$

Exercise

- Find an interpretation and a formula such that the formula is true in that interpretation (or: the interpretation satisfies the formula).
- Find an interpretation and a formula such that the formula is not true in that interpretation (or: the interpretation does not satisfy the formula).
- Find a formula which can't be true in any interpretation (or: no interpretation can satisfy the formula).

Satisfiability and Validity

An interpretation $\mathcal{I}$ is a **model** of ϕ:

$$\mathcal{I} \models \phi$$

A formula ϕ is

- **satisfiable**, if there is some $\mathcal{I}$ that satisfies ϕ,
- **unsatisfiable**, if ϕ is not satisfiable,
- **falsifiable**, if there is some $\mathcal{I}$ that does not satisfy ϕ,
- **valid** (i.e., a **tautology**), if every $\mathcal{I}$ is a model of ϕ.

Two formulas are **logically equivalent** ($\phi \equiv \psi$), if for all $\mathcal{I}$:

$$\mathcal{I} \models \phi \text{ iff } \mathcal{I} \models \psi$$

Exercise

Satisfiable, tautology?

$$(((a \wedge b) \leftrightarrow a) \rightarrow b)$$

$$((\neg\phi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \phi))$$

$$(a \vee b \vee \neg c) \wedge (\neg a \vee \neg b \vee d) \wedge (\neg a \vee b \vee \neg d)$$

Equivalent?

$$\begin{aligned} (\phi \vee (\psi \wedge \chi)) &\equiv ((\phi \vee \psi) \wedge (\psi \wedge \chi)) \\ \neg(\phi \vee \psi) &\equiv \neg\phi \wedge \neg\psi \end{aligned}$$

Consequences

Proposition:

- ϕ is a tautology iff $\neg\phi$ is unsatisfiable
- ϕ is unsatisfiable iff $\neg\phi$ is a tautology.

Proposition: $\phi \equiv \psi$ iff $\phi \leftrightarrow \psi$ is a tautology.

Theorem: If ϕ and ψ are equivalent, and χ' results from replacing ϕ in χ by ψ, then χ and χ' are equivalent.

Entailment

Extension of the entailment relationship to sets of formulas Θ:

$$\mathcal{I} \models \Theta \quad \text{iff} \quad \mathcal{I} \models \phi \ \text{ for all } \ \phi \in \Theta$$

Remember: we want the formula ϕ to be implied by a set Θ, if ϕ is true in all models of Θ (symbolically, $\Theta \models \phi$):

$$\Theta \models \phi \quad \text{iff} \quad \mathcal{I} \models \phi \ \text{ for all models } \mathcal{I} \ \text{ of } \Theta$$

Propositional inference: Enumeration method

Let $\alpha = A \vee B$ and $KB = (A \vee C) \wedge (B \vee \neg C)$

Is it the case that $KB \models \alpha$?

Check all possible models – α must be true wherever KB is true

A	B	C	$A \vee C$	$B \vee \neg C$	KB	α
$False$	$False$	$False$				
$False$	$False$	$True$				
$False$	$True$	$False$				
$False$	$True$	$True$				
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$True$	$False$	$False$	$True$	$True$		
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$False$	$True$	$True$	$True$	$True$	$True$	
$True$	$False$	$False$	$True$	$True$	$True$	
$True$	$False$	$True$	$True$	$False$	$False$	
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$False$	$True$	$False$	$False$	$True$	$False$	$True$
$False$	$True$	$True$	$True$	$True$	$True$	$True$
$True$	$False$	$False$	$True$	$True$	$True$	$True$
$True$	$False$	$True$	$True$	$False$	$False$	$True$
$True$	$True$	$False$	$True$	$True$	$True$	$True$
$True$	$True$	$True$	$True$	$True$	$True$	$True$

Properties of Entailment

- $\Theta \cup \{\phi\} \models \psi$ iff $\Theta \models \phi \rightarrow \psi$

  $\boxed{\text{(Deduction Theorem)}}$

- $\Theta \cup \{\phi\} \models \neg\psi$ iff $\Theta \cup \{\psi\} \models \neg\phi$

  (Contraposition Theorem)

- $\Theta \cup \{\phi\}$ is unsatisfiable iff $\Theta \models \neg\phi$

  (Contradiction Theorem)

Equivalences (I)

Commutativity	$\phi \vee \psi$	$\equiv$	$\psi \vee \phi$
	$\phi \wedge \psi$	$\equiv$	$\psi \wedge \phi$
	$\phi \leftrightarrow \psi$	$\equiv$	$\psi \leftrightarrow \phi$
Associativity	$(\phi \vee \psi) \vee \chi$	$\equiv$	$\phi \vee (\psi \vee \chi)$
	$(\phi \wedge \psi) \wedge \chi$	$\equiv$	$\phi \wedge (\psi \wedge \chi)$
Idempotence	$\phi \vee \phi$	$\equiv$	ϕ
	$\phi \wedge \phi$	$\equiv$	ϕ
Absorption	$\phi \vee (\phi \wedge \psi)$	$\equiv$	ϕ
	$\phi \wedge (\phi \vee \psi)$	$\equiv$	ϕ
Distributivity	$\phi \wedge (\psi \vee \chi)$	$\equiv$	$(\phi \wedge \psi) \vee (\phi \wedge \chi)$
	$\phi \vee (\psi \wedge \chi)$	$\equiv$	$(\phi \vee \psi) \wedge (\phi \vee \chi)$

Equivalences (II)

Tautology	$\phi \vee \top$	$\equiv$	$\top$
Unsatisfiability	$\phi \wedge \bot$	$\equiv$	$\bot$
Negation	$\phi \vee \neg\phi$	$\equiv$	$\top$
	$\phi \wedge \neg\phi$	$\equiv$	$\bot$
Neutrality	$\phi \wedge \top$	$\equiv$	ϕ
	$\phi \vee \bot$	$\equiv$	ϕ
Double Negation	$\neg\neg\phi$	$\equiv$	ϕ
De Morgan	$\neg(\phi \vee \psi)$	$\equiv$	$\neg\phi \wedge \neg\psi$
	$\neg(\phi \wedge \psi)$	$\equiv$	$\neg\phi \vee \neg\psi$
Implication	$\phi \rightarrow \psi$	$\equiv$	$\neg\phi \vee \psi$

Normal Forms

Other approaches to inference use syntactic operations on sentences, often expressed in standardized forms

Conjunctive Normal Form (CNF)

conjunction of $\underbrace{\textit{disjunctions} \text{ of } \textit{literals}}_{\textit{clauses}}$: $\quad \bigwedge_{i=1}^{n}(\bigvee_{j=1}^{m} l_{i,j})$

E.g., $(A \vee \neg B) \wedge (B \vee \neg C \vee \neg D)$

Disjunctive Normal Form (DNF)

disjunction of $\underbrace{\textit{conjunctions} \text{ of } \textit{literals}}_{\textit{terms}}$: $\quad \bigvee_{i=1}^{n}(\bigwedge_{j=1}^{m} l_{i,j})$

E.g., $(A \wedge B) \vee (A \wedge \neg C) \vee (A \wedge \neg D) \vee (\neg B \wedge \neg C) \vee (\neg B \wedge \neg D)$

Normal Forms, cont.

Horn Form (restricted)

conjunction of *Horn clauses* (clauses with ≤ 1 positive literal)

E.g., $(A \vee \neg B) \wedge (B \vee \neg C \vee \neg D)$

Often written as set of implications:

$B \Rightarrow A$ and $(C \wedge D) \Rightarrow B$

Theorem For every formula, there exists an equivalent formula in CNF and one in DNF.

Why Normal Forms?

- We can transform propositional formulas, in particular, we can construct their CNF and DNF.
- DNF tells us something as to whether a formula is satisfiable. If all disjuncts contain $\perp$ or complementary literals, then no model exists. Otherwise, the formula is satisfiable.
- CNF tells us something as to whether a formula is a tautology. If all clauses (= conjuncts) contain $\top$ or complementary literals, then the formula is a tautology. Otherwise, the formula is falsifiable.

But:

- the transformation into DNF or CNF is expensive (in time/space)
- it is only possible for finite sets of formulas

Summary: important notions

- Syntax: formula, atomic formula, literal, clause
- Semantics: truth value, assignment, interpretation
- Formula satisfied by an interpretation
- Logical implication, entailment
- Satisfiability, validity, tautology, logical equivalence
- Deduction theorem, Contraposition Theorem
- Conjunctive normal form, Disjunctive Normal form, Horn form