Free University of Bozen-Bolzano – Faculty of Computer Science
Bachelor in Computer Science and Engineering
Discrete Mathematics and Logic – A.Y. 2017/2018
MidTerm Exam – Discrete Mathematics – 24/November/2017

Prof. Alessandro Artale – *Time: 120 minutes*

`This is a closed book exam: the only resources allowed are blank paper, pens, and your head. Explain your reasoning. Write clearly, in the sense of logic, language and legibility. The clarity of your explanations affects your grade. Write your name and ID in the solution sheet.`

Problem 1. [6 points] **Proofs.**

1. **Universal Statement.** Prove the following statement: $\forall n \in \mathbb{Z}$ *if* n is odd, then, $3n + 5$ is even. [2 POINTS]
2. **Universal Statement.** Disprove the following statement: $\forall m, n \in \mathbb{Z}$ if $2m + n$ is odd, then, m and n are odd. [2 POINTS]
3. **Proof by Contradiction.** Prove by contradiction the following statement: $\forall n, m \in \mathbb{Z}$ *if* $m \times n$ is even, then, m *or* n is even. [2 POINTS]

Problem 2. [12 points] **Induction.**

1. Show that $2 + 5 + 8 + \ldots + (3n - 1) = \dfrac{n(3n + 1)}{2}$, for any $n \geq 1$. [3 POINTS]
2. **Loop Invariant.** The following while loop is annotated with a pre- and a post-condition and also a loop invariant. Furthermore, assume the integer $m \geq 1$ in the guard of the while loop. Use the *loop invariant theorem* to prove the correctness of the loop with respect to the pre- and post-conditions. [6 POINTS]

   [*Pre-condition:* greatest $= A[1]$ and $i = 1$]
   while $i \neq m$ **do**
   i := i + 1
   if($A[i] >$ greatest) **then** greatest := $A[i]$
   end while

   [*Post-condition:* greatest = maximum value of $A[1], \ldots, A[m]$]

   Loop Invariant I(n): greatest is the maximum value of $A[1], A[2], \ldots, A[n + 1]$ and $i = n + 1$.

3. **Structural Induction.** Define a set S recursively as follows:

   I. Base: $\epsilon \in S$ (by ϵ we denote the empty string).

   II. Recursion: If $s \in S$, then a) $bs \in S$; b) $aas \in S$ and c) $saa \in S$.

   III. Restriction: Nothing is in S other than objects defined in I. and II. above.

   Use structural induction to prove that every string in S has an even number of "a". [3 POINTS]

Problem 3. [8 points] **Sets.**

1. Powerset property: Show that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. [4 POINTS]
2. **Halting Problem.** Discuss the Halting Problem. Formulate the Halting problem Theorem and give an idea on how it can be proved. [4 POINTS]

Problem 4. [2 points] **Cardinality.**

1. Determine whether the following set is **finite**, **countably infinite** or **uncountable**. In case the set is countably infinite, show a one-to-one correspondence from the set of positive integers.

   – The set of positive integers multiple of 5.

Problem 5. [6 points] **Relations and Trees.**

1. Let $A = \{a, b, c, d, e\}$ and R the following equivalence relation over A:

   $$R = \{(a, a), (d, b), (b, b), (b, d), (c, c), (a, e), (e, e), (d, d), (e, a)\}$$

   Show the equivalence class of each element in A with respect to R. [2 POINTS]

2. Let R be a partial order relation. Show that R^{-} (the inverse of R) is also a partial order relation. [2 POINTS]

3. Find all non-isomorphic trees with 5 vertices. Provide an explanation with your answer. [2 POINTS]