CHAPTER 6

SET THEORY

Alessandro Artale – UniBZ - http://www.inf.unibz.it/~artale/

SECTION 6.1

Set Theory: Definitions

Set Theory: Definitions and the Element Method of Proof

The words *set* and *element* are undefined terms of set theory just as *sentence*, *true*, and *false* are undefined terms of logic.

The founder of set theory, Georg Cantor (1845, Saint-Petersburg, Russia), suggested imagining a set as a "collection into a whole *M* of definite and separate objects of our intuition or our thought. These objects are called the elements of *M*."

Cantor used the letter *M* because it is the first letter of the German word for set: *Menge*.

The Empty Set

The Empty Set

A set is defined by the elements that compose it. This being so, can there be a set that has no elements? It turns out that it is convenient to allow for such a set.

Because it is unique, we can give it a special name. We call it the **empty set** (or **null set**) and denote it by the symbol $\emptyset$.

Thus $\{1, 3\} \cap \{2, 4\} = \emptyset$ and $\{x \in \mathbf{R} \mid x^2 = -1\} = \emptyset$.

Cartesian Products

Ordered n-tuples

Definition

Let n be a positive integer and let $x_1, x_2, \ldots, x_n$ be (not necessarily distinct) elements. The **ordered n-tuple, $(x_1, x_2, \ldots, x_n)$,** consists of $x_1, x_2, \ldots, x_n$ together with the ordering: first x_1, then x_2, and so forth up to x_n. An ordered 2-tuple is called an **ordered pair,** and an ordered 3-tuple is called an **ordered triple.**

Two ordered n-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are **equal** if, and only if, $x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$.

Symbolically:

$$(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \quad \Leftrightarrow \quad x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n.$$

In particular,

$$(a, b) = (c, d) \quad \Leftrightarrow \quad a = c \text{ and } b = d.$$

Example 13 – *Ordered n-tuples*

a. Is $(1, 2, 3, 4) = (1, 2, 4, 3)$?

b. Is $\left(3, (-2)^2, \frac{1}{2}\right) = \left(\sqrt{9}, 4, \frac{3}{6}\right)$?

Solution:

a. No. By definition of equality of ordered 4-tuples,

$$(1,\ 2,\ 3,\ 4) = (1,\ 2,\ 4,\ 3) \Leftrightarrow 1 = 1,\ 2 = 2,\ 3 = 4,\ \text{and } 4 = 3$$

But $3 \neq 4$, and so the ordered 4-tuples are not equal.

Example 13 – *Solution*

cont'd

b. Yes. By definition of equality of ordered triples,

$$\left(3, (-2)^2, \tfrac{1}{2}\right) = \left(\sqrt{9}, 4, \tfrac{3}{6}\right) \quad \Leftrightarrow \quad 3 = \sqrt{9} \text{ and } (-2)^2 = 4 \text{ and } \tfrac{1}{2} = \tfrac{3}{6}.$$

Because these equations are all true, the two ordered triples are equal.

Cartesian Products

Definition

Given sets $A_1, A_2, \ldots, A_n$, the **Cartesian product** of $A_1, A_2, \ldots, A_n$ denoted $\boldsymbol{A_1 \times A_2 \times \ldots \times A_n}$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$.

Symbolically:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

In particular,

$$A_1 \times A_2 = \{(a_1, a_2) \mid a_1 \in A_1 \text{ and } a_2 \in A_2\}$$

is the Cartesian product of A_1 and A_2.

Example 14 – *Cartesian Products*

Let $A_1 = \{x, y\}$, $A_2 = \{1, 2, 3\}$, and $A_3 = \{a, b\}$.

a. Find $A_1 \times A_2$.

b. Find $(A_1 \times A_2) \times A_3$.

c. Find $A_1 \times A_2 \times A_3$.

Solution:

a. $A_1 \times A_2 = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\}$

b. The Cartesian product of A_1 and A_2 is a set, so it may be used as one of the sets making up another Cartesian product. This is the case for $(A_1 \times A_2) \times A_3$.

Example 14 – *Solution*

cont' d

$$(A_1 \times A_2) \times A_3 = \{(u, v) \mid u \in A_1 \times A_2 \text{ and } v \in A_3\}$$ by definition of Cartesian product

$$= \{((x, 1), a), ((x, 2), a), ((x, 3), a), ((y, 1), a),$$

$$((y, 2), a), ((y, 3), a), ((x, 1), b), ((x, 2), b), ((x, 3), b),$$

$$((y, 1), b), ((y, 2), b), ((y, 3), b)\}$$

c. The Cartesian product $A_1 \times A_2 \times A_3$ is superficially similar to, but is not quite the same mathematical object as, $(A_1 \times A_2) \times A_3$. $(A_1 \times A_2) \times A_3$ is a set of ordered pairs of which one element is itself an ordered pair, whereas $A_1 \times A_2 \times A_3$ is a set of ordered triples.

Example 14 – *Solution*

cont'd

By definition of Cartesian product,

$$A_1 \times A_2 \times A_3 = \{(u, v, w) \mid u \in A_1, v \in A_2, \text{ and } w \in A_3\}$$

$$= \{(x, 1, a), (x, 2, a), (x, 3, a), (y, 1, a), (y, 2, a),$$

$$(y, 3, a), (x, 1, b), (x, 2, b), (x, 3, b), (y, 1, b),$$

$$(y, 2, b), (y, 3, b)\}.$$

Power Sets

Power Sets

There are various situations in which it is useful to consider the set of all subsets of a particular set.

The **power set axiom** guarantees that this is a set.

• Definition

Given a set A, the **power set** of A, denoted $\mathscr{P}(A)$, is the set of all subsets of A.

Example 12 – *Power Set of a Set*

Find the power set of the set $\{x, y\}$. That is, find $\mathscr{P}(\{x, y\})$.

Solution:

$\mathscr{P}(\{x, y\})$ is the set of all subsets of $\{x, y\}$. We know that Ø is a subset of every set, and so $\emptyset \in \mathscr{P}(\{x, y\})$.

Also any set is a subset of itself, so $\{x, y\} \in \mathscr{P}(\{x, y\})$. The only other subsets of $\{x, y\}$ are $\{x\}$ and $\{y\}$, so

$$\mathscr{P}(\{x, y\}) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}.$$

The Number of Subsets of a Set

The Number of Subsets of a Set

The following theorem states the important fact that if a set has n elements, then its power set has 2^n elements.

The proof uses mathematical induction and is based on the following observations. Suppose X is a set and z is an element of X.

1. The subsets of X can be split into two groups: those that do not contain z and those that do contain z.

2. The subsets of X that do not contain z are the same as the subsets of $X - \{z\}$.

The Number of Subsets of a Set

The subsets of X that do not contain z can be matched one to one with the subsets of X that do contain z by matching each subset A that does not contain z to the subset $A \cup \{z\}$ that contains z.

Thus there are as many subsets of X that contain z as there are subsets of X that do not contain z.

The Number of Subsets of a Set

For instance, if $X = \{x, y, z\}$, the following table shows the correspondence between subsets of X that do not contain z and subsets of X that contain z.

Subsets of X That Do Not Contain z		Subsets of X That Contain z
$\emptyset$	$\longleftrightarrow$	$\emptyset \cup \{z\} = \{z\}$
$\{x\}$	$\longleftrightarrow$	$\{x\} \cup \{z\} = \{x, z\}$
$\{y\}$	$\longleftrightarrow$	$\{y\} \cup \{z\} = \{y, z\}$
$\{x, y\}$	$\longleftrightarrow$	$\{x, y\} \cup \{z\} = \{x, y, z\}$

Theorem 6.3.1

For all integers $n \geq 0$, if a set X has n elements, then $\mathscr{P}(X)$ has 2^n elements.

The Number of Subsets of a Set

Proof (by mathematical induction):

Let the property $P(n)$ be the sentence

Any set with n elements has 2^n subsets. $\leftarrow P(n)$

Show that $P(0)$ is true:

To establish $P(0)$, we must show that

Any set with 0 elements has 2^0 subsets. $\leftarrow P(0)$

But the only set with zero elements is the empty set, and the only subset of the empty set is itself.

Thus a set with zero elements has one subset. Since $1 = 2^0$, we have that $P(0)$ is true.

The Number of Subsets of a Set

Show that for all integers $k \geq 0$, if $P(k)$ is true then $P(k + 1)$ is also true: *[Suppose that P(k) is true for a particular but arbitrarily chosen integer k ≥ 0. That is:]*

Suppose that k is any integer with $k \geq 0$ such that

Any set with k elements has 2^k subsets. $\leftarrow P(k)$
inductive hypothesis

[We must show that $P(k + 1)$ *is true. That is:]*

We must show that

Any set with $k + 1$ elements has 2^{k+1} subsets. $\leftarrow P(k+1)$

The Number of Subsets of a Set

Let X be a set with $k + 1$ elements. Since $k + 1 \geq 1$, we may pick an element z in X. Observe that any subset of X either contains z or not.

Any subset A of $X - \{z\}$ can be matched up with a subset B, equal to $A \cup \{z\}$, of X that contains z.

Consequently, there are as many subsets of X that contain z as do not:

> The number of subsets of X *are* twice the number of subsets of $X - \{z\}$.

The Number of Subsets of a Set

But $X - \{z\}$ has k elements, and so

$$\text{the number of subsets of } X - \{z\} = 2^k \qquad \text{by inductive hypothesis.}$$

Therefore,

$$\begin{aligned}\text{the number of subsets of } X &= 2 \cdot (\text{the number of subsets of } X - \{z\}) \\ &= 2 \cdot (2^k) \qquad \text{by substitution} \\ &= 2^{k+1} \qquad \text{by basic algebra.}\end{aligned}$$

[This is what was to be shown.]
[Since we have proved both the basis step and the inductive step, we conclude that the theorem is true.]

Russell's Paradox

(1872–1970)

Russell's Paradox

Russell's Paradox: Most sets are not elements of themselves. For instance, the set of all integers is not an integer and the set of all horses is not a horse.

However, we can imagine the possibility of a set's being an element of itself. For instance, the set of all abstract ideas might be considered an abstract idea.

If we are allowed to use any description of a property as the defining property of a set, we can let *S* be the set of all sets that are not elements of themselves:

$$S = \{A \mid A \text{ is a set and } A \notin A\}.$$

Russell's Paradox

Is S an element of itself? The answer is neither yes nor no.

For if $S \in S$, then S must satisfies the defining property for S, and hence $S \notin S$.

But if $S \notin S$, then S is a set such that $S \notin S$ and so S satisfies the defining property for S, which implies that

$S \in S$.

Thus neither is $S \in S$ nor is $S \notin S$, which is a contradiction.

To help explain his discovery to laypeople, Russell devised a puzzle, the barber puzzle, whose solution exhibits the same logic as his paradox.

Example 3 – *The Barber Puzzle*

In a certain town there is a male barber who shaves all those men, and only those men, who do not shave themselves.

Question: Does the barber shave himself?

Solution:

Neither yes nor no. If the barber shaves himself, he is a member of the class of men who shave themselves.

But no member of this class is shaved by the barber, and so the barber does *not* shave himself.

Example 3 – *Solution*

cont'd

On the other hand, if the barber does not shave himself, he belongs to the class of men who do not shave themselves.

But the barber shaves every man in this class, so the barber *does* shave himself.

Russell's Paradox

So let's accept the fact that the paradox has no easy solution and see where that thought leads. Since the barber neither shaves himself nor doesn't shave himself, the sentence "The barber shaves himself" is neither true nor false.

But the sentence arose in a natural way from a description of a situation. If the situation actually existed, then the sentence would have to be either true or false.

Thus we are forced to conclude that the situation described in the puzzle simply cannot exist in the world as we know it.

Russell's Paradox

In a similar way, the conclusion to be drawn from Russell's paradox itself is that the object *S* is not a set.

Because if it actually were a set, in the sense of satisfying the general properties of sets that we have been assuming, then it either would be an element of itself or not.

One way to avoid this contradiction is to assume the existence of a Universal Set such that every set must be a subset of this Universal Set.

Russell's Paradox -- Solution

Let U be a universal set and suppose that all sets under discussion must be subsets of U. Let

$$S = \{A \mid A \subseteq U \text{ and } A \notin A\}.$$

In Russell's paradox, both implications

$$S \in S \rightarrow S \notin S \quad \text{and} \quad S \notin S \rightarrow S \in S$$

can be proved, and the contradictory conclusion

$$\text{neither } S \in S \quad \text{nor} \quad S \notin S$$

is therefore deduced.

If all sets under discussion are subsets of U, the implication $S \in S \rightarrow S \notin S$ is proved in almost the same way as it is for Russell's paradox: (Suppose $S \in S$. Then by definition of S, $S \subseteq U$ and $S \notin S$. In particular, $S \notin S$.)

Russell's Paradox

On the other hand, from the supposition that $S \notin S$ we can only deduce that the statement "$S \subseteq U$ and $S \notin S$" is false.

By De Morgan's laws, this means that "$S \not\subseteq U$ or $S \in S$." Since $S \in S$ would contradict the supposition that $S \notin S$, we eliminate it and conclude that $S \not\subseteq U$.

In other words, the only conclusion we can draw is that the seeming "definition" of S is faulty—that is, that S is not a set in U.

Alan Turing （1912－ 1954）

The Halting Problem

The Halting Problem

If you have some experience programming computers, you know how badly an infinite loop can tie up a computer system.

It would be useful to be able to preprocess a program and its data set by running it through a checking program that determines whether execution of the given program with the given data set would result in an infinite loop.

Can an algorithm for such a program be written?

The Halting Problem

In other words, can an algorithm be written that will accept any algorithm *X* and any data set *D* as input and will then print "halts" or "loops forever" to indicate whether *X* terminates in a finite number of steps or loops forever when run with data set *D*?

In the 1930s, Turing proved that the answer to this question is no.

Theorem 6.4.2

There is no computer algorithm that will accept any algorithm X and data set D as input and then will output "halts" or "loops forever" to indicate whether or not X terminates in a finite number of steps when X is run with data set D.

The Halting Problem

CheckHalt(*X*, *D*)

if *X* terminates in a finite number of steps when run with data set *D*

then ***return*** "*halts*"

else ***return*** "loops forever"

Note: the sequence of characters making up an algorithm *X* can be regarded as a data set itself.

Test(*X*)

if CheckHalt(*X*, *X*) = "halts" ***then*** **loop forever**

else if CheckHalt(*X*, *X*) = "loops forever" ***then*** **stop**.

The Halting Problem

Run the algorithm Test with input Test: **Test**(Test).

Test(Test) terminates after a finite number of steps, then **CheckHalt**(Test, Test) = “halts” and so **Test**(Test) loops forever, and we have a contradiction.

Test(Test) does not terminate after a finite number of steps, then **CheckHalt**(Test, Test) = “loops forever” and so **Test**(Test) terminates, and again we have a contradiction.

Then we can conclude that such a **Test** algorithm does not exist.