CHAPTER 10

GRAPHS AND TREES

SECTION 10.4

Isomorphisms of Graphs

Isomorphisms of Graphs

The two drawings shown in Figure 10.4.1 both represent the same graph: Their vertex and edge sets are identical, and their edge-endpoint functions are the same. Call this graph *G*.

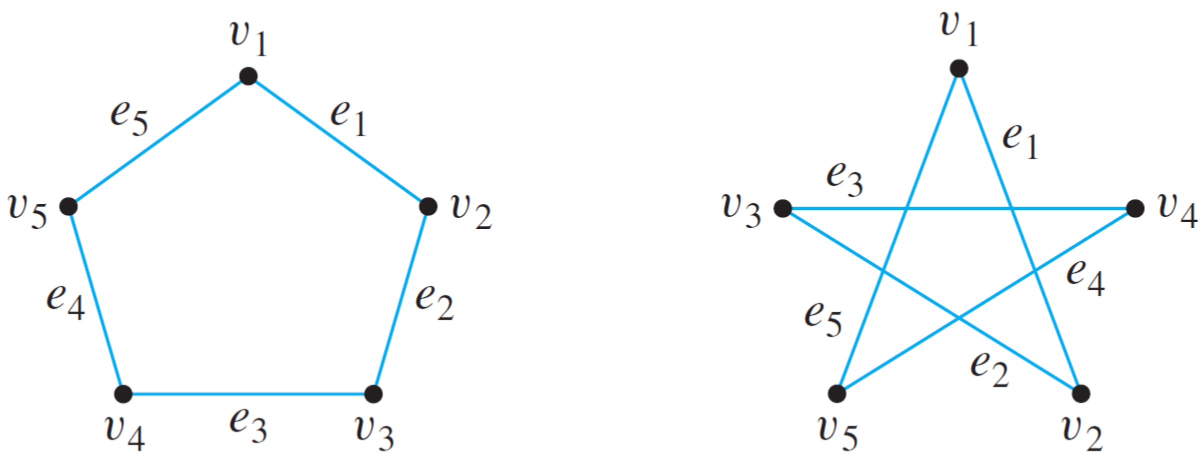


Figure 10.4.1

Isomorphisms of Graphs

Now consider the graph G' represented in Figure 10.4.2.

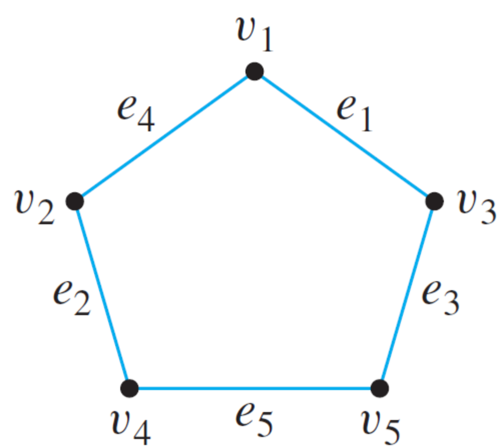


Figure 10.4.2

Observe that G' is a different graph from G (for instance, in G the endpoints of e_1 are v_1 and v_2, whereas in G' the endpoints of e_1 are v_1 and v_3).

Isomorphisms of Graphs

Yet G' is certainly very similar to G. In fact, if the vertices and edges of G' are renamed by the functions shown in Figure 10.4.3, then G' becomes the same as G.

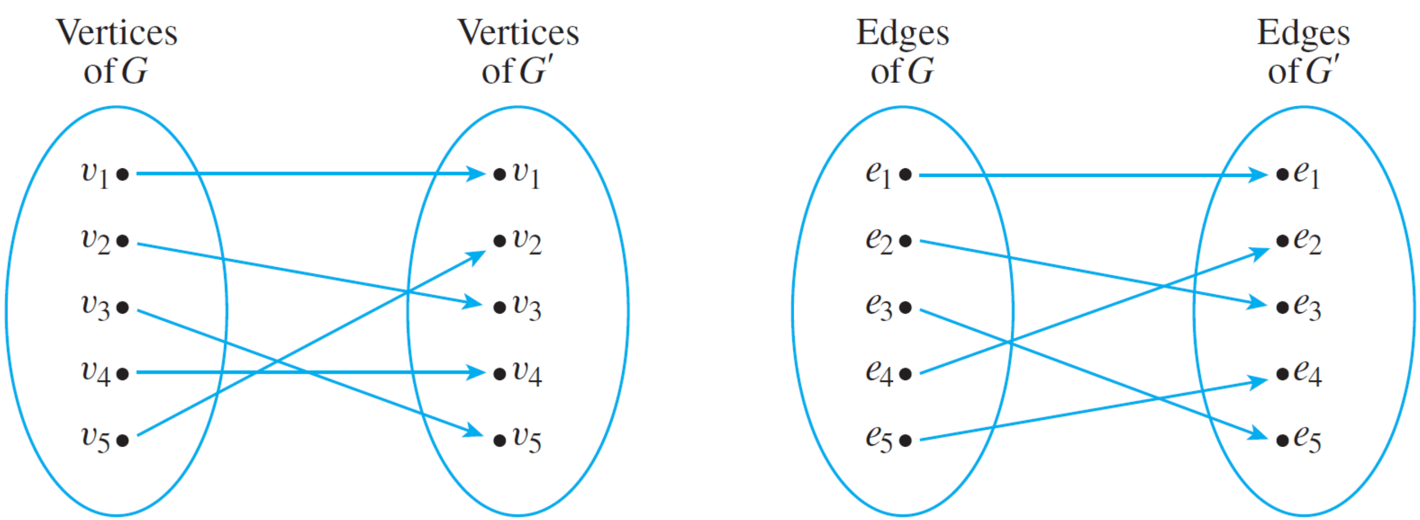


Figure 10.4.3

Note that these renaming functions are one-to-one and onto.

Isomorphisms of Graphs

Two graphs that are the same except for the labeling of their vertices and edges are called ***isomorphic***.

The word *isomorphism* comes from the Greek, meaning "same form." Isomorphic graphs are those that have essentially the same form.

• Definition

Let G and G' be graphs with vertex sets $V(G)$ and $V(G')$ and edge sets $E(G)$ and $E(G')$, respectively. **G is isomorphic to G'** if, and only if, there exist one-to-one correspondences $g: V(G) \to V(G')$ and $h: E(G) \to E(G')$ that preserve the edge-endpoint functions of G and G' in the sense that for all $v \in V(G)$ and $e \in E(G)$,

$$v \text{ is an endpoint of } e \quad \Leftrightarrow \quad g(v) \text{ is an endpoint of } h(e). \qquad 10.4.1$$

Isomorphisms of Graphs

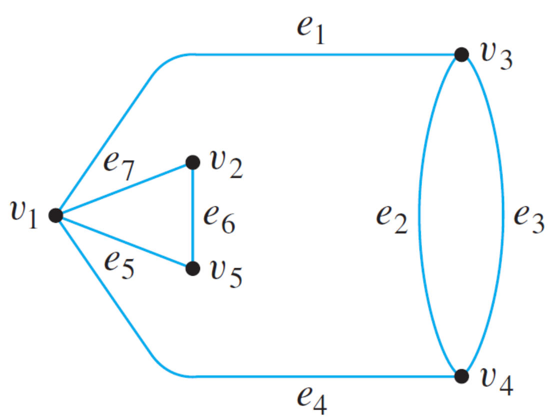
In words, G is isomorphic to G' if, and only if, the vertices and edges of G and G' can be matched up by one-to-one, onto functions such that the edges between corresponding vertices correspond to each other.

It is common in mathematics to identify objects that are isomorphic.

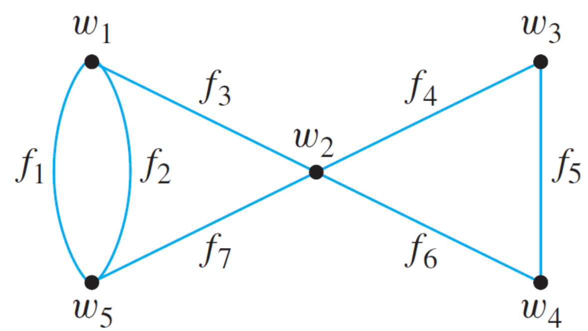
For instance, if we are given a graph G with five vertices such that each pair of vertices is connected by an edge, then we may identify G with K_5, saying that G is K_5 rather than that G is isomorphic to K_5.

Example 1 – *Showing That Two Graphs Are Isomorphic*

Show that the following two graphs are isomorphic.



G



G′

Solution:

To solve this problem, you must find functions
g: $V(G) \to V(G')$ and h: $E(G) \to E(G')$ such that for all
$v \in V(G)$ and $e \in E(G)$, v is an endpoint of e if, and only if, $g(v)$ is an endpoint of $h(e)$.

Example 1 – *Solution*

cont'd

Setting up such functions is partly a matter of trial and error and partly a matter of deduction.

For instance, since e_2 and e_3 are parallel (have the same endpoints), $h(e_2)$ and $h(e_3)$ must be parallel also.
So $h(e_2) = f_1$ and $h(e_3) = f_2$ or $h(e_2) = f_2$ and $h(e_3) = f_1$.

Also, the endpoints of e_2 and e_3 must correspond to the endpoints of f_1 and f_2, and so $g(v_3) = w_1$ and $g(v_4) = w_5$ or $g(v_3) = w_5$ and $g(v_4) = w_1$.

Example 1 – *Solution*

cont'd

Similarly, since v_1 is the endpoint of four distinct edges (e_1, e_7, e_5, and e_4), $g(v_1)$ must also be the endpoint of four distinct edges (because every edge incident on $g(v_1)$ is the image under h of an edge incident on v_1 and h is one-to-one and onto).

But the only vertex in G' that has four edges coming out of it is w_2, and so $g(v_1) = w_2$.

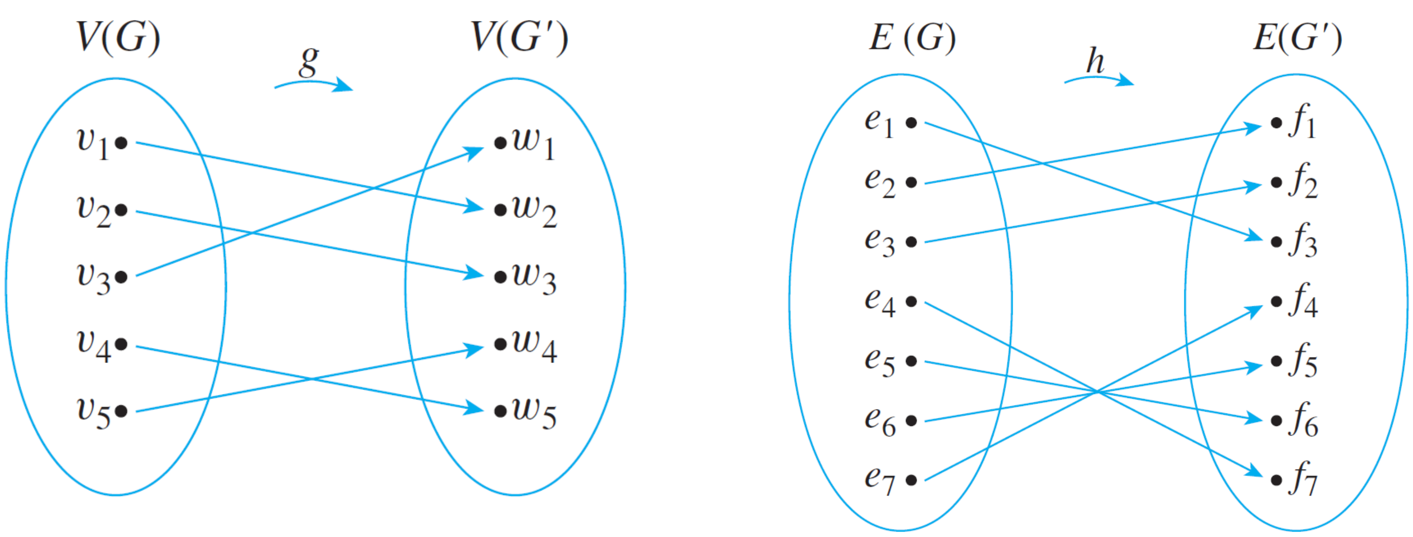
Now if $g(v_3) = w_1$, then since v_1 and v_3 are endpoints of e_1 in G, $g(v_1) = w_2$ and $g(v_3) = w_1$ must be endpoints of $h(e_1)$ in G'. This implies that $h(e_1) = f_3$.

Example 1 – *Solution*

cont'd

By continuing in this way, possibly making some arbitrary choices as you go, you eventually can find functions *g* and *h* to define the isomorphism between *G* and *G*′.

One pair of functions (there are several) is the following:



Isomorphisms of Graphs

It is not hard to show that graph isomorphism is an equivalence relation on a set of graphs; in other words, it is reflexive, symmetric, and transitive.

Theorem 10.4.1 Graph Isomorphism is an Equivalence Relation

Let S be a set of graphs and let R be the the relation of graph isomorphism on S. Then R is an equivalence relation on S.

Now consider the question, "Is there a general method to figure out whether graphs G and G' are isomorphic?"

In other words, is there some algorithm that will accept graphs G and G' as input and produce a statement as to whether they are isomorphic?

Isomorphisms of Graphs

In fact, there is such an algorithm. It consists of generating all one-to-one, onto functions from the set of vertices of G to the set of vertices of G' and from the set of edges of G to the set of edges of G' and checking each pair to determine whether it preserves the edge-endpoint functions of G and G'.

The problem with this algorithm is that it takes an unreasonably long time to perform, even on a high-speed computer.

Isomorphisms of Graphs

If G and G' each have n vertices and m edges, the number of one-to-one correspondences from vertices to vertices is $n!$ and the number of one-to-one correspondences from edges to edges is $m!$, so the total number of pairs of functions to check is $n! \cdot m!$.

For instance, if $m = n = 20$, there would be $20! \cdot 20! \cong 5.9 \times 10^{36}$ pairs to check.

Assuming that each check takes just 1 nanosecond, the total time would be approximately **1.9×10^{20} years**!

Isomorphisms of Graphs

Unfortunately, there is no more efficient general method known for checking whether two graphs are isomorphic.

However, there are some simple tests that can be used to show that certain pairs of graphs are *not* isomorphic.

For instance, if two graphs are isomorphic, then they have the same number of vertices (because of the one-to-one correspondence on vertex sets of the two graphs).

It follows that if you are given two graphs, one with 16 vertices and the other with 17, you can immediately conclude that the two are not isomorphic.

Isomorphisms of Graphs

More generally, a property that is preserved by graph isomorphism is called an *isomorphic invariant*.

For instance, "having 16 vertices" is an isomorphic invariant: If one graph has 16 vertices, then so does any graph that is isomorphic to it.

- **Definition**

A property P is called an **invariant for graph isomorphism** if, and only if, given any graphs G and G', if G has property P and G' is isomorphic to G, then G' has property P.

Isomorphisms of Graphs

Theorem 10.4.2

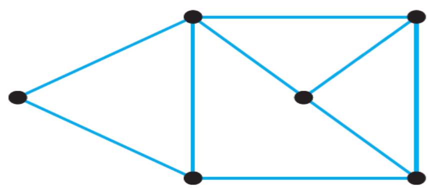
Each of the following properties is an invariant for graph isomorphism, where n, m, and k are all nonnegative integers:

1. has n vertices;
2. has m edges;
3. has a vertex of degree k;
4. has m vertices of degree k;
5. has a circuit of length k;
6. has a simple circuit of length k;
7. has m simple circuits of length k;
8. is connected;
9. has an Euler circuit;
10. has a Hamiltonian circuit.

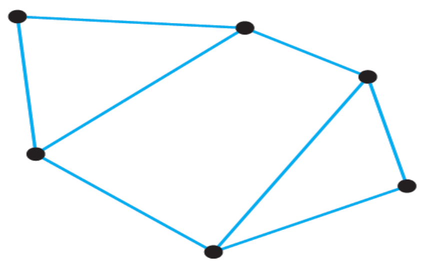
Example 3 – *Showing That Two Graph Are Not Isomorphic*

Show that the following pairs of graphs are not isomorphic by finding an isomorphic invariant that they do not share.

a.

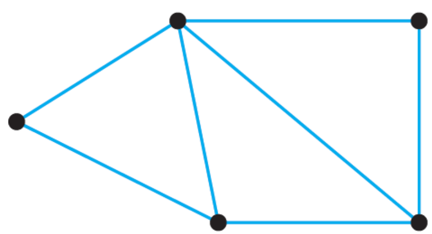


G

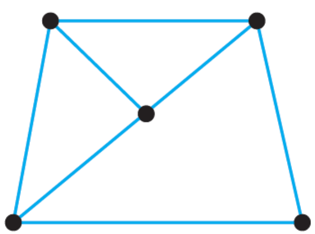


G′

b.



H



H′

Example 3 – *Solution*

a. *G* has nine edges; G' has only eight.

b. *H* has a vertex of degree 4; H' does not.

Graph Isomorphism for Simple Graphs

Graph Isomorphism for Simple Graphs

When graphs G and G' are both simple, the definition of G being isomorphic to G' can be written without referring to the correspondence between the edges of G and the edges of G'.

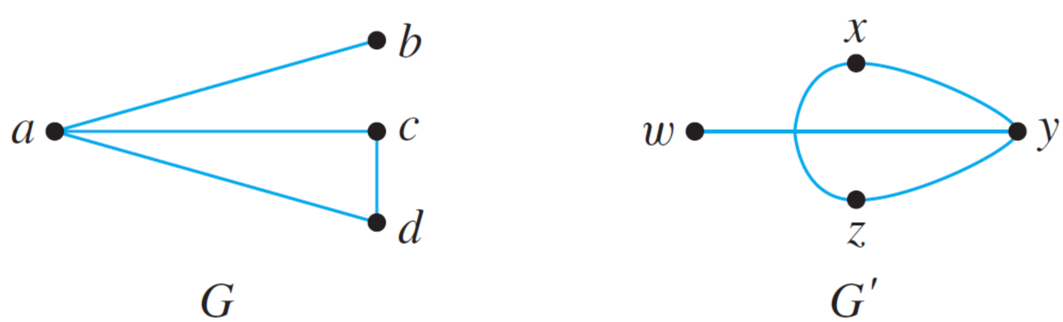
• Definition

If G and G' are simple graphs, then **G is isomorphic to G'** if, and only if, there exists a one-to-one correspondence g from the vertex set $V(G)$ of G to the vertex set $V(G')$ of G' that preserves the edge-endpoint functions of G and G' in the sense that for all vertices u and v of G,

$$\{u, v\} \text{ is an edge in } G \quad \Leftrightarrow \quad \{g(u), g(v)\} \text{ is an edge in } G'. \qquad 10.4.2$$

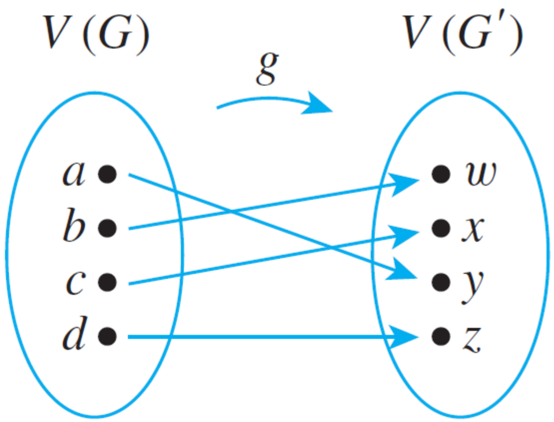
Example 5 – *Isomorphism of Simple Graphs*

Are the two graphs shown below isomorphic? If so, define an isomorphism.



Example 5 – *Solution*

Yes. Define $f : V(G) \rightarrow V(G')$ by the arrow diagram shown below.



Then g is one-to-one and onto by inspection.

Example 5 – *Solution*

cont'd

The fact that g preserves the edge-endpoint functions of G and G' is shown by the following table:

Edges of G	Edges of G'
$\{a, b\}$	$\{y, w\} = \{g(a), g(b)\}$
$\{a, c\}$	$\{y, x\} = \{g(a), g(c)\}$
$\{a, d\}$	$\{y, z\} = \{g(a), g(d)\}$
$\{c, d\}$	$\{x, z\} = \{g(c), g(d)\}$