CHAPTER 8

RELATIONS

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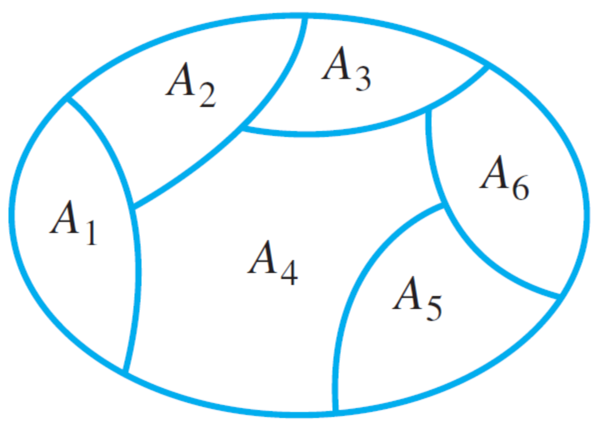
SECTION 8.3

Equivalence Relations

The Relation Induced by a Partition

The Relation Induced by a Partition

A **partition** of a set *A* is a finite or infinite collection of nonempty, mutually disjoint subsets whose union is *A*. The diagram of Figure 8.3.1 illustrates a partition of a set *A* by subsets $A_1, A_2, \ldots, A_6$.



$A_i \cap A_j = \emptyset$, whenever $i \neq j$
$A_i \cup A_2 \cup \cdots \cup A_6 = A$

A Partition of a Set

Figure 8.3.1

The Relation Induced by a Partition

- **Definition**

Given a partition of a set A, the **relation induced by the partition,** R, is defined on A as follows: For all $x, y \in A$,

$$x \; R \; y \quad \Leftrightarrow \quad \text{there is a subset } A_i \text{ of the partition such that both } x \text{ and } y \text{ are in } A_i.$$

Example 1 – *Relation Induced by a Partition*

Let $A = \{0, 1, 2, 3, 4\}$ and consider the following partition of A:

$$\{0, 3, 4\}, \{1\}, \{2\}.$$

Find the relation R induced by this partition.

Solution:

Since $\{0, 3, 4\}$ is a subset of the partition,

$0\ R\ 3$ because both 0 and 3 are in $\{0, 3, 4\}$,

$3\ R\ 0$ because both 3 and 0 are in $\{0, 3, 4\}$,

Example 1 – *Solution*

cont'd

$0\ R\ 4$ because both 0 and 4 are in {0, 3, 4},

$4\ R\ 0$ because both 4 and 0 are in {0, 3, 4},

$3\ R\ 4$ because both 3 and 4 are in {0, 3, 4}, and

$4\ R\ 3$ because both 4 and 3 are in {0, 3, 4}.

Also, $0\ R\ 0$ because both 0 and 0 are in {0, 3, 4}

$3\ R\ 3$ because both 3 and 3 are in {0, 3, 4}, and

$4\ R\ 4$ because both 4 and 4 are in {0, 3, 4}.

Example 1 – *Solution*

cont' d

Since {1} is a subset of the partition,

1 R 1 because both 1 and 1 are in {1},

and since {2} is a subset of the partition,

2 R 2 because both 2 and 2 are in {2}.

Hence

R = {(0, 0), (0, 3), (0, 4), (1, 1), (2, 2), (3, 0), (3, 3), (3, 4), (4, 0), (4, 3), (4, 4)}.

The Relation Induced by a Partition

The fact is that a relation induced by a partition of a set satisfies all three properties: reflexivity, symmetry, and transitivity.

Theorem 8.3.1

Let A be a set with a partition and let R be the relation induced by the partition. Then R is reflexive, symmetric, and transitive.

Definition of an Equivalence Relation

Definition of an Equivalence Relation

A relation on a set that satisfies the three properties of reflexivity, symmetry, and transitivity is called an ***equivalence relation***.

• Definition

Let A be a set and R a relation on A. R is an **equivalence relation** if, and only if, R is reflexive, symmetric, and transitive.

Thus, according to Theorem 8.3.1, the relation induced by a partition is an equivalence relation.

Example 2 – *An Equivalence Relation on a Set of Subsets*

Let X be the set of all nonempty subsets of $\{1, 2, 3\}$. Then

$$X = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Define a relation **R** on X as follows: For all A and B in X,

A **R** $B \Leftrightarrow$ the least element of A equals the least element of B.

Prove that **R** is an equivalence relation on X.

Example 2 – *Solution*

R is reflexive: Suppose *A* is a nonempty subset of {1, 2, 3}. *[We must show that A* **R** *A.]*

It is true to say that the least element of *A* equals the least element of *A*. Thus, by definition of *R*, *A* **R** *A*.

R is symmetric: Suppose *A* and *B* are nonempty subsets of {1, 2, 3} and *A* **R** *B*. *[We must show that B* **R** *A.]*

Since *A* **R** *B*, the least element of *A* equals the least element of *B*.

But this implies that the least element of *B* equals the least element of *A*, and so, by definition of **R**, *B* **R** *A*.

Example 2 – *Solution*

cont'd

R is transitive: Suppose A, B, and C are nonempty subsets of {1, 2, 3}, A **R** B, and B **R** C. *[We must show that A **R** C.]*

Since A **R** B, the least element of A equals the least element of B and since B **R** C, the least element of B equals the least element of C.

Thus the least element of A equals the least element of C, and so, by definition of **R**, A **R** C.

Equivalence Classes of an Equivalence Relation

Equivalence Classes of an Equivalence Relation

Suppose there is an equivalence relation, R, on a certain set. If *a* is any particular element of the set, then one can ask, "What is the set of elements that are R-related to *a*?" This set is called the ***equivalence class of a***.

• Definition

Suppose A is a set and R is an equivalence relation on A. For each element a in A, the **equivalence class of a**, denoted **[a]** and called the **class of a** for short, is the set of all elements x in A such that x is related to a by R.

In symbols:

$$[a] = \{x \in A \mid x \; R \; a\}$$

• Definition

Suppose R is an equivalence relation on a set A and S is an equivalence class of R. A **representative** of the class S is any element a such that $[a] = S$.

Equivalence Classes of an Equivalence Relation

When several equivalence relations on a set are under discussion, the notation $[a]_R$ is often used to denote the equivalence class of a under R.

The procedural version of this definition is

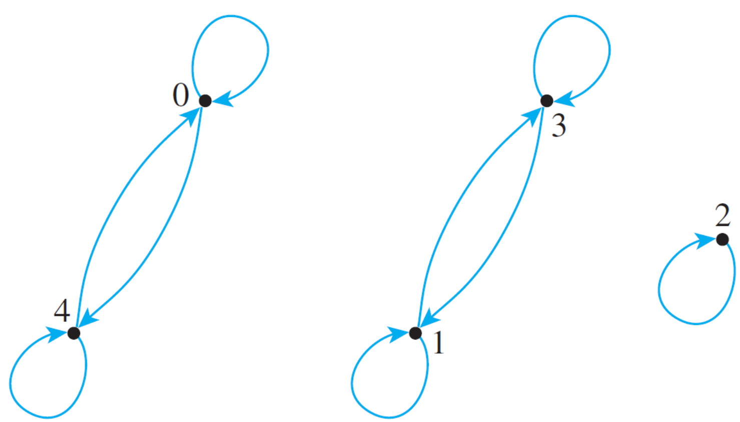
$$\boxed{\text{for all } x \in A, \quad x \in [a] \quad \Leftrightarrow \quad x\ R\ a.}$$

Example 5 – *Equivalence Classes of a Relation Given as a set of Ordered Pairs*

Let $A = \{0, 1, 2, 3, 4\}$ and define a relation R on A as follows:

$$R = \{(0, 0), (0, 4), (1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (4, 0), (4, 4)\}.$$

The directed graph for R is as shown below. As can be seen by inspection, R is an equivalence relation on A. Find the distinct equivalence classes of R.



Example 5 – *Solution*

First find the equivalence class of every element of *A*.

$$[0] = \{x \in A \mid x\ R\ 0\} = \{0, 4\}$$

$$[1] = \{x \in A \mid x\ R\ 1\} = \{1, 3\}$$

$$[2] = \{x \in A \mid x\ R\ 2\} = \{2\}$$

$$[3] = \{x \in A \mid x\ R\ 3\} = \{1, 3\}$$

$$[4] = \{x \in A \mid x\ R\ 4\} = \{0, 4\}$$

Note that [0] = [4] and [1] = [3]. Thus the *distinct* equivalence classes of the relation are

{0, 4}, {1, 3}, and {2}.

Equivalence Classes of an Equivalence Relation

The following lemma says that if two elements of *A* are related by an equivalence relation *R*, then their equivalence classes are the same.

Lemma 8.3.2

Suppose A is a set, R is an equivalence relation on A, and a and b are elements of A. If $a \; R \; b$, then $[a] = [b]$.

Proof. This lemma says that if a certain condition is satisfied, then [*a*] = [*b*]. Now [*a*] and [*b*] are *sets*, and two sets are equal if, and only if, each is a subset of the other.

Equivalence Classes of an Equivalence Relation

Hence the proof of the lemma consists of two parts: first, a proof that $[a] \subseteq [b]$ and second, a proof that $[b] \subseteq [a]$.

The second lemma says that any two equivalence classes are either mutually disjoint or identical.

Lemma 8.3.3

If A is a set, R is an equivalence relation on A, and a and b are elements of A, then

$$\text{either} \quad [a] \cap [b] = \emptyset \quad \text{or} \quad [a] = [b].$$

Equivalence Classes of an Equivalence Relation

The statement of Lemma 8.3.3 has the form

if *p* then (*q* or *r*),

where *p* is the statement "*A* is a set, *R* is an equivalence relation on *A*, and *a* and *b* are elements of *A*," *q* is the statement "$[a] \cap [b] = \emptyset$," and *r* is the statement "$[a] = [b]$."

Theorem 8.3.4 The Partition Induced by an Equivalence Relation

If *A* is a set and *R* is an equivalence relation on *A*, then the distinct equivalence classes of *R* form a partition of *A*; that is, the union of the equivalence classes is all of *A*, and the intersection of any two distinct classes is empty.

Congruence Modulo *n*

Example 10 – *Equivalence Classes of Congruence Modulo 3*

Let *R* be the relation of congruence modulo 3 on the set **Z** of all integers. That is, for all integers *m* and *n*,

$$\boxed{m \; R \; n \quad \Leftrightarrow \quad 3 \mid (m - n) \quad \Leftrightarrow \quad m \equiv n \pmod 3.}$$

Describe the distinct equivalence classes of *R*.

Solution:

For each integer *a*,

$$[a] = \{x \in \mathbf{Z} \mid x \; R \; a\}$$

$$= \{x \in \mathbf{Z} \mid 3 \mid (x - a)\}$$

Example 10 – *Solution*

cont' d

$$= \{x \in \mathbf{Z} \mid x - a = 3k, \text{ for some integer } k\}.$$

Therefore,

$$[a] = \{x \in \mathbf{Z} \mid x = 3k + a, \text{ for some integer } k\}.$$

In particular, $[0] = \{x \in \mathbf{Z} \mid x = 3k + 0, \text{ for some integer } k\}$

$$= \{x \in \mathbf{Z} \mid x = 3k, \text{ for some integer } k\}$$

$$= \{\ldots -9, -6, -3, 0, 3, 6, 9, \ldots\},$$

$$[1] = \{x \in \mathbf{Z} \mid x = 3k + 1, \text{ for some integer } k\}$$

$$= \{\ldots -8, -5, -2, 1, 4, 7, 10, \ldots\},$$

Example 10 – *Solution*

cont'd

$$[2] = \{x \in \mathbf{Z} \mid x = 3k + 2, \text{ for some integer } k\}$$

$$= \{\ldots -7, -4, -1, 2, 5, 8, 11, \ldots\}.$$

Now since 3 *R* 0, then by Lemma 8.3.2,

$$[3] = [0].$$

More generally, by the same reasoning,

$$[0] = [3] = [-3] = [6] = [-6] = \ldots, \text{ and so on.}$$

Similarly,

$$[1] = [4] = [-2] = [7] = [-5] = \ldots, \text{ and so on.}$$

Example 10 – *Solution*

cont'd

And

$$[2] = [5] = \lceil -1 \rceil = [8] = [-4] = \ldots, \text{ and so on.}$$

Notice that every integer is in class [0], [1], or [2]. Hence the distinct equivalence classes are

$$\{x \in \mathbf{Z} \mid x = 3k, \text{ for some integer } k\},$$

$$\{x \in \mathbf{Z} \mid x = 3k + 1, \text{ for some integer } k\}, \quad \text{and}$$

$$\{x \in \mathbf{Z} \mid x = 3k + 2, \text{ for some integer } k\}.$$

Example 10 – *Solution*

cont'd

In words, the three classes of congruence modulo 3 are (1) the set of all integers that are divisible by 3, (2) the set of all integers that leave a remainder of 1 when divided by 3, and (3) the set of all integers that leave a remainder of 2 when divided by 3.

Congruence Modulo *n*

Definition

Let m and n be integers and let d be a positive integer. We say that $\boldsymbol{m}$ **is congruent to** $\boldsymbol{n}$ **modulo** $\boldsymbol{d}$ and write

$$\boldsymbol{m \equiv n \ (\mathrm{mod}\ d)}$$

if, and only if, $d \mid (m - n)$.

Symbolically: $m \equiv n \ (\mathrm{mod}\ d) \quad \Leftrightarrow \quad d \mid (m - n)$

Example 11 – *Evaluating Congruences*

Determine which of the following congruences are true and which are false.

a. $12 \equiv 7 \pmod{5}$ $\quad$ $6 \equiv -8 \pmod{4}$ $\quad$ $3 \equiv 3 \pmod{7}$

Solution:

a. True. 12 – 7 = 5 = 5 · 1. Hence 5 | (12 – 7), and so 12 ≡ 7 (mod 5).

b. False. 6 – (–8) = 14, and $4 \nmid 14$ because 14 ≠ 4 · *k* for any integer *k*. Consequently, $6 \not\equiv -8 \pmod{4}$.

c. True. 3 – 3 = 0 = 7 · 0. Hence 7 | (3 – 3), and so 3 ≡ 3 (mod 7).

SECTION 8.5

Partial Order Relations

Antisymmetry

Antisymmetry

We have defined three properties of relations: reflexivity, symmetry, and transitivity. A fourth property of relations is called ***antisymmetry***.

In terms of the arrow diagram of a relation, saying that a relation is antisymmetric is the same as saying that whenever there is an arrow going from one element to another *distinct* element, there is *not* an arrow going back from the second to the first.

Antisymmetry

- **Definition**

Let R be a relation on a set A. R is **antisymmetric** if, and only if,

$$\text{for all } a \text{ and } b \text{ in } A, \quad \text{if } a\ R\ b \text{ and } b\ R\ a \text{ then } a = b.$$

By taking the negation of the definition, you can see that a relation *R* is ***not* antisymmetric** if, and only if,

there are elements a and b in A such that $a\ R\ b$, $b\ R\ a$ and $a \neq b$.

Example 2 – *Testing for Antisymmetry of "Divides" Relations*

Let R_1 be the "divides" relation on the set of all positive integers, and let R_2 be the "divides" relation on the set of all integers.

$$\text{For all } a, b \in Z^+, \qquad a \; R_1 \; b \quad \Leftrightarrow \quad a \mid b.$$

$$\text{For all } a, b \in Z, \qquad a \; R_2 \; b \quad \Leftrightarrow \quad a \mid b.$$

a. Is R_1 antisymmetric? Prove or give a counterexample.

b. Is R_2 antisymmetric? Prove or give a counterexample.

Example 2 – *Solution*

a. R_1 is antisymmetric.

Proof:
Suppose a and b are positive integers such that $a\ R_1\ b$ and $b\ R_1\ a$. *[We must show that a = b.]* By definition of R_1, $a \mid b$ and $b \mid a$.

Thus, by definition of divides, there are integers k_1 and k_2 with $b = k_1a$ and $a = k_2b$. It follows that

$$b = k_1a = k_1(k_2b) = (k_1k_2)b.$$

Dividing both sides by b gives

$$k_1k_2 = 1.$$

Example 2 – *Solution*

cont' d

Now since *a* and *b* are both positive integers k_1 and k_2 are both positive integers, too.

But the only product of two positive integers that equals 1 is 1 • 1.

Thus

$$k_1 = k_2 = 1$$

and so $$a = k_2 b = 1 \cdot b = b.$$

[This is what was to be shown.]

Example 2 – *Solution*

cont'd

b. R_2 is not antisymmetric.

Counterexample:

Let $a = 2$ and $b = -2$. Then $a \mid b$ *[since* $-2 = (-1) \cdot 2$*]* and $b \mid a$ *[*since $2 = (-1)(-2)$*]*.

Hence $a\ R_2\ b$ and $b\ R_2\ a$ but $a \neq b$.

Partial Order Relations

Partial Order Relations

A relation that is reflexive, antisymmetric, and transitive is called a ***partial order***.

> **• Definition**
>
> Let R be a relation defined on a set A. R is a **partial order relation** if, and only if, R is reflexive, antisymmetric, and transitive.

Two fundamental partial order relations are the "less than or equal" relation on a set of real numbers and the "subset" relation on a set of sets.

These can be thought of as models, or paradigms, for general partial order relations.

Partial Order Relations

- **Notation**

Because of the special paradigmatic role played by the $\leq$ relation in the study of partial order relations, the symbol $\preceq$ is often used to refer to a general partial order relation, and the notation $x \preceq y$ is read "x is less than or equal to y" or "y is greater than or equal to x."

Hasse Diagrams

Hasse Diagrams

It is possible to associate a graph, called a **Hasse diagram** (after Helmut Hasse, a twentieth-century German number theorist), with a partial order relation defined on a finite set.

To obtain a Hasse diagram, proceed as follows:

Start with a directed graph of the relation, placing vertices on the page so that all arrows point upward. Then eliminate

1. the loops at all the vertices,
2. all arrows whose existence is implied by the transitive property,
3. the direction indicators on the arrows.

Example 7 – *Constructing a Hasse Diagram*

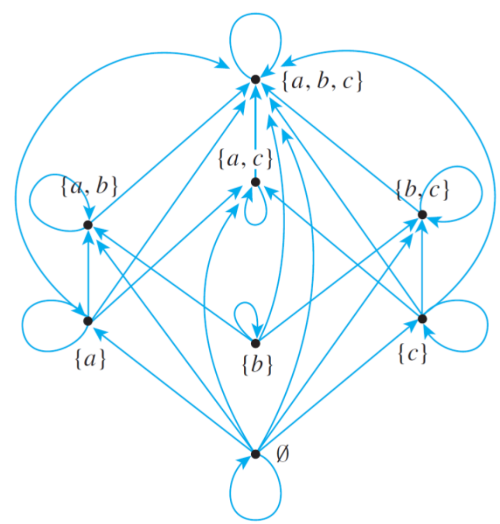
Consider the "subset" relation, ⊆, on the set $\mathscr{P}(\{a, b, c\})$. That is, for all sets *U* and *V* in $\mathscr{P}(\{a, b, c\})$,

$$\boxed{U \subseteq V \quad \Leftrightarrow \quad \forall x, \text{ if } x \in U \text{ then } x \in V.}$$

Construct the Hasse diagram for this relation.

Solution:

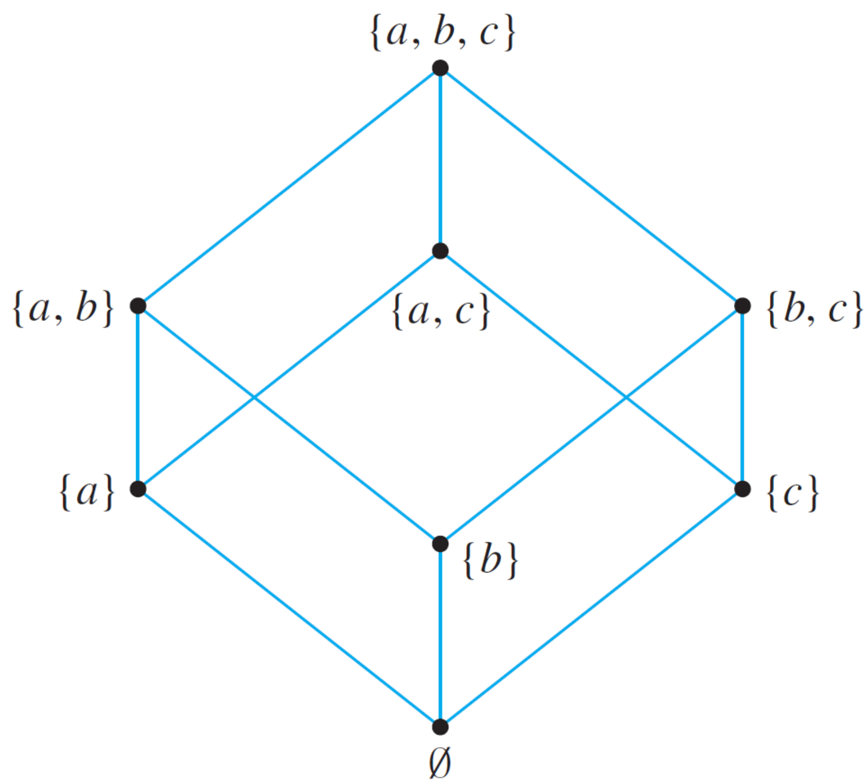
Draw the directed graph of the relation in such a way that all arrows except loops point upward.



Example 7 – *Solution*

cont'd

Then strip away all loops, unnecessary arrows, and direction indicators to obtain the Hasse diagram.



Partially and Totally Ordered Sets

Partially and Totally Ordered Sets

Given any two real numbers x and y, either $x \le y$ or $y \le x$.

In a situation like this, the elements x and y are said to be *comparable*.

On the other hand, given two subsets A and B of $\{a, b, c\}$, it may be the case that neither $A \subseteq B$ nor $B \subseteq A$.

For instance, let $A = \{a, b\}$ and $B = \{b, c\}$. Then $A \nsubseteq B$ and $B \nsubseteq A$.

In such a case, A and B are said to be *non-comparable*.

Partially and Totally Ordered Sets

• Definition

Suppose $\preceq$ is a partial order relation on a set A. Elements a and b of A are said to be **comparable** if, and only if, either $a \preceq b$ or $b \preceq a$. Otherwise, a and b are called **noncomparable.**

When all the elements of a partial order relation are comparable, the relation is called ***a total order***.

• Definition

If R is a partial order relation on a set A, and for any two elements a and b in A either $a\ R\ b$ or $b\ R\ a$, then R is a **total order relation** on A.

Partially and Totally Ordered Sets

The "less than or equal to" relation on real numbers is a total order relation.

Many important partial order relations have elements that are not comparable and are, therefore, not total order relations.

For instance, the subset relation on $\mathscr{P}(\{a, b, c\})$ is not a total order relation because, as shown previously, the subsets $\{a, b\}$ and $\{a, c\}$ of $\{a, b, c\}$ are not comparable.

Partially and Totally Ordered Sets

A set A is called a **partially ordered set** (or **poset**) with respect to a relation $\preceq$ if, and only if, $\preceq$ is a partial order relation on A.

For instance, the set of real numbers is a partially ordered set with respect to the "less than or equal to" relation **≤**, and a set of sets is partially ordered with respect to the "subset" relation $\subseteq$.

A set A is called a **totally ordered set** with respect to a relation $\preceq$ if, and only if, A is partially ordered with respect to $\preceq$ and $\preceq$ is a total order.

Partially and Totally Ordered Sets

A set that is partially ordered but not totally ordered may have totally ordered subsets. Such subsets are called ***chains***.

• Definition

Let A be a set that is partially ordered with respect to a relation $\preceq$. A subset B of A is called a **chain** if, and only if, the elements in each pair of elements in B is comparable. In other words, $a \preceq b$ or $b \preceq a$ for all a and b in A. The **length of a chain** is one less than the number of elements in the chain.

Observe that if *B* is a chain in *A*, then *B* is a totally ordered set with respect to the "restriction" of $\preceq$ to *B*.

Example 9 – *A Chain of Subsets*

The set $\mathscr{P}(\{a, b, c\})$ is partially ordered with respect to the subset relation. Find a chain of length 3 in $\mathscr{P}(\{a, b, c\})$.

Solution:

Since $\emptyset \subseteq \{a\} \subseteq \{a, b, \} \subseteq \{a, b, c\}$, the set

$$S = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$$

is a chain of length 3 in $\mathscr{P}(\{a, b, c\})$.

Partially and Totally Ordered Sets

A *maximal element* in a partially ordered set is an element that is greater than or equal to every element to *which it is comparable*. (There may be many elements to which it is *not* comparable.)

A *greatest element* in a partially ordered set is an element that is greater than or equal to *every* element in the set (so it is comparable to every element in the set).

Minimal and least elements are defined similarly.

Partially and Totally Ordered Sets

- **Definition**

Let a set A be partially ordered with respect to a relation $\preceq$.

1. An element a in A is called a **maximal element of A** if, and only if, for all b in A, either $b \preceq a$ or b and a are not comparable.
2. An element a in A is called a **greatest element of A** if, and only if, for all b in A, $b \preceq a$.
3. An element a in A is called a **minimal element of A** if, and only if, for all b in A, either $a \preceq b$ or b and a are not comparable.
4. An element a in A is called a **least element of A** if, and only if, for all b in A, $a \preceq b$.

Partially and Totally Ordered Sets

A greatest element is maximal, but a maximal element need not be a greatest element.

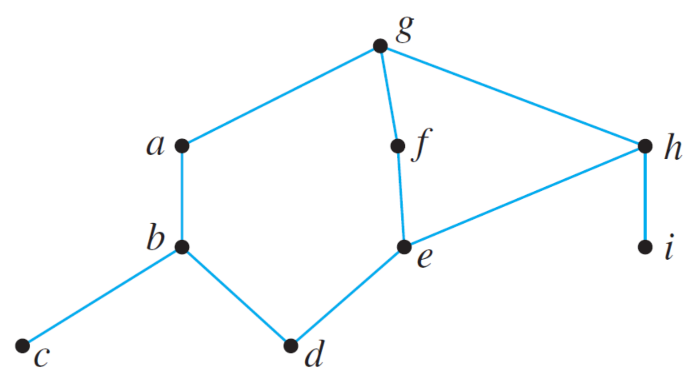
Similarly, a least element is minimal, but a minimal element need not be a least element.

Every finite subset of a totally ordered set has both a least element and a greatest element.

A partially ordered set can have at most one greatest element and one least element, but it may have more than one maximal or minimal element.

Example 10 – *Maximal, Minimal, Greatest, and Least Elements*

Let $A = \{a, b, c, d, e, f, g, h, i\}$ have the partial ordering $\preceq$ defined by the following Hasse diagram. Find all maximal, minimal, greatest, and least elements of A.



Solution:

There is just one maximal element, g, which is also the greatest element. The minimal elements are c, d, and i, and there is no least element.

Topological Sorting

Topological Sorting

Is it possible to input the sets of $\mathscr{P}(\{a, b, c\})$ into a computer in a way that is *compatible* with the subset relation $\subseteq$ in the sense that if set *U* is a subset of set *V*, then *U* is input before *V*?

The answer, as it turns out, is yes. For instance, the following input order satisfies the given condition:

$$\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}.$$

Another input order that satisfies the condition is

$$\emptyset, \{a\}, \{b\}, \{a, b\}, \{c\}, \{b, c\}, \{a, c\}, \{a, b, c\}.$$

Topological Sorting

• Definition

Given partial order relations $\preceq$ and $\preceq'$ on a set A, $\preceq'$ is **compatible** with $\preceq$ if, and only if, for all a and b in A, if $a \preceq b$ then $a \preceq' b$.

Given an arbitrary partial order relation $\preceq$ on a set *A*, is there a total order $\preceq'$ on *A* that is compatible with $\preceq$?

If the set on which the partial order is defined is finite, then the answer is yes. A total order that is compatible with a given order is called a ***topological sorting***.

• Definition

Given partial order relations $\preceq$ and $\preceq'$ on a set A, $\preceq'$ is a **topological sorting** for $\preceq$ if, and only if, $\preceq'$ is a total order that is compatible with $\preceq$.

Topological Sorting

Constructing a Topological Sorting

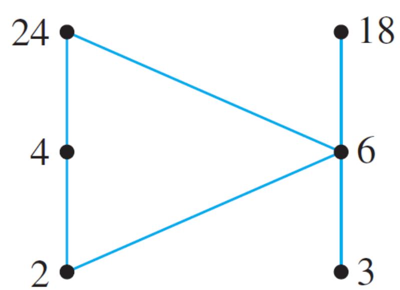
Let $\preceq$ be a partial order relation on a nonempty finite set A. To construct a topological sorting,

1. Pick any minimal element x in A. *[Such an element exists since A is nonempty.]*
2. Set $A' := A - \{x\}$.
3. Repeat steps a–c while $A' \neq \emptyset$.
   a. Pick any minimal element y in A'.
   b. Define $x \preceq' y$.
   c. Set $A' := A' - \{y\}$ and $x := y$.
   [Completion of steps 1–3 of this algorithm gives enough information to construct the Hasse diagram for the total ordering $\preceq'$. We have already shown how to use the Hasse diagram to obtain a complete directed graph for a relation.]

Example 11 – *A Topological Sorting*

Consider the set $A = \{2, 3, 4, 6, 18, 24\}$ ordered by the "divides" relation $|$.

The Hasse diagram of this relation is the following:



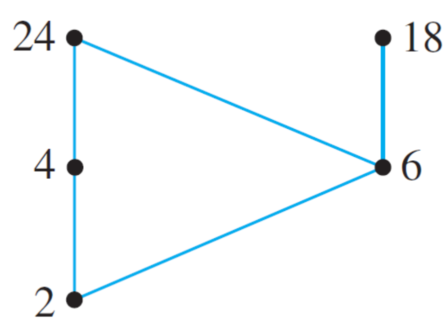
The ordinary "less than or equal to" relation $\leq$ on this set is a topological sorting for it since for positive integers a and b, if $a \mid b$ then $a \leq b$. Find another topological sorting for this set.

Example 11 – *Solution*

The set has two minimal elements: 2 and 3. Either one may be chosen; say you pick 3. The beginning of the total order is

total order: 3.

Set $A' = A - \{3\}$. You can indicate this by removing 3 from the Hasse diagram as shown below.



Example 11 – *Solution*

cont'd

Next choose minimal element from $A' - \{3\}$. Only 2 is minimal, so you must pick it. The total order thus far is

total order: $3 \preceq 2$.

Set $A' = (A - \{3\}) - \{2\} = A - \{3, 2\}$.

You can indicate this by removing 2 from the Hasse diagram, as is shown below.

Choose a minimal element from $A' - \{3, 2\}$.

Example 11 – *Solution*

cont'd

Again you have two choices: 4 and 6. Say you pick 6. The total order for the elements chosen thus far is

$$\text{total order: } 3 \preceq 2 \preceq 6.$$

You continue in this way until every element of *A* has been picked. One possible sequence of choices gives

$$\text{total order: } 3 \preceq 2 \preceq 6 \preceq 18 \preceq 4 \preceq 24.$$

Example 11 – *Solution*

cont'd

You can verify that this order is compatible with the "divides" partial order by checking that for each pair of elements a and b in A such that $a \mid b$, then $a \preceq b$.

Note that it is *not* the case that if $a \preceq b$ then $a \mid b$.

An Application

An Application

Consider the set of university courses. The following defines a partial order relation on the set of courses required for a university degree:

For all required courses *x* and *y*,

$$x \preceq y \quad \Leftrightarrow \quad x = y \quad \text{or} \quad x \text{ is a prerequisite for } y$$

The Hasse diagram for the relation can be used to answer some interesting questions.

An Application

For instance, consider the Hasse diagram for the pre-requisite courses at a particular university.

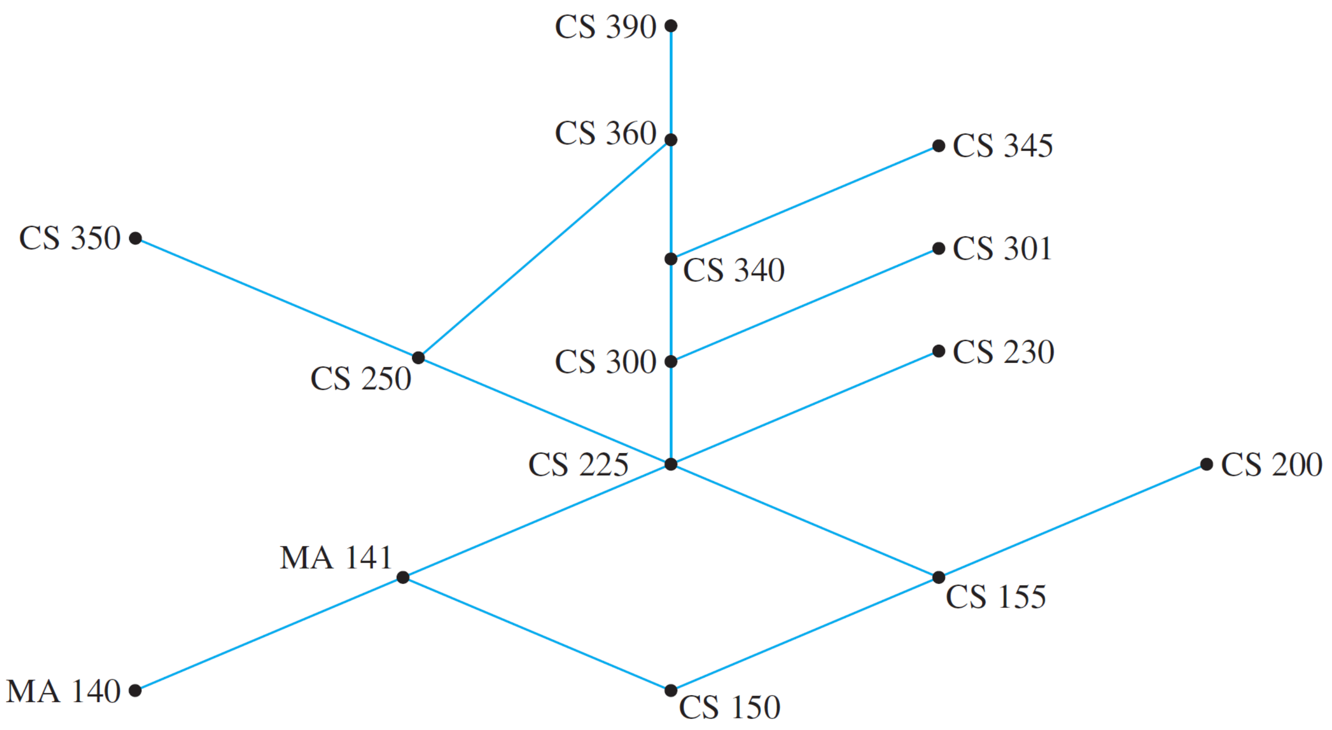


Figure 8.5.1

An Application

The minimum number of school terms needed to complete the requirements is the size of a longest chain, which is 7 (150, 155, 225, 300, 340, 360, 390, for example).

The maximum number of courses that could be taken in the same term (assuming the university allows it) is the maximum number of non-comparable courses, which is 6 (350, 360, 345, 301, 230, 200, for example).

An Application

A student could take the courses in a sequence determined by constructing a topological sorting for the set.

One such sorting is:

140, 150, 141, 155, 200, 225, 230, 300, 250, 301, 340, 345, 350, 360, 390.

There are many others.

PERT and CPM

PERT and CPM

Two important and widely used applications of partial order relations are **PERT** (Program Evaluation and Review Technique) and **CPM** (Critical Path Method).

These techniques came into being in the 1950s as planners came to grips with the complexities of scheduling the individual activities needed to complete very large projects, and although they are very similar, their developments were independent.

PERT and CPM

PERT was developed by the U.S.

PERT was developed by the U.S. Navy to help organize the construction of the Polaris submarine.

CPM was developed by the E. I. Du Pont de Nemours company for scheduling chemical plant maintenance.

Here is a somewhat simplified example of the way the techniques work.

Example 12 – *A Job Scheduling Problem*

At an automobile assembly plant, the job of assembling an automobile can be broken down into these tasks:

1. Build frame.

2. Install engine, power train components, gas tank.

3. Install brakes, wheels, tires.

4. Install dashboard, floor, seats.

Example 12 – *A Job Scheduling Problem* cont'd

5. Install electrical lines.

6. Install gas lines.

7. Install brake lines.

8. Attach body panels to frame.

9. Paint body.

Certain of these tasks can be carried out at the same time, whereas some cannot be started until other tasks are finished.

Example 12 – *A Job Scheduling Problem* cont'd

Table 8.5.1 summarizes the order in which tasks can be performed and the time required to perform each task.

Task	Immediately Preceding Tasks	Time Needed to Perform Task
1		7 hours
2	1	6 hours
3	1	3 hours
4	2	6 hours
5	2, 3	3 hours
6	4	1 hour
7	2, 3	1 hour
8	4, 5	2 hours
9	6, 7, 8	5 hours

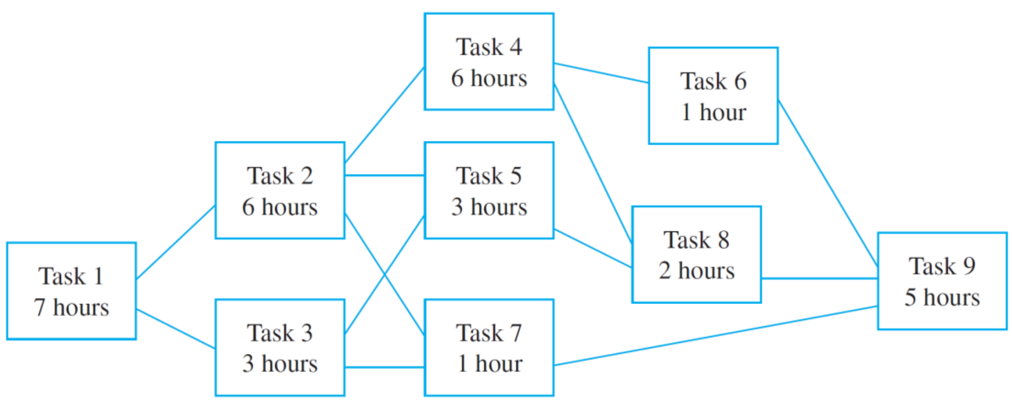
Table 8.5.1

Example 12 – *A Job Scheduling Problem* cont’d

Let *T* be the set of all tasks, and consider the partial order relation $\preceq$ defined on *T* as follows: For all tasks *x* and *y* in *T*,

$$x \preceq y \quad \Leftrightarrow \quad x = y \text{ or } x \text{ precedes } y.$$

If the Hasse diagram of this relation is turned sideways (as is customary in PERT and CPM analysis), it has the appearance shown below.



Example 12 – *A Job Scheduling Problem* cont’d

What is the minimum time required to assemble a car?

You can determine this by working from left to right across the diagram, noting for each task (say, just above the box representing that task) the minimum time needed to complete that task starting from the beginning of the assembly process.

For instance, you can put a 7 above the box for task 1 because task 1 requires 7 hours.

Task 2 requires completion of task 1 (7 hours) plus 6 hours for itself, so the minimum time required to complete task 2, starting at the beginning of the assembly process, is
$7 + 6 = 13$ hours.

Example 12 – *A Job Scheduling Problem* cont'd

You can put a 13 above the box for task 2.

Similarly, you can put a 10 above the box for task 3 because 7 + 3 = 10.

Now consider what number you should write above the box for task 5.

The minimum times to complete tasks 2 and 3, starting from the beginning of the assembly process, are 13 and 10 hours respectively.

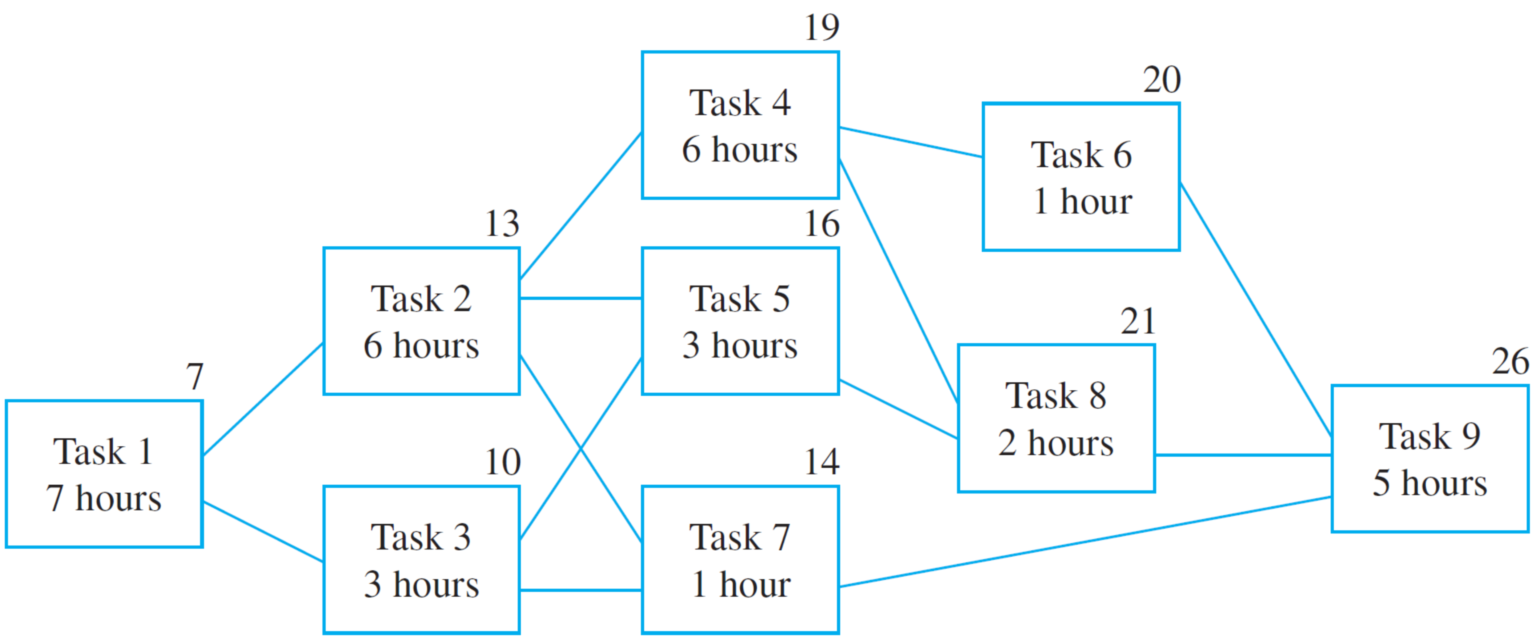
Example 12 – *A Job Scheduling Problem* cont'd

Since *both* tasks must be completed before task 5 can be started, the minimum time to complete task 5, starting from the beginning, is the time needed for task 5 itself (3 hours) plus the *maximum* of the times to complete tasks 2 and 3 (13 hours), and this equals 3 + 13 = 16 hours.

Thus you should place the number 16 above the box for task 5. The same reasoning leads you to place a 14 above the box for task 7.

Example 12 – *A Job Scheduling Problem* cont'd

Similarly, you can place a 19 above the box for task 4, a 20 above the box for task 6, a 21 above the box for task 8, and a 26 above the box for task 9, as shown below.



Example 12 – *A Job Scheduling Problem* cont'd

This analysis shows that at least 26 hours are required to complete task 9 starting from the beginning of the assembly process. When task 9 is finished, the assembly is complete, so 26 hours is the minimum time needed to accomplish the whole process.

Note that the minimum time required to complete tasks 1, 2, 4, 8, and 9 in sequence is exactly 26 hours.

This means that a delay in performing any one of these tasks causes a delay in the total time required for assembly of the car.

For this reason, the path through tasks 1, 2, 4, 8, and 9 is called a **critical path.**