Exercises for Discrete Maths

Discrete Maths

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http://www.inf.unibz.it/~artale/DML/dml.htm

Week 6

Computer Science

Free University of Bozen-Bolzano

Graph Theory

Exercise Set 10.2: Euler Circuits

In the following cases, answer to this question: Does the graph have an Euler circuit? Justify your answers.

Recall that: An *Euler circuit* for G is a *circuit* that contains every vertex and every edge of G. (That is, an Euler circuit for G is a sequence of adjacent vertices and edges in G that has at least one edge, starts and ends at the same vertex, uses every vertex of G at least once, and uses every edge of G exactly once.)

Theorem 10.2.4. A graph G has an Euler circuit if, and only if, G is connected and every vertex of G has positive even degree.

Exercise 9.

a. G is a connected graph with five vertices of degrees 2, 2, 3, 3, and 4.

Answer. No, since there are vertices with odd degrees.

b. G is a connected graph with five vertices of degrees $2, 2, 4, 4$, and 6.

Answer. Yes, since all degrees are even and the graph is connected.

c. G is a graph with five vertices of degrees $2, 2, 4, 4$, and 6.

Answer. Not necessarily. Take a graph G with vertices A, B, C, D, E, each with a loop. Let G have two edges connecting A and B, and two other edges connecting A and C. Then G has five vertices of degrees $2, 2, 4, 4$, 6, and it is not connected. Thus it does not have an Euler circuit.

c' (sligthly modified). G is a graph without parallel edges and with five vertices of degrees $2, 2, 4, 4$, and 6.

Answer. Since the graph has no parallel edges, the vertex of degree 6 must have 4 distinct connected vertices, and a loop. Then the graph is connected with vertices of positive degrees only. Therefore the graph has an Euler circuit.

Exercise 12 & 13 (Homework). Check if the graphs in Fig. 1 and Fig. 2 have an Euler circuit. If the graph does not have an Euler circuit, explain why not. If it does have an Euler circuit, describe one.

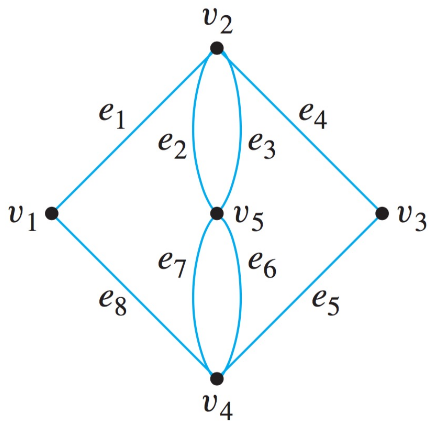

FIGURE 1. Graph for Exercise 12

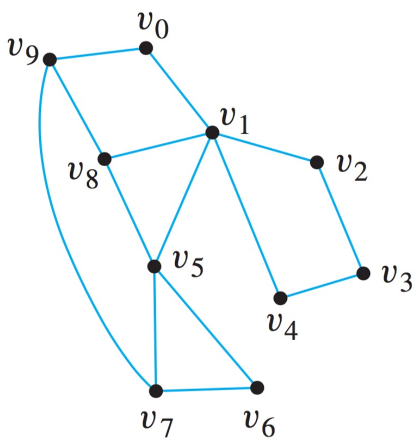

FIGURE 2. Graph for Exercise 13

Answer. Ex.12: Yes. One Euler circuit is $e4e5e6e3e2e7e8e1$.

Answer. Ex.13: No, since it has vertices with odd degree.

EXERCISE SET 10.4: GRAPH ISOMORPHISMS

Determine whether the graphs shown are isomorphic or not.

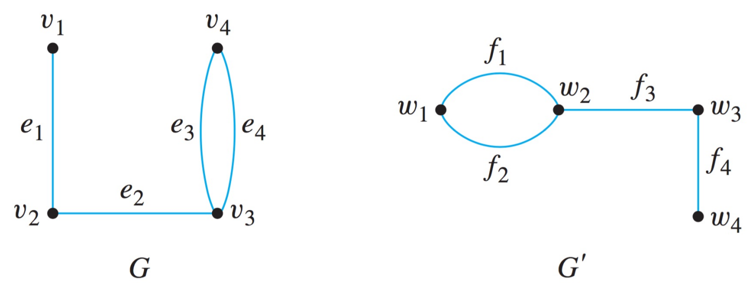

FIGURE 3. Graph for Exercise 1

Exercise 1. **Answer.** They are isomorphic.

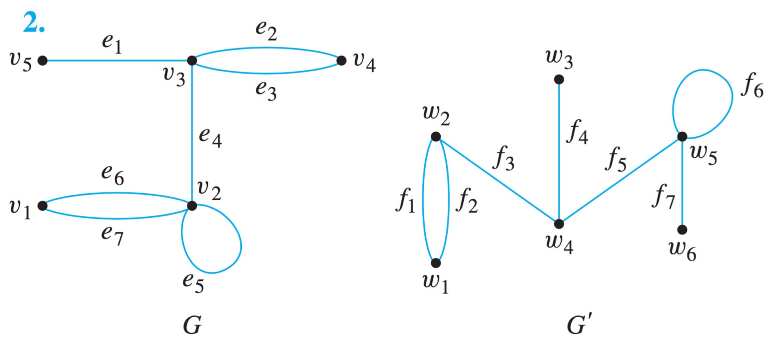

FIGURE 4. Graph for Exercise 2

Exercise 2. **Answer.** They are not since deg is an invariant and $deg(v2) = 5$ and there is not w in G' with such a degree.

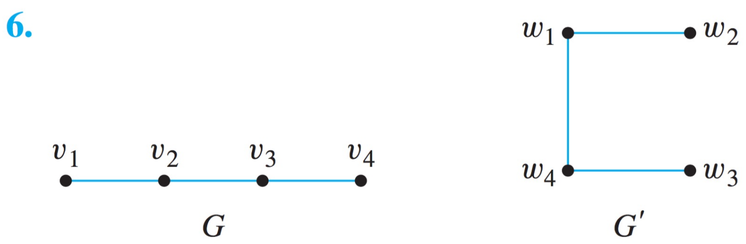

FIGURE 5. Graph for Exercise 6

Exercise 6 (Homework). **Answer.** Yes they are isomorphic.

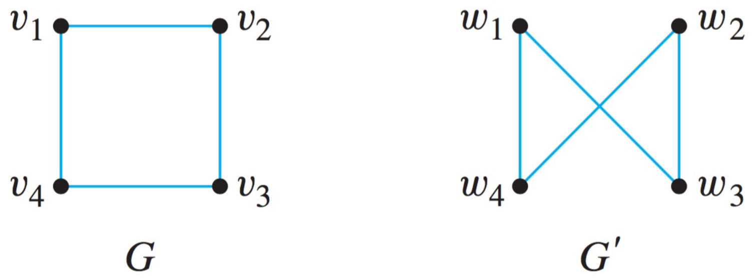

FIGURE 6. Graph for Exercise 7

Exercise 7 (Homework). **Answer.** Yes they are. Pull w_2 below w_3. Take the mapping $v1 \mapsto w1$, $v2 \mapsto w3$, $v4 \mapsto w4$, $v3 \mapsto w2$. It preserves the edge relation, i.e., if there is an edge between v_n and v_m then there is an edge between the images of v_n and v_m.

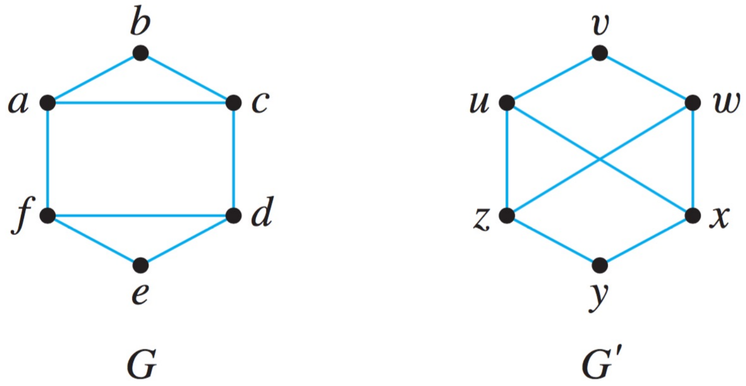

FIGURE 7. Graph for Exercise 8

Exercise 8. **Answer.** The two graphs are not isomorphic. G has a circuit of length 3 while G' does not.

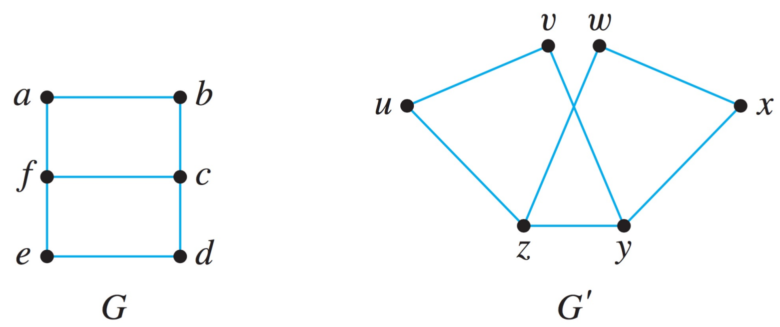

FIGURE 8. Graph for Exercise 9

Exercise 9 (Homework). **Answer.** Yes. The function is the following:

$a \mapsto u$

$b \mapsto v$

$f \mapsto z$

$c \mapsto y$

$e \mapsto w$

$d \mapsto x$

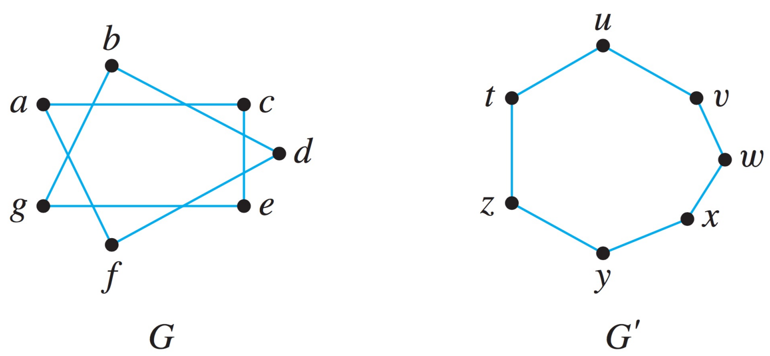

FIGURE 9. Graph for Exercise 10

Exercise 10 (Homework). **Answer.** The graph are isomorphic. Firstly pull a and c down. Then move e to the left of g. The mapping is now easier to spot.

Exercise 14. Draw all non-isomorphic simple graphs with three vertices.

Solution. See Figure 10.

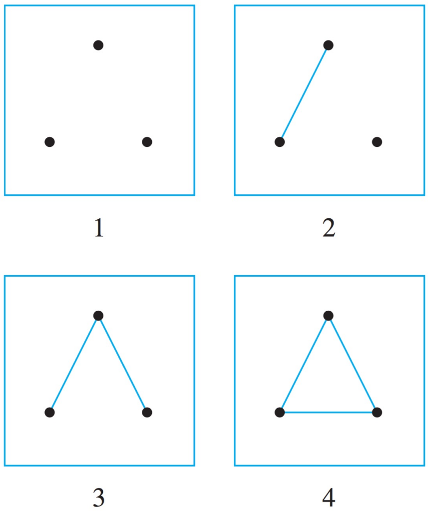

FIGURE 10. Solution for Exercise 14

EXERCISE SET 10.5: TREES

Exercise 3. What is the total degree of a tree with n vertices? Why?

Solution. $2n-2$ (For any $n \in \mathbb{N}$, any tree with n vertices has $n-1$ edges; the degree of a tree/graph is 2$\cdot$ number of edges).

Exercise 8. Is there a tree with 9 vertices and 9 edges?

Answer. No, due to the previous theorem: any tree with n vertices has $n-1$ edges.

Exercise 12 (Homework). Is there a tree with 5 vertices and total degree 8?

Answer. The tree has $4 = 5-1$ edges, and hence total degree $2 \cdot 4 = 8$. Yes, there can be such a tree. For instance, one has two leaves (each with degree 1). The other vertices are internal of degree 2 and connected to each other and the other two vertices.

EXERCISE SET 10.6: BINARY TREES

In each of the following exercises, either draw a graph with the given specifications, or explain why no such graph exists.

Exercise 11. A binary tree with height 3 and 9 terminal vertices.

Solution. Remember the following: if T is any binary tree with height h and t terminal vertices, then $9 = t \leq 2^h = 8$. So no.

Exercise 12. A full binary tree, 8 internal vertices (k), and 7 terminal vertices.

Solution. Remember the following: If T is a full binary tree with $k > 0$ internal vertices, then T has a total of $2k + 1$ vertices and has $k + 1$ terminal vertices. So the number of terminal vertices is $k + 1 = 8 + 1$. So no.

Exercise 18 (Homework). Tree, 5 vertices, total degree 10.

Solution. No. By Ex. 3, Set 10.5, the total degree of a tree with n vertices is 2n-2. Hence $10 = 2 \cdot 5 - 2$ which is false.