Exercises for Discrete Maths

Discrete Maths

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http://www.inf.unibz.it/~artale/DML/dml.htm

Week 5

Computer Science

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Exercise Set 8.5: Partial Orders

Exercise 3. Let S be the set of all strings of a's and b's. Let $\mathrm{l}(s)$ define the lenght of a string s of S. Define a binary relation R on S as follows: for all s, $t \in S$, sRt iff $\mathrm{l}(s) \leq \mathrm{l}(t)$. Is R a partial order? No, because it is not antisymmetric. Show this, by giving a counterexample.

Proof. R is not antisymmetric. Take 2 distinct strings, s, t, with $\mathrm{l}(s) = \mathrm{l}(t)$. Then, both sRt and tRs hold, but s and t are distinct.

Exercise 5. Let $\mathbb{R}$ be the set of all real numbers, and define a binary relation R on $\mathbb{R} \times \mathbb{R}$: for all (a, b), $(c, d) \in \mathbb{R} \times \mathbb{R}$, $(a, b)R(c, d)$ iff either $a < c$ or both $a = c$ and $b \leq d$. Prove that R is a partial order relation.

Proof. R is reflexive, antisymmetric and transitive. For a proof, see the book.

Exercise 8 (Homework). Define a relation R on the set $\mathbb{Z}$ of all integers as follows: for all $m, n \in \mathbb{Z}$, mRn iff $m + n$ is even. Is R a partial order relation? Prove or give a counterexample.

Proof. No. The relation is symmetric, hence it cannot be a partial order.

Exercise 31. Let $A = \{a, b, c, d\}$, and let R be the relation defined as follows:

$$R = \{(a, a), (b, b), (c, c), (d, d), (c, a), (a, d), (c, d), (b, c), (b, d), (b, a)\}.$$

Is R a total order on A? Justify your answer.

Proof. R is reflexive. R is antisymmetric. R is transitve. Therefore R is a partial order. Since all elements are comparable, R is a total order.

Exercise 34. Suppose that R is a partial order relation on a set A and that B is a subset of A. Show that the restriction of R to B, that is, R_B, is also a partial order.

Proof. It follows from R being a partial order relation.

Exercise 50. A set S of jobs can be ordered by writing $x \leq y$ to mean that either $x = y$ or x must be done before y, for all x and y in S. Given the Hasse diagram for this relation for a particular set S of jobs (see Figure 1 below), show the following:

(1) minimal, least, maximal, and greatest elements;
(2) a topological sort.

Solution. Minimal $= \{1, 2, 9\}$. Least does not exist. Maximal = Greatest = 3. A topological sort requires to iteratively choose one of the minimal elements as least, e.g.,

$$1 \leq 9 \leq 2 \leq 10 \leq 6 \leq 8 \leq 5 \leq 7 \leq 4 \leq 3.$$

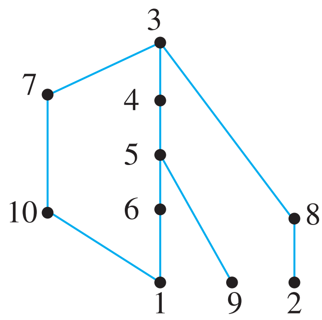

FIGURE 1. Hasse diagram of Exercise 50

EXERCISE SET 10.1: GRAPHS

Exercise 15. A graph has vertices of degrees 0, 2, 2, 3, and 9. How many edges does the graph have?

Solution. Theorem 10.1.1 (The Handshake Theorem) states that the sum of the degrees of the verteces of a graph (that is, the degree of the graph) is always twice the number of edges of the graph. Therefore the graph has $\frac{0+2+2+3+9}{2} = 8$ edges.

Exercise 17. Decide whether there exists a graph with 5 vertices of degree 1, 2, 3, 3, and 5, respectively.

Solution. Yes. See Figure 2.

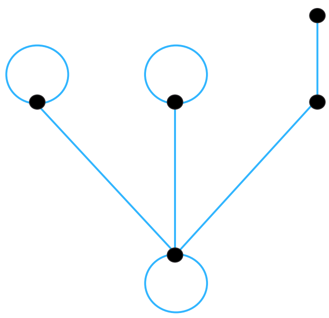

FIGURE 2. Graph of Exercise 17

Exercise 18 (Homework). Decide whether there exists a graph with four vertices of degrees 1, 2, 3, and 3.

Solution. The graph does not exist because $1+2+3+3 = 9$ and, by the Handshake Theorem (Theorem 10.1.1), the sum of the degrees of the verteces of a graph (that is, the total degree of the graph) is always even number (being twice the number of edges of the graph).
Alternative proof: there is not such a graph because, by Proposition 10.1.3, there is an *even* number of vertices of odd degree in any graph, hence there cannot be 3.

Exercise 21. Is there a simple graph G with four vertices of degrees $1, 2, 3$, and 4?

Answer. Such a graph does not exist. A vertex v with $deg(v) = 4$ needs to be connected to 4 distinct vertices, since a simple graph is not allowed to have loops or parallel edges.

Exercise 24 (Homework). Simple graph with six edges and all vertices of degree 3.

Answer. For having all vertices of degree 3, the graph should have 4 vertices with two diagonals.

Exercise 29 (Homework). Is there a simple graph, each of whose vertices has even degree? Explain.

Solution. Yes. Consider a graph that forms a geometric figure, e.g., a triangle. This is a simple circuit and each vertex has degree 2.

Exercise 33 (Homework). Recall that K_n denotes a complete graph on n vertices, that is, a simple graph with n vertices and exactly 1 edge between each pair of distinct vertices. Show that for all integers $n \geq 1$, the number of edges of K_n is: $\frac{n\cdot(n-1)}{2}$.

Proof. The statement can be proved by induction, since K_{n+1} can be obtained starting from K_n and by adding a vertex and connecting it to the other n vertices.

K_1 has 1 vertex and 0 edges $= \frac{(1\cdot 0)}{2}$.

Assume that K_n has $\frac{n\cdot(n-1)}{2}$ edges. K_{n+1} is obtained by K_n adding an $(n+1)$th vertex, and connecting it with all the other n vertices through n distinct edges. Therefore K_{n+1} has $n + n\frac{\cdot(n-1)}{2}$ edges, that is

$$(2n + n^2 - n)/2 = (n^2 + n)/2 = (n + 1) \cdot n/2$$

QED.

Alternatively, use the Handshake Theorem: 2 times number of edges of $G = deg(G) = \sum_{i=1}^{n} deg(v_i)$. Since, by definition, v_i has $(n-1)$ edges (1 for each of the other $(n-1)$ vertices), then, for each $i = 1 \ldots n$, $deg(v_i) = (n-1)$. Therefore, 2 times the number of edges of $G = n \cdot (n-1)$, that is, the number of edges of $G = n \cdot (n-1)/2$.

Exercise Set 10.2: Paths, Trails, Walks and Circuits

Exercise 4. Consider the graph G in the textbook, reported in Figure 3.

a. How many paths are there from $v1$ to $v4$?
b. How many trails are there from $v1$ to $v4$?
c. How many walks are there from $v1$ to $v4$?

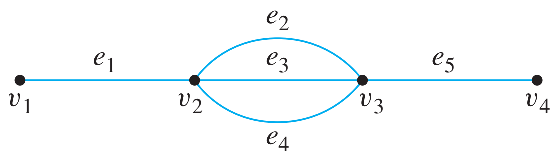

Figure 3. Graph of Exercise 4

Solution.
Remember what follows:

(1) A walk from a vertex v to a vertex w is a finite alternating sequence of adjacent vertices and edges of G.
(2) A trail from a vertex v to a vertex w is a walk from v to w that does not contain a repeated edge.
(3) A path from a vertex v to a vertex w is a trail from v to w that does not contain a repeated vertex.

a. G has 3 paths;
b. G has $3 + 3!$ trails;
c. G has infinitely many walks.