Exercises for Discrete Maths

Discrete Maths

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Week 4

Computer Science

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Exercise Set 7.4, p. 440: Cardinality and Computability

Exercise 26. Prove that any infinite set A contains a countably infinite subset.

Proof. We construct inductively a function $f : \mathbb{N} \mapsto A$.

Basis Step: Pick an arbitrary[1] element $a_1 \in A$. Let $f(1) = a_1$.

Inductive Step: Assume that $f(n)$ has been defined for $n \geq 1$. Now, $A - \{f(1), \ldots, f(n)\} \neq \emptyset$ because A is infinite. Pick an arbitrary $b \in A - \{f(1), \ldots, f(n)\}$. Define $f(n+1) = b$.

Next we prove that f is injective. If $1 \leq m < n$ then $f(m) \in \{f(1), \ldots, f(n-1)\}$ whereas $f(n) \in A - \{f(1), \ldots, f(n-1)\}$. Thus $f(n) \neq f(m)$, that is, f is bijective from $\mathbb{N}$ to $f(\mathbb{N})$. Thus $f(\mathbb{N})$ is countable by definition of countable set.

Exercise Set 8.1, p. 449: Relations

Exercise 17. Let $A = \{2, 3, 4, 5, 6, 7, 8\}$ and R a relation over A. Draw the directed graph of R, after realizing that xRy iff $x - y = 3n$ for some $n \in \mathbb{Z}$. Check that R is an equivalence relation.

Solution. The relation is:

$$\begin{aligned} R = \{&(2,2), (3,3), (4,4), (5,5), (6,6), (7,7), (8,8), \\ &(8,5), (8,2), (7,4), (6,3), (5,2) \\ &(5,8), (2,8), (4,7), (3,6), (2,5)\}\,. \end{aligned}$$

Exercise Set 8.2, pp. 458–459: Properties of Relations

Exercise 1. Let $A = \{0, 1, 2, 3\}$ and R a relation over A:

$$R = \{(0,0), (0,1), (0,3), (1,1), (1,0), (2,3), (3,3)\}$$

Draw the directed graph of R. Check whether R is an equivalence relation. Give a counterexample in each case in which the relation does not satisfy one of the properties of being an equivalence relation.

Solution.
R is not reflexive because $(2,2) \notin R$. It is not symmetric because $(3,2) \notin R$. It is not transitive because $(1,0)$ and $(0,3)$ are in R but $(1,3) \notin R$.

Exercise 20. Let $X = \{a, b, c\}$ and 2^X be the power set of X. A relation R is defined on 2^X as follows: For all $A, B \in 2^X, (A, B) \in R$ *iff* the number of elements in A equals the number of elements in B. Show that R is an equivalence relation.

[1] Formally, the Axiom of Choice allows us to do so.

Exercise 21. Let $X = \{a, b, c\}$ and 2^X be the power set of X. A relation R is defined on 2^X as follows: For all $A, B \in 2^X, (A, B) \in R$ *iff* the number of elements in A is less than the number of elements in B. Show that R is not an equivalence relation.

Exercise 37. If R and S are reflexive, then $R \cap S$ is so. Explain why.

Exercise 38. If R and S are symmetric, then $R \cap S$ is so. Explain why.

Exercise 39. If R and S are transitive, then $R \cap S$ is so. Explain why.

Exercise 40. If R and S are reflexive, then $R \cup S$ is so. Explain why.

Exercise 41. If R and S are symmetric, then $R \cup S$ is so.
Proof. Let $(x, y) \in R \cup S$. Then either $(x, y) \in R$ and then $(y, x) \in R$, or $(x, y) \in S$ and then $(y, x) \in S$. Thus, $(y, x) \in R \cup S$.

Exercise 42. If R and S are transitive, then $R \cup S$ is not necessarily so.
Counter-example: $R = \{(a, b)\}$ and $S = \{(b, c)\}$.

Exercise 51. Let $R = \{(0, 1), (0, 2), (1, 1), (1, 3), (2, 2), (3, 0)\}$. Find its transitive closure R^t, after drawing the directed graph of R.

Exercise Set 8.3, p. 475–477: Equivalence Relations

Exercise 2. A relation R induced by a partition is an equivalence relation—reflexive, symmetric, transitive. See Theorem 8.3.1.

a) Let $A = \{0, 1, 2, 3, 4\}$ and let a partition be $P = \{\{0, 2\}, \{1\}, \{3, 4\}\}$. Find the ordered pairs in R.

Solution.

Then equivalence classes are:

$$\begin{aligned} \{0, 2\} &= [0] = [2] \\ \{1\} &= [1] \\ \{3, 4\} &= [3] = [4] \end{aligned}$$

and hence

$$R = \{(0, 0), (2, 2), (0, 2), (2, 0), (1, 1), (3, 3), (4, 4), (3, 4), (4, 3)\}.$$

b) Let $A = \{0, 1, 2, 3, 4\}$ and let a partition be $P = \{\{0\}, \{1, 3, 4\}, \{2\}\}$.

Solution.

Reasoning as above,

$$R = \{(0, 0), (1, 1), (3, 3), (4, 4), (1, 3), (3, 1), (1, 4), (4, 1), (3, 4), (4, 3), (2, 2)\}.$$

c) Let $A = \{0, 1, 2, 3, 4\}$ and let a partition be $P = \{\{0\}, \{1, 2, 3, 4\}\}$.
Solution.
Reasoning as above,

$$\begin{aligned} R = \{&(0,0),(1,1),(2,2),(3,3),(4,4),(1,2),\\ &(2,1),(1,3),(3,1),(1,4),(4,1),(2,3),(3,2),(2,4),(4,2),(3,4),(4,3)\}. \end{aligned}$$

Exercise 8. Consider the powerset of $X = \{a, b, c\}$ and define R on the powerset as follows: URV iff U and V have the same cardinality. Find the equivalence classes of R.
Solution.
The equivalence classes are:
$[\{\emptyset\}] = \{\emptyset\}$; $[\{a\}] = \{\{a\}, \{b\}, \{c\}\}$; $[\{a, b\}] = \{\{a, b\}, \{a, c\}, \{b, c\}\}$;
$[\{a, b, c\}] = \{\{a, b, c\}\}$.

Exercise 28. Consider the following relation I over reals: xIy iff $(x - y) \in \mathbb{Z}$. Prove that it is an equivalence and characterize its equivalence classes. See the book solution.

Exercise 46. Let R be a relation on a set A and suppose R is symmetric and transitive. Prove the following: If for every x in A there exists a y in A such that xRy, then R is an equivalence relation.

Proof. For every x in A there is a y in A such that xRy, then, by symmetry, yRx, and by transivity, xRx. Thus R is also reflexive and so it is an equivalence relation.