Exercises for Discrete Maths

Discrete Maths

Teacher: Alessandro Artale
Teaching Assistants: Elena Botoeva, Daniele Porello
http://www.inf.unibz.it/~artale/DML/dml.htm

Week 2

Computer Science

Free University of Bozen-Bolzano

EXERCISES FOR SECTION 5.3 (P. 266)

Induction

Example 5.3.2. $\forall n \geq 3$ we have that $2n + 1 < 2^n$

5.3.16. Show that $2^n < (n+1)!$, for $n \geq 2$.

Proof. The base step is as follows:

Base Step: $n = 2$: we have to check that $2^2 < 3!$, but $4 < 6$, so this is true.

Inductive Step: Assume $2^k < (k+1)!$, for $k \geq 2$. We have

$$2^{k+1} = 2 \cdot 2^k < 2 \cdot (k+1)!$$

Since $k \geq 2$, we have $(k+2) > 2$. Therefore:

$$2 \cdot (k+1)! < (k+2) \cdot (k+1)! = (k+2)!$$

This completes the inductive step.

5.3.19. Show that $n^2 < 2^n$, for $n \geq 5$.

Proof. The base step is as follows:

Base Step: $n = 5$: $5^2 = 25 < 2^5 = 32$, which is true.

Inductive Step: Assume $k^2 < 2^k$, for $k \geq 5$ We have:

$$(k+1)^2 = k^2 + 2k + 1 < 2^k + 2k + 1$$

It suffices to show that (*): $2k + 1 < 2^k$, for $k \geq 5$. That was proved for Example 5.3.2. This completese the inductive proof.

Loop Invariants

Exercise 5.5.6. Pre-condition: m nonnegative integer, x is a real number, $i = 0$, and $exp = 1$

Program:
while $(i \neq m)$
1. $exp := exp \cdot x$
2. $i := i + 1$
end while

Post-condition: $exp = x^m$

Loop-invariant: $I(n)$ is '$i = n$ and $exp = x^n$'

Proof. :

Basis Property: $n = 0$, then $i = 0$, $exp = x^0 = 1$.

Inductive Property:
$i \neq m$ and $I(k)$, then:
by 1.: $exp_{new} = exp_{old} \cdot x = x^k \cdot x = x^{k+1}$
by 2.: $i_{new} = i_{old} + 1 = k + 1$

Eventual Falsity of the Guard: At each iteration, $i = i + 1$, and $i = 0$ at the start, so after m iterations we have $i = m$.

Correctness of the Post-Condition: Guard false implies $i = m$ after m iterations and $I(m) = x^m$.

Structural Induction

Exercise 5.9.5. Define a set S recursively as follows:

I.: Base: $1 \in S$
II.: Recursion: If $s \in S$, then a) $0s \in S$ and b) $1s \in S$.
III.: Restriction: Nothing is in S other than objects defined in I. and II. above.

Use structural induction to prove that every string in S ends in a 1.

Basis Property: '1' ends in 1.

Inductive Property: Assume that s ends with '1'. Then:

a) $0s$ ends in 1 and b) $1s$ ends in 1.

Exercise 5.9.10. Define a set S recursively as follows:

I.: Base: $0 \in S$ and $5 \in S$
II.: Recursion: If $s \in S$ and $t \in S$, then a) $s + t \in S$ and b) $s - t \in S$.
III.: Restriction: Nothing is in S other than objects defined in I. and II. above.

Use structural induction to prove that every integer in S is divisible by 5.

Basis Property: 0 and 5 are both divisible by 5.

Inductive Property: Assume that s and t are divisible by 5, i.e. there are integers k and q such that $s = 5 \cdot k$ and t $= 5 \cdot q$. Then:

a) $s + t = 5 \cdot k + 5 \cdot q = 5(k + q)$,
b) $s - t = 5 \cdot k - 5 \cdot q = 5(k - q)$,
so $s + t$ and $s - t$ are both divisible by 5.