Exercises for the Logic Course

First Order Logic

Course Web Page
http://www.inf.unibz.it/~artale/DML/dml.htm

Computer Science

Free University of Bozen-Bolzano
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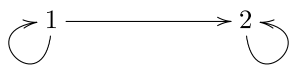
1 Exercises

1.1 Formalisation

1.1.1 Graph properties

Formalise properties of the below graph first using the language with only the binary relation symbol R, and then using the language with the binary relation symbol R, and the constant symbols c_1 and c_2.



1.1.2 Program properties *(Homework)*

Write a first-order formula expressing that an array of size 3 is sorted in decreasing order.

Solution. A sentence expressing this is

$$\forall i \left(\Big((\mathsf{0} = i \vee \mathsf{0} < i) \wedge i < \mathsf{3} \Big) \to \Big(arr\big(i\big) < arr\big(s(i)\big) \Big) \right)$$

with $\mathcal{L} = \langle <; s, arr; \mathsf{3}, \mathsf{0} \rangle$ where $\mathsf{0}$ is a constant symbol which stands for 0, $\mathsf{3}$ is a constant symbol which stands for 3 (the size of the array), s is a unary function symbol and $s(i)$ stands for "successor of i", arr is a unary function symbol and $arr(i)$ stands for "i-th element of the array", $<$ is a binary relation symbol which stands for the standard linear order over naturals (we wrote $s < t$ instead of $<(s,t)$ for keeping the sentence readable).

1.2 Substitutions

Consider a language with a constant symbol c, binary predicate r and unary predicate p. Compute the following simultaneous substitutions of the constant symbol for all the *free* occurrences of a variable:

- $\forall y\, r(x, y)[x/c]$ (read: "substitution of c for x");
- $\forall x\, p(x)[x/c]$;
- $\forall y\, \big(\forall x\, r(x, y) \wedge r(x, y)\big)[x/c]$;
- $\big(r(x, y) \wedge \forall y\, r(x, y)\big)[x/c, y/d]$.

1.3 Semantics and Informal Semantic Arguments

1.3.1 Model Checking

Read Definition 7.19 in your textbook (p. 137) and Theorem 7.20 (p. 138) for tackling the following exercises.

(1) Consider a first order language with a binary predicate r, and a unary predicate p. Consider the interpretation $\mathcal{I}$ with domain $\{0, 1\}$, and

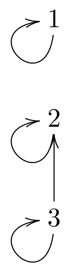
$$\begin{aligned} p^{\mathcal{I}} &= \{0, 1\}\,; \\ r^{\mathcal{I}} &= \{(0, 0), (0, 1)\}\,. \end{aligned}$$

Check whether $\mathcal{I}$ is a model or a counter-model of the following three formulae, and justify your answers:

$$\begin{aligned} &\forall x p(x); \\ &\forall x \exists y\, r(x, y); \\ &\exists x \forall y\, r(x, y). \end{aligned}$$

(Visualisation aid: depict the interpretation as a (directed) graph with nodes 0, 1 *and* $p^{\mathcal{I}}$ *predicating that both nodes are, say, pink coloured. Then check the above properties* also *pictorially.)*

(2) (Homework) Consider the graph on the right and the first-order language with only the binary predicate r.



– Define in set-theoretic terms the interpretation $\mathcal{I}$ with: domain equal to the set of nodes of the graph; $r^{\mathcal{I}}$ equal to the set of edges of the graph.

– Determine which of the following formulas $\mathcal{I}$ satisfies: (i) $\exists x \neg r(x, x)$; (ii) $\forall x \exists y (r(x, y) \wedge \neg r(y, x))$.

– Define a closed formula different than the above two and of which $\mathcal{I}$ is a model.

Justify your answers.

1.3.2 Satisfiability: Model or Countermodel Building

(1) Consider a first-order language with one binary predicate, r, and one constant, c. Let φ be the following closed formula:

$$\forall x \exists y \big(r(x, y) \wedge r(y, c)\big).$$

Define two interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$, each with underlying domain $\{0, 1\}$, such that $\mathcal{I}_1 \models \varphi$ (i.e., $\mathcal{I}_1$ is a model of φ) and $\mathcal{I}_2 \not\models \varphi$ (i.e., $\mathcal{I}_2$ is a counter-model of φ).

Solution. Take $\mathcal{I}_1$ and $\mathcal{I}_2$ as follows: both $r^{\mathcal{I}_1}$ and $r^{\mathcal{I}_2}$ are $\geq$; $c^{\mathcal{I}_1}$ is 0, whereas $c^{\mathcal{I}_2}$ is 1. Then $\mathcal{I}_1 \models \varphi$ but $\mathcal{I}_2 \not\models \varphi$ (prove it as in 1.3.1)

(2) (Homework) Consider the language with one binary predicate, r. Let φ be the following closed formula:

$$\forall x \exists y \Big(r(x, y) \wedge \forall z \big(r(y, z) \rightarrow \neg r(x, z)\big)\Big).$$

Define two interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ for φ so that: both have domains equal to the set of all natural numbers, $\mathbb{N}$, but $\mathcal{I}_1$ is a model of φ, and $\mathcal{I}_2$ is a counter-model of φ.

Solution. Define $r^{\mathcal{I}_1}$ as the successor relation over natural numbers and $r^{\mathcal{I}_2}$ as $<$.

(3) Consider a language with a unary predicate p and constant c. Find an interpretation that falsifies (a.k.a., is a countermodel of) $\exists x p(x) \rightarrow p(c)$.

1.3.3 Validity and Unsatisfiability

Consider a first-order language with at least a constant c and a predicate p. Consider the following closed formulae of the language:

1. $\neg\forall x\varphi \leftrightarrow \exists x\neg\varphi$;

2. $\neg\exists x\varphi \leftrightarrow \forall x\neg\varphi$;

3. $\forall x(\varphi(x) \wedge \psi(x)) \leftrightarrow (\forall x\varphi(x) \wedge \forall x\psi(x))$ *(homework)*;

4. $\exists x(\varphi(x) \vee \psi(x)) \leftrightarrow (\exists x\varphi(x) \vee \exists x\psi(x))$ *(homework)*;

5. $\forall x\varphi \wedge \exists x\neg\varphi$.

Using an informal semantic argument, prove that (1)–(4) are valid, and (5) is unsatisfiable.

Solution. Let us prove that $\models \forall x(\varphi(x) \wedge \psi(x)) \leftrightarrow (\forall x\varphi(x) \wedge \forall x\psi(x))$.

This means that, for all interpretations $\mathfrak{I}$, we have $\mathfrak{I} \models \forall x(\varphi(x)\wedge\psi(x)) \leftrightarrow (\forall x\varphi(x)\wedge\forall x\psi(x))$, that is, $\mathfrak{I} \models \forall x(\varphi(x) \wedge \psi(x))$ iff $\mathfrak{I} \models \forall x\varphi(x) \wedge \forall x\psi(x)$. We prove the following claim: (1) we assume $\mathfrak{I} \models \forall x(\varphi(x) \wedge \psi(x))$ and prove $\mathfrak{I} \models \forall x\varphi(x) \wedge \forall x\psi(x)$; (2) then we assume $\mathfrak{I} \models \forall x\varphi(x) \wedge \forall x\psi(x)$ and prove $\mathfrak{I} \models \forall x(\varphi(x) \wedge \psi(x))$.

(1) $\mathfrak{I} \models \forall x(\varphi(x)\wedge\psi(x))$ means that, for every d in the domain in $\mathfrak{I}$, we have $\mathfrak{I}, [x/d] \models \varphi(x)\wedge\psi(x)$ (this is a shortand for: for all assignments v, we have $\mathfrak{I}, v_{[x\leftarrow d]} \models \varphi(x) \wedge \psi)$. Therefore $\mathfrak{I}, [x/d] \models \varphi(x)$ and $\mathfrak{I}, [x/d] \models \psi(x)$. This amounts to saying that, for all d in the domain of $\mathfrak{I}$, $\mathfrak{I} \models \varphi(x)$ and $\mathfrak{I} \models \psi(x)$, that is, $\mathfrak{I} \models \forall x\varphi(x) \wedge \forall x\psi(x)$.

(2) $\mathfrak{I} \models \forall x\varphi(x) \wedge \forall x\psi(x)$ means that $\mathfrak{I} \models \forall x\varphi(x)$ and $\mathfrak{I} \models \forall x\psi(x)$. This means that, for every d in the domain in $\mathfrak{I}$, we have $\mathfrak{I}, [x/d] \models \varphi(x)$ and $\mathfrak{I}, [x/d] \models \psi(x)$. Therefore $\mathfrak{I}, [x/d] \models \varphi(x)\wedge\psi(x)$, for every d in the domain in $\mathfrak{I}$, and hence $\mathfrak{I} \models \forall x(\varphi(x) \wedge \psi(x))$.

Let us prove that $\not\models \forall x\varphi(x) \wedge \exists x\neg\varphi(x)$. We prove a stronger claim, that is, the formula is unsatisfiable: no interpretation satisfies it. We reason by contradition and assume that there exists $\mathfrak{I}$ so that $\mathfrak{I} \models \forall x\varphi(x) \wedge \exists x\neg\varphi(x)$. Then $\mathfrak{I} \models \forall x\varphi(x)$ (*) and $\mathfrak{I} \models \exists x\neg\varphi(x)$ (**). Now, (*) means that, for every d in the domain of $\mathfrak{I}$, we have $\mathfrak{I}, [x/d] \models \varphi(x)$. Instead (**) means that there exists d' in the domain of $\mathfrak{I}$ so that $\mathfrak{I}, [x/d'] \models \neg\varphi(x)$, that is, $\mathfrak{I}, [x/d'] \not\models \varphi(x)$. Such two statements are contraditory.

1.4 The Tableau Calculus and Procedure

1.4.1 Scaffolding Exercises

Detect what's wrong in the following tableau procedures, constructed as in the lecture slides, and fix them to prove what is required.

1 Prove that $\exists xp(x) \wedge \exists xq(x)$ is satisfiable. What's wrong with the following tableau for proving it? Fix it.

$$\begin{array}{c} \exists xp(x) \wedge \exists xq(x) \\ | \\ [\exists xp(x)] \\ [\exists xq(x)] \\ | \\ p(c) \\ | \\ q(c) \\ \text{open} \end{array}$$

2 Prove that $\forall x\varphi(x) \wedge \exists x\neg\varphi(x)$ is unsatisfiable. What's wrong with the following tableau for proving it? Fix it.

$$
\begin{array}{c}
\forall x\varphi(x) \wedge \exists x\neg\varphi(x) \\
| \\
\forall x\varphi(x) \\
[\exists x\neg\varphi(x)] \\
| \\
\neg\varphi(d) \\
| \\
\varphi(c) \\
\forall x\varphi(x) \\
\text{open}
\end{array}
$$

3 Prove that $\forall xp(x) \rightarrow \exists xp(x)$ is valid. What's wrong with the following tableau for proving it? Fix it.

$$
\begin{array}{c}
\forall xp(x) \rightarrow \exists xp(x) \equiv \exists x\neg p(x) \vee \exists xp(x) \\
\begin{array}{cc}
\diagup & \diagdown \\
\exists x\neg p(x) & \exists xp(x) \\
| & | \\
\neg p(c) & p(d) \\
\text{open} & \text{open}
\end{array}
\end{array}
$$

1.4.2 For Satisfiability

1. Use the tableau procedure to prove that $\forall xr(x, x)$ is satisfiable, where r is a binary predicate symbol. Use the tableau to construct a model for the formula.

Solution. We build an **open** tableau for $\forall xr(x, x)$ for proving this to be satisfiable and defining a model for it.

$$
\begin{array}{c}
\forall xr(x, x) \\
| \\
r(a, a) \\
\forall xr(x, x)
\end{array}
$$

A model $\mathfrak{I}$ is given by the (single) open branch: it has domain $D = \{\mathbf{a}\}$ and $r^{\mathfrak{I}} = \{(\mathbf{a}, \mathbf{a})\}$.

2. Use the tableau procedure to prove that $\forall x\forall y(r(x, y) \rightarrow r(y, x))$ is satisfiable, where r is a binary predicate symbol. Use the tableau to construct a model for the formula.

Solution. We first transform the formula into *negated normal form* (NNF) and then build an **open** tableau for this to prove that $\forall x\forall y(r(x, y) \rightarrow r(y, x))$ is satisfiable and define a model for it. The NNF is the equivalent formula $\forall x\forall y(\neg r(x, y) \vee r(y, x))$.

$$
\begin{array}{c}
\forall x \forall y(\neg r(x,y) \vee r(y,x)) \\
| \\
\forall y(\neg r(a,y) \vee r(y,a)) \\
\forall x \forall y(\neg r(x,y) \vee r(y,x)) \\
| \\
[\neg r(a,a) \vee r(a,a)] \\
\forall y(\neg r(a,y) \vee r(y,a)) \\
\swarrow \quad \searrow \\
\neg r(a,a) \qquad r(a,a) \\
\text{open} \qquad \text{open}
\end{array}
$$

The two open branches give two models, both with the same domain $D = \{\mathbf{a}\}$; in one $r^{\mathfrak{I}} = \emptyset$; in the other $r^{\mathfrak{I}} = \{(\mathbf{a}, \mathbf{a})\}$.

1.4.3 For Validity

1. Consider the following closed formulae of a first-order language with only predicates and constants. Use tableaux to prove that they are valid.

$$\neg\forall x p(x) \vee \exists x p(x); \tag{1}$$
$$\forall y p(y) \rightarrow \forall x p(x); \tag{2}$$
$$\exists x(p(x) \wedge p'(x)) \rightarrow \exists x p(x); \tag{3}$$
$$\forall x(p(x) \wedge p'(x)) \rightarrow \forall x p(x); \tag{4}$$
$$\forall x(p(x) \wedge p'(x)) \rightarrow \forall x(p(x) \vee p'(x)). \tag{5}$$

Solution. To prove that (1) and (2) are **valid**, we first negate them and transform the negated formulae into NNF; we finally build a **closed** tableau for each of the resulting NNF formulae.

(1) $\neg(\neg\forall x p(x) \vee \exists x p(x)) \equiv \forall x p(x) \wedge \forall x \neg p(x)$.

$$
\begin{array}{c}
\forall x p(x) \wedge \forall x \neg p(x) \\
| \\
\forall x p(x) \\
\forall x \neg p(x) \\
| \\
p(c) \\
\forall x p(x) \\
| \\
\neg p(c) \\
\forall x \neg p(x) \\
\text{closed}
\end{array}
$$

(2) $\neg(\forall y p(y) \rightarrow \forall x p(x)) \equiv \forall y p(y) \wedge \exists x \neg p(x)$.

$$
\begin{array}{c}
\forall y p(y) \land \exists x \neg p(x) \\
| \\
\forall y p(y) \\
[\exists x \neg p(x)] \\
| \\
\forall y p(y) \\
\neg p(c) \\
| \\
p(c) \\
\forall y p(y) \\
\text{closed}
\end{array}
$$

2. Consider the following closed formulae of a first-order language with only a binary predicate: φ is $\forall x r(x, x)$, ψ is $\forall x \forall y (r(x, y) \to r(y, x))$. Using a semantic argument *or* the tableau procedure, verify whether the following two formulae

1. $\varphi \to \psi$
2. $\psi \to \varphi$

are valid. In case the given formulae are not valid, define counter-models for them.

Solution. To prove that the given formulae are **not valid**, we negate them and transform the negated formulae into NNF; we finally build an **open** tableau for each of the resulting NNF formulae.

The negation of the first formula is equivalent to $\forall x r(x, x) \land \exists x \exists y (r(x, y) \land \neg r(y, x))$. A tableau for it is as follows:

$$
\begin{array}{c}
\forall x r(x, x) \land \exists x \exists y (r(x, y) \land \neg r(y, x)) \\
| \\
\forall x r(x, x) \\
[\exists x \exists y (r(x, y) \land \neg r(y, x))] \\
| \\
[\exists y (r(a, y) \land \neg r(y, a))] \\
| \\
[r(a, b) \land \neg r(b, a)] \\
| \\
r(a, b) \\
\neg r(b, a) \\
| \\
r(a, a) \\
r(b, b) \\
\forall x r(x, x) \\
\text{open}
\end{array}
$$

An interpretation $\mathfrak{I}$ that is a counter-model of the formula is found along the open branch of the tableau: $r^{\mathfrak{I}} = \{(\mathbf{a}, \mathbf{b}), (\mathbf{a}, \mathbf{a}), (\mathbf{b}, \mathbf{b})\}$.

The negation of the second formula is equivalent to $\exists x \neg r(x, x) \land \forall x \forall y (\neg r(x, y) \lor r(y, x))$. A tableau for it is as follows:

$$
\begin{array}{c}
\exists x \neg r(x,x) \wedge \forall x \forall y(\neg r(x,y) \vee r(y,x)) \\
| \\
[\exists x \neg r(x,x)] \\
\forall x \forall y(\neg r(x,y) \vee r(y,x)) \\
| \\
\neg r(a,a) \\
| \\
\forall y(\neg r(a,y) \vee r(y,a)) \\
\forall x \forall y(\neg r(x,y) \vee r(y,x)) \\
| \\
\neg r(a,a) \vee r(a,a) \\
\forall y(\neg r(a,y) \vee r(y,a)) \\
\begin{array}{cc}
/ & \backslash \\
\neg r(a,a) & r(a,a) \\
\text{open} & \text{closed}
\end{array}
\end{array}
$$

An interpretation $\mathfrak{I}$ that is a counter-model of the formula is found along the open branch of the tableau: $r^{\mathfrak{I}} = \emptyset$.