Free University of Bozen-Bolzano – Faculty of Computer Science
Bachelor in Computer Science and Engineering
Discrete Mathematics and Logic – A.Y. 2016/2017
Mid-Term Exam – Discrete Mathematics – 05/12/2015
Prof. Alessandro Artale – *Time: 120 minutes*

`This is a closed book exam: the only resources allowed are blank papers, pens, and your head. Explain your reasoning. Write clearly, in the sense of logic, language and legibility. The clarity of your explanations affects your grade. Write your name and ID in the solution sheet.`

Problem 1 [6 points] **Induction.**

- Show that $2^{2n} - 1$ is divisible by 3, for all $n \geq 0$.
  Note. An integer m is divisible by 3 iff $m = 3r$, for some integer r.
- **Loop Invariant.** The following while loop is annotated with a pre- and a post-condition and also a loop invariant. Use the *loop invariant theorem* to prove the correctness of the loop with respect to the pre- and post-conditions. [4 POINTS]

  [*Pre-condition:* m is a non-negative integer, x is a real number, $i = 0$ and $exp = 1$]
  while $i \neq m$ **do**
  i := i + 1
  $exp = exp \cdot x$
  end while

  [*Post-condition:* $exp = x^m$]

  Loop Invariant I(n): $exp = x^n$ and $i = n$.

Problem 2 [8 points] **Sets.**

- Show the $\mathcal{P}(\{a, b, c, d\})$, i.e., the power set of the set $\{a, b, c, d\}$. [1 POINT]
- Given the following sets: $A = \{1, 2, 3\}$, $B = \{c, d, e\}$. Show the Cartesian Product $A \times B$. [1 POINT]
- Prove that, for all sets A, B and C, if $A \subseteq C$ and $B \subseteq C$ then $A \cup B \subseteq C$. [2 POINTS]
- **Halting Problem.** Discuss the Halting Problem. Formulate the halting problem Theorem and give an idea on how it can be proved. [4 POINTS]

Problem 3 [6 points] **Cardinality.**

- Give the definition of **2 sets have the same cardinality** and also the definition of **a set being countably infinite**. [2 POINTS]
- Determine whether each of this sets is **finite**, **countably infinite** or **uncountable**. In case the set is countably infinite, show a one-to-one correspondence from the set of positive integers. [4 POINTS]

  1. The set of real numbers between 0 and 10.
  2. The set of negative integers greater than $-10.000.000$;
  3. The set of positive integers multiple of 3;

Problem 4 [12 points] **Relations.**

- Say which of the following relations is an equivalence relation. In case it is not, say what is the missing property. [2 POINTS]

  1. $R1 = \{(0,0),(0,4),(1,1),(1,3),(2,2),(3,1),(3,3),(4,0),(4,4)\}$;
  2. $R2 = \{(a,a),(a,b),(a,c),(c,c),(b,a),(b,c),(c,a),(b,b)\}$.

- Let $A = \{a,b,c,d,e\}$ and R the following equivalence relation over A:

  $$R = \{(a,a),(a,e),(b,b),(b,d),(a,d),(d,d),(d,a),(d,b),(e,a),(e,e),(e,d),(c,c),(d,e)\}$$

  Show the equivalence class of each element in A with respect to R. [4 POINTS]

- Say which of the following relations is a partial order relation. In case it is not, say what is the missing property. [2 POINTS]

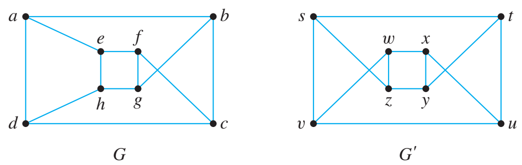
  1. $R1 = \{(0,0),(1,1),(0,3),(1,0),(1,3),(2,2),(3,3),(3,2)\}$;
  2. $R2 = \{(a,a),(b,a),(c,c),(b,c),(b,d),(c,a),(b,b),(c,b),(d,c),(d,a),(d,d)\}$.

- Given the following set $A = \{\{1\},\{1,2\},\{1,3\},\{1,4\},\{1,2,3\},\{1,3,4\},\{1,2,3,4\},\{1,3,4,6\}\}$ and the subset relation on A, say $\subseteq_A$. Show the following concerning the poset $(A, \subseteq_A)$: [4 POINTS]

  1. The Hasse diagram;
  2. The minimal and maximal elements;
  3. A topological sort.

Problem 5 [4 points] **Graphs and Trees.**

- Find all non-isomorphic trees with 5 vertices. Provide an explanation with your answer.
- Given the following non-isomorphic graphs:



Describe an *invariant* property for graph isomorphism that they do not share.