Free University of Bozen-Bolzano – Faculty of Computer Science
Bachelor in Computer Science and Engineering
Discrete Mathematics and Logic – A.Y. 2015/2016
Final Exam – Discrete Mathematics – 29/June/2016

Prof. Alessandro Artale – *Time: 60 minutes*

`This is a closed book exam: the only resources allowed are blank paper, pens, and your head. Explain your reasoning. Write clearly, in the sense of logic, language and legibility. The clarity of your explanations affects your grade.`

Problem 1 [12 points] **Induction.**

- Show that $n^3 \leq 2^n$, for all $n \geq 10$. [4 POINTS]
  Hint: Use the fact that $(n+1)^3 \leq (n+n)^3$, for all $n \geq 10$.
- **Loop Invariant.** The following while loop is annotated with a pre- and a post-condition and also a loop invariant. Use the *loop invariant theorem* to prove the correctness of the loop with respect to the pre- and post-conditions. [8 POINTS]

  [*Pre-condition:* min $= A[1]$ and $i = 1$]
  while $i \neq m$ **do**
  i := i + 1
  if($A[i] <$ min) **then** min := $A[i]$
  end while

  [*Post-condition:* min = minimum value of $A[1], \ldots, A[m]$]

  Loop Invariant I(n): min is the minimum value of $A[1], A[2], \ldots, A[n+1]$ and $i = n+1$.

Problem 2 [6 points] **Sets.**

- Show $\mathcal{P}(\{a,b,c\})$, i.e., the power set of the set $\{a,b,c\}$. [1 POINT]
- Given the following sets: $A = \{a,b,c\}$, $B = \{1,2\}$. Show the Cartesian Product $A \times B$. [1 POINT]
- Show that, if $\mathcal{P}(A) = \mathcal{P}(B)$ then $A = B$. [4 POINTS]
  Hint: Suppose that $\mathcal{P}(A) = \mathcal{P}(B)$ and $A \neq B$ and reach a contradiction.

Problem 3 [6 points] **Cardinality.**

- Give the definition of the following notions: [2 POINTS]
  1. **2 sets have the same cardinality**, and
  2. **a set being countably infinite**.
- Let A be an infinite set. Show that it contains a countably infinite subset. [4 POINTS]

Problem 4 [9 points] **Relations and Trees.**

- Say which of the following relations is an equivalence relation. In case it is not, say what is the missing property. [2 POINTS]
  1. $R1 = \{(a,a),(a,c),(c,d),(c,c),(c,a),(d,c),(d,d)\}$;
  2. $R2 = \{(5,4),(3,4),(3,3),(5,5),(3,5),(4,5),(4,4),(6,6),(4,3)\}$.
- Let $A = \{a,b,c,d,e\}$ and R the following equivalence relation over A:
  $$R = \{(a,a),(a,e),(b,b),(b,d),(a,d),(d,d),(d,a),(d,b),(e,a),(e,e),(a,b),(b,a),(e,d),(d,e)\}$$
  Show the equivalence class of each element in A with respect to R. [4 POINTS]
- Find all non-isomorphic trees with 5 vertices. Provide an explanation with your answer. [3 POINTS]