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KRDB Research Centre Technical Report:

Do You Need Infinite Time?

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Linear temporal logic over finite traces is used as a formalism for temporal specification in automated planning, process modelling and (runtime) verification. In this paper, we investigate first-order temporal logic over finite traces, lifting some known results to a more expressive setting. Satisfiability in the two-variable monodic fragment is shown to be ExpSpace-complete, as for the infinite trace case, while it decreases to NExpTime when we consider finite traces bounded in the number of instants. This leads to new complexity results for temporal description logics over finite traces. We further investigate satisfiability and equivalences of formulas under a model-theoretic perspective, providing a set of semantic conditions that characterise when the distinction between reasoning over finite and infinite traces can be blurred. Finally, we apply these conditions to planning and verification.

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Abstract

Linear temporal logic over finite traces is used as a formalism for temporal specification in automated planning, process modelling and (runtime) verification. In this paper, we investigate first-order temporal logic over finite traces, lifting some known results to a more expressive setting. Satisfiability in the two-variable monodic fragment is shown to be EXPSPACE-complete, as for the infinite trace case, while it decreases to NEXPTIME when we consider finite traces bounded in the number of instants. This leads to new complexity results for temporal description logics over finite traces. We further investigate satisfiability and equivalences of formulas under a model-theoretic perspective, providing a set of semantic conditions that characterise when the distinction between reasoning over finite and infinite traces can be blurred. Finally, we apply these conditions to planning and verification.

1 Introduction

Since the introduction of *linear temporal logic* (LTL), several propositional and first-order LTL-based formalisms have been developed for applications such as automated planning [Bacchus and Kabanza, 2000; Baier and McIlraith, 2006], process modelling [van der Aalst and Pesic, 2006; Maggi *et al.*, 2011] and verification of programs [Manna and Pnueli, 1995]. Related research in knowledge representation has focused on decidable fragments of first-order temporal logic, $T_{\mathcal{U}}\mathcal{QL}$ [Hodkinson *et al.*, 2000; Gabbay *et al.*, 2003], and in particular on *temporal description logics* [Wolter and Zakharyaschev, 1998; Artale and Franconi, 2005] that combine LTL operators with description logics (DLs). These logics usually lie within the two-variable monodic fragment of $T_{\mathcal{U}}\mathcal{QL}$, $T_{\mathcal{U}}\mathcal{QL}^2_{\boxed{1}}$, obtained by restricting to formulas having at most two variables, and so that the temporal operators are applied only to subformulas with at most one free variable. The complexity of the satisfiability problem ranges from EXPSPACE-complete, for $T_{\mathcal{U}}\mathcal{QL}^2_{\boxed{1}}$ [Gabbay *et al.*, 2003; Hodkinson *et al.*, 2003], down to NEXPTIME- or EXPTIME-complete, for temporal extensions of $\mathcal{ALC}$ without temporalised roles and with restrictions on the application of temporal operators [Lutz *et al.*, 2008; Baader *et al.*, 2012], or even in NP or NLOGSPACE, by modifying further both the temporal and the DL components, as in some temporal extensions of *DL-Lite* for conceptual data modelling [Artale *et al.*, 2014].

Besides the usual LTL semantics defined over infinite linear structures, attention has been devoted also to *finite traces*, which are linear temporal structures with only a finite number of time points [Cerrito *et al.*, 1999; De Giacomo and Vardi, 2013; Fionda and Greco, 2016]. Indeed, the finiteness of the time dimension is a fairly natural restriction. In automated planning, or when modelling business processes with a declarative formalism, we consider finite action plans and terminating services, often within a given temporal bound [Bauer and Haslum, 2010; De Giacomo *et al.*, 2014a; De Giacomo *et al.*, 2014b; Camacho *et al.*, 2017]. In runtime verification, only the current finite behaviour of the system is taken into account. Infinite models are then considered to check whether a given requirement is satisfied in the infinite extensions of this finite trace [Giannakopoulou and Havelund, 2001; Bauer *et al.*, 2010]. Although some recent work has considered temporal extensions of DLs in the context of runtime verification [Baader and Lippmann, 2014] and business process modelling [van der Aalst *et al.*, 2017], the proposals developed so far are based on the usual infinite trace semantics and are limited in expressivity.

This work focuses on first-order temporal logic over finite traces, $T^f_{\mathcal{U}}\mathcal{QL}$ [Cerrito *et al.*, 1999], defined by extending the first-order language with temporal operators interpreted over finite traces. We obtain the following main results.

- The complexity of the two-variable monodic fragment, $T^f_{\mathcal{U}}\mathcal{QL}^2_{\boxed{1}}$, remains EXPSPACE-complete, but lowers down to NEXPTIME if we restrict to traces with a bound k on the number of instants. Moreover, $T^f_{\mathcal{U}}\mathcal{QL}^2_{\boxed{1}}$ has the bounded model property.
- Similar results (including the bounded model property) also hold for the temporal DL $T^f_{\mathcal{U}}\mathcal{ALC}$, where the complexity further reduces to EXPTIME if only global axioms and finite traces with a fixed bound on instants are allowed.
- We establish semantic and syntactic conditions which characterise when the distinction between reasoning over finite and infinite traces can be completely blurred, providing connections with planning and verification.

Concerning this last point, several approaches have been considered to preserve satisfiability of formulas from the finite to the infinite case, so to reuse algorithms developed for the infinite case [Bauer and Haslum, 2010; De Giacomo *et al.*, 2014b]. Here, we determine conditions that preserve satisfiability also in the other direction, from infinite to finite traces, allowing for the application of dedicated finite traces reasoners [Li *et al.*, 2014]. We provide a uniform framework for these notions, bridging finite and infinite trace semantics. Proofs are available at http://publish.lycos.com/ijcai-2019/ijcai-19/.

2 First-order Temporal Logics

The *first-order temporal language*, $T_{\mathcal{U}}\mathcal{QL}$ [Gabbay *et al.*, 2003], is obtained by extending the usual first-order language with the temporal operator *until* ($\mathcal{U}$) interpreted over discrete linear structures, called *traces*.

Syntax The *alphabet* of $T_{\mathcal{U}}\mathcal{QL}$ consists of countably infinite and pairwise disjoint sets of *predicates* $\mathsf{N_P}$ (with $\mathsf{ar}(P) \in \mathbb{N}$ being the arity of $P \in \mathsf{N_P}$), *constants* (or *individual names*) $\mathsf{N_I}$, and *variables* Var; the *logical operators* $\neg, \wedge$; the *existential quantifier* $\exists$, and the *temporal operator* $\mathcal{U}$ (*until*). The *formulas* of $T_{\mathcal{U}}\mathcal{QL}$ are of the form:

$$\varphi, \psi ::= P(\bar{\tau}) \mid \neg\varphi \mid \varphi \wedge \psi \mid \exists x\varphi \mid \varphi\,\mathcal{U}\,\psi,$$

where $P \in \mathsf{N_P}$, $\bar{\tau} = (\tau_1, \dots, \tau_{\mathsf{ar}(P)})$ is a tuple of *terms*, i.e., constants or variables, and $x \in \mathsf{Var}$. Formulas without the until operator are called *non-temporal*. We write $\varphi(x_1, \dots, x_m)$ to indicate that the free variables of a formula φ are in $\{x_1, \dots, x_m\}$. For $p \in \mathbb{N}$, the *p-variable fragment* of $T_{\mathcal{U}}\mathcal{QL}$, denoted by $T_{\mathcal{U}}\mathcal{QL}^p$, consists of $T_{\mathcal{U}}\mathcal{QL}$ formulas with at most p variables ($T_{\mathcal{U}}\mathcal{QL}^0$ is simply propositional *LTL*). The *monodic fragment* of $T_{\mathcal{U}}\mathcal{QL}$, denoted by $T_{\mathcal{U}}\mathcal{QL}_{\boxed{1}}$, consists of formulas such that all subformulas of the form $\varphi\,\mathcal{U}\,\psi$ have *at most one* free variable.

Semantics A *first-order temporal interpretation* is a structure $\mathfrak{M} = (\Delta^{\mathfrak{M}}, (I_n)_{n\in\mathfrak{T}})$, where $\mathfrak{T}$ is an interval of the form $[0, \infty)$ or $[0, l]$, with $l \in \mathbb{N}$, and each I_n is a classical first-order interpretation with domain $\Delta^{\mathfrak{M}}$ (or simply Δ): we have $P^{I_n} \subseteq \Delta^{\mathsf{ar}(P)}$, for each $P \in \mathsf{N_P}$, and $a^{I_i} = a^{I_j} \in \Delta$ for all $a \in \mathsf{N_I}$ and $i, j \in \mathbb{N}$, i.e., constants are *rigid designators* (with fixed interpretation, denoted simply by a^I). The stipulation that all time points share the same domain Δ is called the *constant domain assumption* (meaning that objects are not created or destroyed over time), and it is the most general choice in the sense that increasing, decreasing, and varying domains can all be reduced to it [Gabbay *et al.*, 2003]. An *assignment* in $\mathfrak{M}$ is a function $\mathfrak{a}$ from terms to Δ: $\mathfrak{a}(\tau) = \mathfrak{a}(x)$, if $\tau = x$, and $\mathfrak{a}(\tau) = a^I$, if $\tau = a \in \mathsf{N_I}$ (given a tuple of m terms $\bar{\tau} = (\tau_1, \dots, \tau_m)$, we set $\mathfrak{a}(\bar{\tau}) = (\mathfrak{a}(\tau_1), \dots, \mathfrak{a}(\tau_m))$. *Satisfaction* of a formula φ in $\mathfrak{M}$ at time point $n \in \mathfrak{T}$ under assignment $\mathfrak{a}$ (written $\mathfrak{M}, n \models^{\mathfrak{a}} \varphi$) is inductively defined as:

$$\begin{array}{lll}
\mathfrak{M}, n \models^{\mathfrak{a}} P(\bar{\tau}) & \text{iff} & \mathfrak{a}(\bar{\tau}) \in P^{I_n}, \\
\mathfrak{M}, n \models^{\mathfrak{a}} \neg\varphi & \text{iff} & \text{not } \mathfrak{M}, n \models^{\mathfrak{a}} \varphi, \\
\mathfrak{M}, n \models^{\mathfrak{a}} \varphi \wedge \psi & \text{iff} & \mathfrak{M}, n \models^{\mathfrak{a}} \varphi \text{ and } \mathfrak{M}, n \models^{\mathfrak{a}} \psi, \\
\mathfrak{M}, n \models^{\mathfrak{a}} \exists x\varphi & \text{iff} & \mathfrak{M}, n \models^{\mathfrak{a}'} \varphi \text{ for some assignment } \mathfrak{a}' \\
 & & \text{that can differ from } \mathfrak{a} \text{ only on } x, \\
\mathfrak{M}, n \models^{\mathfrak{a}} \varphi\,\mathcal{U}\,\psi & \text{iff} & \exists m \in \mathfrak{T}, m > n\colon \mathfrak{M}, m \models^{\mathfrak{a}} \psi \text{ and} \\
 & & \forall i \in (n, m)\colon \mathfrak{M}, i \models^{\mathfrak{a}} \varphi.
\end{array}$$

We say that φ is *satisfied in* $\mathfrak{M}$ (and $\mathfrak{M}$ is a *model* of φ), writing $\mathfrak{M} \models \varphi$, if $\mathfrak{M}, 0 \models^{\mathfrak{a}} \varphi$, for some $\mathfrak{a}$. Moreover, φ is said to be *satisfiable* if it is satisfied in some $\mathfrak{M}$. A formula φ *logically implies* a formula ψ if every $\mathfrak{M}$ that satisfies φ satisfies also ψ, and we write $\varphi \models \psi$. We say that φ and ψ are *equivalent*, writing $\varphi \equiv \psi$, if $\varphi \models \psi$ and $\psi \models \varphi$.

In the following, we call *finite trace* a first-order temporal interpretation with $\mathfrak{T} = [0, l]$, often denoted by $\mathfrak{F} = (\Delta^{\mathfrak{F}}, (\mathcal{F}_n)_{n\in[0,l]})$, while *infinite traces*, based on $\mathfrak{T} = [0, \infty)$, will be denoted by $\mathfrak{I} = (\Delta^{\mathfrak{I}}, (\mathcal{I}_n)_{n\in[0,\infty)})$. Thus, by $T^i_{\mathcal{U}}\mathcal{QL}$ we denote the language interpreted over *infinite traces*, $T^f_{\mathcal{U}}\mathcal{QL}$ is the language interpreted over *finite traces*, while, for a fixed $k \in \mathbb{N}, k > 0$, the semantics of $T^k_{\mathcal{U}}\mathcal{QL}$ is restricted to *finite traces* based on $\mathfrak{T} = [0, l]$, *with* $l \leq k - 1$.

Let $\mathfrak{F} = (\Delta^{\mathfrak{F}}, (\mathcal{F}_n)_{n\in[0,l]})$ and $\mathfrak{I} = (\Delta^{\mathfrak{I}}, (\mathcal{I}_n)_{n\in[0,\infty)})$ be, respectively, a finite and an infinite trace s.t. $\Delta^{\mathfrak{F}} = \Delta^{\mathfrak{I}}$ (writing Δ) and $a^{\mathcal{F}} = a^{\mathcal{I}}$, for all $a \in \mathsf{N_I}$. We denote by $\mathfrak{F} \cdot \mathfrak{I} = (\Delta^{\mathfrak{F}\cdot\mathfrak{I}}, (\mathcal{F} \cdot \mathcal{I}_n)_{n\in[0,\infty)})$ the *extension of* $\mathfrak{F}$ *with* $\mathfrak{I}$, defined as the infinite trace with $\Delta^{\mathfrak{F}\cdot\mathfrak{I}} = \Delta$, $a^{\mathcal{F}\cdot\mathcal{I}} = a^{\mathcal{F}}$, for all $a \in \mathsf{N_I}$, and for $P \in \mathsf{N_P}$, $n \in \mathbb{N}$:

$$P^{\mathcal{F}\cdot\mathcal{I}_n} = \begin{cases} P^{\mathcal{F}_n}, & \text{if } n \in [0, l] \\ P^{\mathcal{I}_{n-(l+1)}}, & \text{if } n \in [l+1, \infty). \end{cases}$$

In addition to the standard Boolean equivalences, we will use the following equivalences for formulas: $\bot \equiv \exists x(P(x) \wedge \neg P(x))$, $\top \equiv \neg\bot$; $\bigcirc\varphi \equiv \bigcirc^1\varphi \equiv \bot\,\mathcal{U}\,\varphi$; $\bigcirc^q\varphi \equiv \bigcirc\bigcirc^{q-1}\varphi$, with $q > 1$; $\Diamond\varphi \equiv \top\,\mathcal{U}\,\varphi$; $\Box\varphi \equiv \neg\Diamond\neg\varphi$; $\Diamond^+\varphi \equiv \varphi \vee \Diamond\varphi$; and $\Box^+\varphi \equiv \neg\Diamond^+\neg\varphi$. Over finite traces, *last* $\equiv \neg\bigcirc\top$ is satisfiable in the *last* time point.

3 Satisfiability over Finite Traces

In the following we show how to reduce the formula satisfiability problem over finite traces to the same problem over infinite traces. Similar to the encoding proposed in [De Giacomo and Vardi, 2013] for propositional *LTL*, to capture the finiteness of the temporal dimension, we introduce a fresh predicate E (standing for the *end of time*) with the following properties: *(i)* there is a least one instant before the end of time; *(ii)* the end of time comes for all objects; *(iii)* the end of time comes at the same time for every object; *(iv)* the end of time never goes away. We axiomatise these properties as follows:

$$\begin{array}{ll}
\psi_{f_1} = \forall x \neg E(x) & \text{(Point } (i)), \\
\psi_{f_2} = (\forall x \neg E(x))\,\mathcal{U}\,(\forall x E(x)) & \text{(Points } (ii), (iii)), \\
\psi_{f_3} = \Box\forall x(E(x) \rightarrow \bigcirc E(x)) & \text{(Point } (iv)).
\end{array}$$

We now characterise models satisfying the *end of time formula* $\psi_f = \psi_{f_1} \wedge \psi_{f_2} \wedge \psi_{f_3}$. Let $\mathfrak{F} = (\Delta, (\mathcal{F}_n)_{n\in[0,l]})$ and $\mathfrak{I} = (\Delta, (\mathcal{I}_n)_{n\in[0,\infty)})$ be, respectively, a finite and an infinite trace with the same domain Δ and such that $a^{\mathcal{F}} = a^{\mathcal{I}}$, for all $a \in \mathsf{N_I}$. We denote by $\mathfrak{F} \cdot_E \mathfrak{I}$ the *end extension of* $\mathfrak{F}$ *with* $\mathfrak{I}$, defined as the extension $\mathfrak{F} \cdot \mathfrak{I}$, for all $P \in \mathsf{N_P} \setminus \{E\}$, such that:

$$E^{\mathcal{F}\cdot_E\mathcal{I}_n} = \begin{cases} \emptyset, & \text{if } n \in [0, l] \\ \Delta, & \text{if } n \in [l+1, \infty) \end{cases}$$

Clearly, the extension of E characterises the satisfiability of ψ_f. We formalise this in the next lemma.

Lemma 1. *For every infinite trace $\mathfrak{I}$, $\mathfrak{I} \models \psi_f$ iff $\mathfrak{I} = \mathfrak{F} \cdot_E \mathfrak{I}'$, for some finite trace $\mathfrak{F}$ and some infinite trace $\mathfrak{I}'$.*

We now introduce a translation $\cdot^\dagger$ from $T^f_{\mathcal{U}}\mathcal{QL}$ to $T^i_{\mathcal{U}}\mathcal{QL}$ formulas, such that, together with the end of time formula, ψ_f, it captures those formulas satisfiable over finite traces. More formally, a $T^f_{\mathcal{U}}\mathcal{QL}$ formula φ is satisfiable if and only if its translation $\varphi^\dagger$ is satisfiable in a $T^i_{\mathcal{U}}\mathcal{QL}$ model that also satisfies the formula ψ_f. The translation $\cdot^\dagger$ is defined as:

$$(P(\bar{\tau}))^\dagger \mapsto P(\bar{\tau}), \qquad (\neg\varphi)^\dagger \mapsto \neg\varphi^\dagger,$$
$$(\varphi \wedge \psi)^\dagger \mapsto \varphi^\dagger \wedge \psi^\dagger, \qquad (\exists x\varphi)^\dagger \mapsto \exists x\varphi^\dagger,$$
$$(\varphi\,\mathcal{U}\,\psi)^\dagger \mapsto \varphi^\dagger\,\mathcal{U}\,(\psi^\dagger \wedge \psi_{f_1}).$$

Lemma 2 states the correctness of $\cdot^\dagger$.

Lemma 2. *Let $\mathfrak{F} \cdot_E \mathfrak{I}$ be an end extension of a finite trace $\mathfrak{F}$. For every $T^f_{\mathcal{U}}\mathcal{QL}$ formula φ and every assignment $\mathfrak{a}$, $\mathfrak{F} \models^{\mathfrak{a}} \varphi$ iff $\mathfrak{F} \cdot_E \mathfrak{I} \models^{\mathfrak{a}} \varphi^\dagger$.*

From the previous lemmas, we obtain a reduction of the $T^f_{\mathcal{U}}\mathcal{QL}$ satisfiability problem to the same problem for $T^i_{\mathcal{U}}\mathcal{QL}$.

Theorem 3. *Let φ be a $T^f_{\mathcal{U}}\mathcal{QL}$ formula. Then φ is satisfiable iff $\varphi^\dagger \wedge \psi_f$ is a satisfiable $T^i_{\mathcal{U}}\mathcal{QL}$ formula.*

We use the translation to transfer the EXPSPACE upper bound for $T_{\mathcal{U}}\mathcal{QL}^2_{\boxed{1}}$ over infinite traces to finite traces [Hodkinson *et al.*, 2003]. The lower bound can be proved using similar ideas as those used to prove hardness of $T_{\mathcal{U}}\mathcal{QL}^2_{\boxed{1}}$.

Theorem 4. *Satisfiability in $T^f_{\mathcal{U}}\mathcal{QL}^2_{\boxed{1}}$ is* EXPSPACE-*complete.*

We now study the satisfiability in $T^k_{\mathcal{U}}\mathcal{QL}^2_{\boxed{1}}$, where we restrict the problem to traces with at most k time points, with k given in binary, as part of the input. We show that in this case the complexity of the satisfiability problem decreases from EXPSPACE to NEXPTIME. Hardness follows from the fact that: (1) one can translate $\mathcal{ALC}$-LTL with rigid concepts to $T^k_{\mathcal{U}}\mathcal{QL}^2_{\boxed{1}}$; and (2) satisfiability in $\mathcal{ALC}$-LTL with rigid concepts is NEXPTIME-hard [Baader *et al.*, 2012, Lemma 6.2]. For the upper bound, we resort to a classical abstraction of models called *quasimodels* [Gabbay *et al.*, 2003]. One can show that there is a model with at most k time points iff there is a quasimodel with a sequence of states (sets of subformulas with certain constraints) of length at most k. Then, our upper bound is obtained by guessing an exponential size sequence of states which serves as a certificate for the existence of a quasimodel (and therefore a model) for the input formula.

Theorem 5. *Satisfiability in $T^k_{\mathcal{U}}\mathcal{QL}^2_{\boxed{1}}$ is* NEXPTIME-*complete.*

We end this section by establishing that $T^f_{\mathcal{U}}\mathcal{QL}^2_{\boxed{1}}$ enjoys the bounded model property and the bounded domain property. If there is a finite trace which satisfies a $T^f_{\mathcal{U}}\mathcal{QL}^2_{\boxed{1}}$ formula φ then there is a finite trace with at most k time points, with k double exponentially large w.r.t. the size of φ. This bound follows from the fact that (1) if there is a quasimodel for φ then there is a quasimodel for φ where there is no repetition of states [Gabbay *et al.*, 2003], except for the last state; and (2) the fact that the length of the finite non-repeting sequence of states is correlated with the number of time points in a finite trace. The number of distinct states of a $T^f_{\mathcal{U}}\mathcal{QL}^2_{\boxed{1}}$ formula is the same as for a $T_{\mathcal{U}}\mathcal{QL}^2_{\boxed{1}}$ formula, which is known to be double exponential [Gabbay *et al.*, 2003]. This directly implies that $T^f_{\mathcal{U}}\mathcal{QL}^2_{\boxed{1}}$ enjoys the *bounded trace property*.

Theorem 6. *Satisfiability of φ in $T^f_{\mathcal{U}}\mathcal{QL}^2_{\boxed{1}}$ implies satisfiability of φ in $T^k_{\mathcal{U}}\mathcal{QL}^2_{\boxed{1}}$ with k double exponential in $|\varphi|$.*

We now establish that a $T^k_{\mathcal{U}}\mathcal{QL}^2_{\boxed{1}}$ formula has the *bounded domain property*: if it is satisfiable, then there is a model with the size of the domain exponential in k (meaning that it is double exponential in the binary representation of k).

Theorem 7. *Satisfiability of φ in $T^k_{\mathcal{U}}\mathcal{QL}^2_{\boxed{1}}$ implies the existence of a model with domain size exponential in k and φ.*

Since satisfiability in $T^f_{\mathcal{U}}\mathcal{QL}^2_{\boxed{1}}$ implies satisfiability in $T^k_{\mathcal{U}}\mathcal{QL}^2_{\boxed{1}}$ for some $k > 0$, the formula

$$\Box^+\forall x(P(x) \rightarrow \bigcirc\Box^+(\neg P(x) \wedge \exists y R(x,y) \wedge P(y))), \quad (*)$$

which only admits models with an infinite domain [Lutz *et al.*, 2008], is unsatisfiable over finite traces.

4 Finite vs. Infinite Traces

While certain formulas, such as $\Box\top$, are satisfiable both over finite and infinite traces, others, e.g., $\Diamond$*last* and $\Box^+\bigcirc\top$, are only satisfiable over finite traces and over infinite traces, respectively. One then wonders, when does satisfiability over finite and infinite traces coincide so that solvers can simply stop trying to build the lasso of an infinite trace when a finite trace is built? A similar question can be posed for the problem of equivalences between formulas. For example, $\Diamond\Box(\varphi \vee \psi)$ and $(\Diamond\Box\varphi) \vee (\Diamond\Box\psi)$ are equivalent over finite traces but not over infinite traces [Bauer and Haslum, 2010]. Moreover, $\Box^+\Diamond^+\varphi$ and $\Diamond^+\Box^+\varphi$ are not equivalent over infinite traces, whereas over finite traces they are both equivalent to $\Diamond^+(\mathit{last} \wedge \varphi)$ [De Giacomo and Vardi, 2013]. Conversely, $\bot$ and *last* are only equivalent over infinite traces.

In this section we address these questions and investigate the distinction between reasoning over finite and over infinite traces. More specifically, we propose semantic properties which guarantee that formula satisfiability and equivalences between formulas are preserved, and thus, the distinction can be blurred. Given a finite trace $\mathfrak{F}$, we define the set of *extensions* of $\mathfrak{F}$ as the set $\mathit{Ext}(\mathfrak{F}) = \{\mathfrak{I} \mid \mathfrak{I} = \mathfrak{F} \cdot \mathfrak{I}', \text{ for some } \mathfrak{I}'\}$. Instead, given an infinite trace $\mathfrak{I}$, the set of *prefixes* of $\mathfrak{I}$ is the set $\mathit{Pre}(\mathfrak{I}) = \{\mathfrak{F} \mid \mathfrak{I} = \mathfrak{F} \cdot \mathfrak{I}', \text{ for some } \mathfrak{I}'\}$. For a $T_{\mathcal{U}}\mathcal{QL}$ formula φ and a quantifier $Q \in \{\exists, \forall\}$, we say that φ *is* F_Q if, for all finite traces $\mathfrak{F}$ and all assignments $\mathfrak{a}$, it satisfies the *finite trace property*:

$$\mathfrak{F} \models^{\mathfrak{a}} \varphi \Leftrightarrow Q\mathfrak{I} \in \mathit{Ext}(\mathfrak{F}).\mathfrak{I} \models^{\mathfrak{a}} \varphi,$$

and, similarly, φ *is* I_Q if, for all infinite traces $\mathfrak{I}$ and all assignments $\mathfrak{a}$, it satisfies the *infinite trace property*:

$$\mathfrak{I} \models^{\mathfrak{a}} \varphi \Leftrightarrow Q\mathfrak{F} \in \mathit{Pre}(\mathfrak{I}).\mathfrak{F} \models^{\mathfrak{a}} \varphi.$$

Property	Formulas
$\mathsf{F}_\exists$	$\Diamond^+ \mathit{last} \vee \Diamond P(x)$; $\Diamond^+ \mathit{last} \vee \Diamond P(x) \vee (*)$.
$\mathsf{F}_\forall$	$\Diamond^+ P(x)$; $\Diamond^+ P(x) \vee (*)$.
$\mathsf{I}_\exists$	$\Box \bigcirc \top \vee \mathit{last}$; $\Box \bigcirc \top \vee \bigcirc \mathit{last}$.
$\mathsf{I}_\forall$	$\Box^+ P(x) \vee \Diamond^+ (P(x) \wedge \mathit{last})$; $\Box^+ P(x) \vee \Diamond^+ (P(x) \wedge \bigcirc \mathit{last})$.

Table 1: Formulas satisfying exactly one of the properties (Formula (*) is from Section 3).

Examples of formulas satisfying $\mathsf{F}_\exists$ and $\mathsf{I}_\forall$ are formulas of the form $\Box^+\varphi$, where φ is a formula without temporal operators (formulas of the form $\Box\varphi$ are also $\mathsf{I}_\forall$ but not $\mathsf{F}_\exists$ because of, e.g., $\Box\bot$). On the other hand, the semantic properties $\mathsf{F}_\forall$ and $\mathsf{I}_\exists$ capture formulas of the form $\Diamond^+\varphi$, where φ is again a formula without temporal operators (formulas of the form $\Diamond\varphi$ are also $\mathsf{I}_\exists$ but not $\mathsf{F}_\forall$ because of, e.g., $\Diamond\top$).

The semantic properties, F_Q and I_Q, capture different classes of $T_\mathcal{U}\mathcal{QL}$ formulas, as we illustrate in Table 1. We formalise this statement with the following theorem.

Theorem 8. *Let $T_\mathcal{U}\mathcal{QL}(P)$ denote the set of $T_\mathcal{U}\mathcal{QL}$ formulas which satisfy property P. The sets $T_\mathcal{U}\mathcal{QL}(\mathsf{F}_\exists)$, $T_\mathcal{U}\mathcal{QL}(\mathsf{F}_\forall)$, $T_\mathcal{U}\mathcal{QL}(\mathsf{I}_\exists)$, and $T_\mathcal{U}\mathcal{QL}(\mathsf{I}_\forall)$ are mutually incomparable.*

One can also restrict to the 'one directional' version of the above properties. We denote by $\mathsf{F}_{\circ Q}$ and $\mathsf{I}_{\circ Q}$, where $\circ \in \{\Rightarrow, \Leftarrow\}$, the corresponding '$\Rightarrow$' and '$\Leftarrow$' directions of the F_Q and I_Q properties, respectively. On the relationship between these one directional properties, we have the following. Given two different quantifiers Q and Q' (among $\exists$ and $\forall$), a $T_\mathcal{U}\mathcal{QL}$ formula φ is $\mathsf{F}_{\Rightarrow Q}$ if, and only if, its negation $\neg\varphi$ is $\mathsf{F}_{\Leftarrow Q'}$. Similarly, φ is $\mathsf{I}_{\Rightarrow Q}$ if, and only if, $\neg\varphi$ is $\mathsf{I}_{\Leftarrow Q'}$. Moreover, if a $T_\mathcal{U}\mathcal{QL}$ formula φ is $\mathsf{F}_{\Rightarrow\forall}$, then it is also $\mathsf{F}_{\Rightarrow\exists}$, and if φ is $\mathsf{I}_{\Rightarrow\forall}$, it is $\mathsf{I}_{\Rightarrow\exists}$ as well. However, the sets of formulas satisfying $\mathsf{F}_{\Rightarrow\forall}$ and $\mathsf{I}_{\Rightarrow\exists}$ are incomparable. Indeed, the formula $\Box^+\bigcirc\top$ is $\mathsf{F}_{\Rightarrow\forall}$ and *not* $\mathsf{I}_{\Rightarrow\exists}$. On the other hand, the sets of formulas satisfying $\mathsf{I}_{\Rightarrow\exists}$ and $\mathsf{F}_{\Rightarrow\forall}$ are also incomparable. To see this, consider $\Diamond\mathit{last}$, which is $\mathsf{I}_{\Rightarrow\exists}$ but *not* $\mathsf{F}_{\Rightarrow\forall}$.

In Theorem 3, we have proved that $T^f_\mathcal{U}\mathcal{QL}$ formulas can be translated into equisatisfiable $T^i_\mathcal{U}\mathcal{QL}$ formulas. Such translation is not always needed, since for some classes of formulas satisfiability is already preserved. Indeed, for a $T_\mathcal{U}\mathcal{QL}$ formula φ: if φ is $\mathsf{F}_{\Rightarrow\exists}$, then it is $T^f_\mathcal{U}\mathcal{QL}$ satisfiable only if it is $T^i_\mathcal{U}\mathcal{QL}$ satisfiable; moreover, if φ is $\mathsf{I}_{\Rightarrow\exists}$, then it is $T^i_\mathcal{U}\mathcal{QL}$ satisfiable only if it is $T^f_\mathcal{U}\mathcal{QL}$ satisfiable.

We now consider the problem of formula equivalence, by showing under which semantic properties equivalence between formulas can be blurred. Given $T_\mathcal{U}\mathcal{QL}$ formulas φ and ψ, we write $\varphi \equiv_\mathfrak{I} \psi$ if φ and ψ are $T^i_\mathcal{U}\mathcal{QL}$ equivalent, and $\varphi \equiv_\mathfrak{F} \psi$ if they are $T^f_\mathcal{U}\mathcal{QL}$ equivalent. The following theorem provides sufficient conditions to preserve formula equivalence from the infinite to the finite case (cf. the notion of *LTL compliance* in [Bauer *et al.*, 2010]).

Theorem 9. *Given $T_\mathcal{U}\mathcal{QL}$ formulas φ and ψ, $\varphi \equiv_\mathfrak{I} \psi$ implies $\varphi \equiv_\mathfrak{F} \psi$ whenever both φ and ψ are (1) $\mathsf{F}_\exists$; or (2) $\mathsf{F}_\forall$; or (3) $\mathsf{F}_{\Rightarrow\exists}$ and $\mathsf{I}_{\Rightarrow\forall}$.*

Theorem 9 does not hold for formulas that satisfy only $\mathsf{I}_\exists$ or $\mathsf{I}_\forall$. Consider the formulas $\Box\bigcirc\top \vee \mathit{last}$ and $\Box\bigcirc\top \vee \bigcirc\mathit{last}$, which are $\mathsf{I}_\exists$. These formulas are equivalent only over infinite traces. Also, $\Box^+ P(x) \vee \Diamond^+(P(x) \wedge \mathit{last})$ and $\Box^+ P(x) \vee \Diamond^+(P(x) \wedge \bigcirc\mathit{last})$ are $\mathsf{I}_\forall$, and equivalent over infinite but not over finite traces. This example also shows that the condition $\mathsf{I}_{\Rightarrow\forall}$ alone is not sufficient for Theorem 9. Moreover, $\mathsf{F}_{\Rightarrow\exists}$ alone is also not sufficient. To see this, consider, e.g., $\Box^+\Diamond\top \vee (P(x) \wedge \Diamond\mathit{last})$ and $\Box^+\Diamond\top \vee (\Diamond\mathit{last})$, which are $\mathsf{F}_{\Rightarrow\exists}$ but are equivalent only over infinite traces. We now present sufficient conditions to preserve equivalences from the finite to the infinite case.

Theorem 10. *Given $T_\mathcal{U}\mathcal{QL}$ formulas φ and ψ, $\varphi \equiv_\mathfrak{F} \psi$ implies $\varphi \equiv_\mathfrak{I} \psi$ whenever both φ and ψ are (1) $\mathsf{I}_\exists$; or (2) $\mathsf{I}_\forall$; or (3) $\mathsf{F}_{\Rightarrow\forall}$ and $\mathsf{I}_{\Rightarrow\exists}$.*

We point out that $\mathsf{F}_\exists$ or $\mathsf{F}_\forall$ are not sufficient to ensure that formula equivalence over finite traces implies formula equivalence over infinite traces. To illustrate this, consider for example the formulas $\Phi = \Diamond^+\mathit{last} \vee \Diamond P(x)$, and the union of Φ and Formula (*) from Section 3. These formulas are $\mathsf{F}_\exists$, however, they are only equivalent over finite traces. Moreover, if we take $\Diamond^+ P(x)$ and the union of $\Diamond^+ P(x)$ with Formula (*), we have that they are both $\mathsf{F}_\forall$, though equivalent only over finite traces. This example also shows that the condition $\mathsf{F}_{\Rightarrow\forall}$ alone is not sufficient for Theorem 10. We now argue that $\mathsf{I}_{\Rightarrow\exists}$ alone is also not sufficient. To see this, consider, e.g., $(P(x) \wedge \Box^+\Diamond\top) \vee \Diamond\mathit{last}$ and $\Box^+\Diamond\top \vee \Diamond\mathit{last}$, which are $\mathsf{I}_{\Rightarrow\exists}$ but are equivalent only over finite traces.

From Theorems 9 and 10 we have that if both φ and ψ are $\mathsf{F}_\exists$ or $\mathsf{F}_\forall$, and $\mathsf{I}_\exists$ or $\mathsf{I}_\forall$ $T_\mathcal{U}\mathcal{QL}$ formulas, then, $\varphi \equiv_\mathfrak{F} \psi$ iff $\varphi \equiv_\mathfrak{I} \psi$. In particular, the above examples show that if, from a given pair of conditions F_Q and $\mathsf{I}_{Q'}$, we remove any of the two properties, then formula equivalences over finite and infinite traces may not coincide. We now analyse syntactic features of the properties introduced so far, providing a class of formulas satisfying them.

Theorem 11. *All non-temporal $T_\mathcal{U}\mathcal{QL}$ formulas satisfy the finite/infinite trace properties $\mathsf{F}_\exists$, $\mathsf{F}_\forall$, $\mathsf{I}_\exists$, and $\mathsf{I}_\forall$.*

$\Diamond^+$-formulas φ, ψ are built according to (with $P \in \mathsf{N_P}$):

$$\Diamond^+\varphi \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \exists x\varphi \mid P(\vec{\tau}) \mid \neg P(\vec{\tau})$$

We now show that the language generated by the grammar rule for $\Diamond^+$-formulas contains only formulas which are $\mathsf{F}_\forall$ and $\mathsf{I}_\exists$. We call $\Diamond$-formulas the set of formulas generated by the result of further allowing $\Diamond\varphi$ in the grammar rule for $\Diamond^+$-formulas; and call $\Diamond^+\forall$-formulas the result of allowing $\forall x\varphi$ in the grammar rule for $\Diamond^+$-formulas.

Theorem 12. *All $\Diamond^+$-formulas are $\mathsf{F}_\forall$ and $\mathsf{I}_\exists$. Moreover, all $\Diamond^+\forall$-formulas are $\mathsf{F}_\forall$ and all $\Diamond$-formulas are $\mathsf{I}_\exists$.*

The results of Theorem 12 are tight in the sense that we cannot extend the grammar rule for $\Diamond^+$-formulas with $\forall x\varphi$; and we cannot extend the grammar rule for $\Diamond^+\forall$-formulas with $\Diamond\varphi$. Simple counterexamples are $\forall x\Diamond^+ P(x)$ and $\Diamond\top$, which are not $\mathsf{I}_\exists$ and $\mathsf{F}_\forall$, respectively. To see that $\forall x\Diamond^+ P(x)$

is not $\mathsf{I}_\exists$ consider the model with an infinite (and countable) domain, where one element is in P exactly at time point $n \in \mathbb{N}$, another one exactly at time point $n+1$ and so on. There is no finite prefix where $\forall x \Diamond^+ P(x)$ holds.

$\Box^+$-formulas φ, ψ are built according to (with $P \in \mathsf{N_P}$):

$$\Box^+\varphi \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \forall x \varphi \mid P(\vec{\tau}) \mid \neg P(\vec{\tau})$$

We call $\Box$-formulas the set of formulas generated by the result of further allowing $\Box\varphi$ in the grammar rule for $\Box^+$-formulas; and call $\Box^+\exists$-formulas the result of allowing $\exists x \varphi$ in the grammar rule for $\Box^+$-formulas.

Theorem 13. *All $\Box^+$-formulas are $\mathsf{F}_\exists$ and $\mathsf{I}_\forall$. Moreover, all $\Box^+\exists$-formulas are $\mathsf{F}_\exists$ and all $\Box$-formulas are $\mathsf{I}_\forall$.*

The results of Theorem 13 are also tight in the sense that we cannot extend the grammar rule for $\Box^+$-formulas with $\exists x \varphi$; and we cannot extend the grammar rule for $\Box^+\exists$-formulas with $\Box\varphi$. Simple counterexamples are $\exists x \Box^+ \neg P(x)$ and $\Box\bot$, which are not $\mathsf{I}_\forall$ and $\mathsf{F}_\exists$, respectively. To see that $\exists x \Box^+ \neg P(x)$ is not $\mathsf{I}_\forall$, consider again the model described above with an infinite (and countable) domain, where each element is in P at a specific time point $n \in \mathbb{N}$. The formula $\exists x \Box^+ \neg P(x)$ holds in every finite prefix but it does not hold in this infinite trace.

It follows from our results that there is no distinction between reasoning over finite and infinite traces whenever a formula is either a $\Diamond^+$- or a $\Box^+$-formula. As already pointed out, $\Diamond^+\Box^+\varphi$ and $\Box^+\Diamond^+\varphi$ are only equivalent over finite traces, and so, the distinction between reasoning over finite and infinite traces cannot be blurred for the class of formulas that allow both $\Diamond^+$ and $\Box^+$.

5 Applications

The problem considered so far, to determine when the differences between finite and infinite traces can be safely blurred, is of interest for several applications. Here we show how it can be related to planning and verification. Moreover, for knowledge representation scenarios, we introduce temporal DLs over finite traces, providing complexity results for the satisfiability problem.

Planning. In automated planning, the sequence of states generated by actions is usually finite [Cerrito and Mayer, 1998; Bauer and Haslum, 2010; De Giacomo and Vardi, 2013; De Giacomo *et al.*, 2014b]. To reuse temporal logics based on infinite traces for specifying plan constraints, one approach, developed by De Giacomo *et al.* 2014b for LTL^f, is based on the notion of *insensitivity to infiniteness*. This property is meant to capture those formulas that can be equivalently interpreted over infinite traces, provided that, from a certain instant, these traces satisfy an end event forever and falsify all other atomic propositions. The motivation for this comes from the fact that propositional letters represents atomic tasks/actions that cannot be performed anymore after the end of a process.

In order to lift this notion of insensitivity to our first-order temporal setting, and to provide a characterisation analogous to the propositional one, we introduce the following definitions. Let $\mathfrak{F} = (\Delta^{\mathfrak{F}}, (\mathcal{F}_n)_{n\in[0,l]})$ be a finite trace, and let $\mathfrak{E} = (\Delta^{\mathfrak{E}}, (\mathcal{E}_n)_{n\in[0,\infty)})$ be the infinite trace such that $\Delta^{\mathfrak{E}} = \Delta^{\mathfrak{F}}$ (we write just Δ), $a^{\mathcal{E}} = a^{\mathcal{F}}$ for all $a \in \mathsf{N_I}$, and for all $P \in \mathsf{N_P} \setminus \{E\}$, $P^{\mathcal{E}_n} = \emptyset$, while $E^{\mathcal{E}_n} = \Delta$, where $n \in [0,\infty)$. The end extension (see Section 2) of $\mathfrak{F}$ with $\mathfrak{E}$, $\mathfrak{F} \cdot_E \mathfrak{E}$, will be called the *insensitive extension of* $\mathfrak{F}$. A $T_{\mathcal{U}}\mathcal{QL}$ formula φ is *insensitive to infiniteness* (or simply *insensitive*) if, for every finite trace $\mathfrak{F}$ and all assignments $\mathfrak{a}$, $\mathfrak{F} \models^{\mathfrak{a}} \varphi$ iff $\mathfrak{F} \cdot_E \mathfrak{E} \models^{\mathfrak{a}} \varphi$. Clearly, all insensitive $T_{\mathcal{U}}\mathcal{QL}$ formulas are also $\mathsf{F}_{\Rightarrow\exists}$.

Now, let Σ be a finite subset of $\mathsf{N_P}$. Assume w.l.o.g. that the $T^f_{\mathcal{U}}\mathcal{QL}$ formulas we mention in this subsection have predicates in Σ, and that Σ contains the end of time predicate E. Recalling the definition of ψ_f, we define $\chi_f = \psi_f \wedge \chi_{f_1}$, with

$$\chi_{f_1} = \Box\forall x \forall \bar{y}(E(x) \rightarrow \bigwedge_{P\in\Sigma\setminus\{E\}} \neg P(x,\bar{y})).$$

The next characterisation result extends Theorem 4 in [De Giacomo *et al.*, 2014b] to $T_{\mathcal{U}}\mathcal{QL}$.

Theorem 14. *A $T_{\mathcal{U}}\mathcal{QL}$ formula φ is insensitive to infiniteness iff the $T^i_{\mathcal{U}}\mathcal{QL}$ logical implication $\chi_f \models \varphi \leftrightarrow \varphi^\dagger$ holds.*

Insensitive formulas allow us to blur the distinction between finite and infinite traces as soon as infinite traces satisfy χ_f. Thus, we can check satisfiability of insensitive $T^f_{\mathcal{U}}\mathcal{QL}^2_{\Box}$ (or other decidable languages) formulas by using satisfiability algorithms for the infinite case without the need to apply the $\cdot^\dagger$ translation.

We now analyse some syntactic features of this property. Firstly, non-temporal $T_{\mathcal{U}}\mathcal{QL}$ formulas, are insensitive. Moreover, this property is preserved under non-temporal operators. We generalise Theorem 5 in [De Giacomo *et al.*, 2014b] in our setting as follows.

Theorem 15. *Let φ, ψ be insensitive $T_{\mathcal{U}}\mathcal{QL}$ formulas. Then $\neg\varphi$, $\exists x\varphi$, and $\varphi \wedge \psi$ are insensitive.*

Concerning temporal operators, in [De Giacomo *et al.*, 2014b] it is shown how several standard temporal patterns derived from the declarative process modelling language DECLARE [van der Aalst and Pesic, 2006] are insensitive. On the other hand, negation affects the insensitivity of temporal formulas. For instance, given a $T_{\mathcal{U}}\mathcal{QL}$ formula $P(x)$, we have that $\Diamond^+P(x)$ is insensitive while $\Diamond^+\neg P(x)$ is not. Dually, $\Box^+\neg P(x)$ is insensitive, while $\Box^+P(x)$ is not. Therefore, if a $T_{\mathcal{U}}\mathcal{QL}$ formula φ is insensitive, it cannot be concluded that formulas of the form $\Diamond^+\varphi$ or $\Box^+\varphi$ are insensitive.

Finally, as a consequence of Theorem 14, we show that insensitivity is sufficient to ensure that if formulas are equivalent over infinite traces then they are equivalent over finite traces.

Theorem 16. *Let φ and ψ be insensitive $T_{\mathcal{U}}\mathcal{QL}$ formulas. Then $\varphi \equiv_{\mathfrak{I}} \psi$ implies $\varphi \equiv_{\mathfrak{F}} \psi$.*

However, the viceversa does not hold. Consider, e.g., Formula (*), which is trivially insensitive: this formula is equivalent to $\bot$ only over finite traces. We can obtain the converse direction using Theorem 10. For instance, $\Diamond^+P(x) \vee \Diamond^+R(x)$ and $\Diamond^+(P(x) \vee R(x))$ are insensitive and $\mathsf{I}_\exists$ formulas for which equivalence over finite and infinite traces coincides.

Verification. In this section we show how our comparison between finite and infinite traces can be related to the literature on temporal logics for verification. We point out some connections between the finite/infinite trace properties and: (i)

the definition of safety in LTL^i (over infinite traces); (ii) some notions related to the construction of monitoring functions in runtime verification.

(i) *Safety*. Recall that a safety property intuitively asserts that *bad things* never happen during the execution of a program. In verification, LTL^i is often used as a specification language for such properties, and the notion of safety is defined accordingly over infinite traces [Sistla, 1994]. Let φ be a $T_{\mathcal{U}}\mathcal{QL}$ formula asserting that some "bad thing" never happens. According to the literature, we say that φ denotes a safety property if, whenever φ does not hold for an infinite run of a program, then it must be violated already after a finite execution. That is, the infinite trace contains a finite prefix in which the bad thing (the violation of φ) has already happened. More formally, we say that a $T_{\mathcal{U}}\mathcal{QL}$ formula φ *expresses a safety property* iff, for an infinite trace $\mathfrak{I}$ and an assignment $\mathfrak{a}$:

$$(\forall \mathfrak{F} \in Pre(\mathfrak{I}).\mathfrak{F} \models^{\mathfrak{a}} \varphi) \Rightarrow \mathfrak{I} \models^{\mathfrak{a}} \varphi.$$

In other words, safety is captured in $T_{\mathcal{U}}\mathcal{QL}$ by $\mathsf{I}_{\Leftarrow\forall}$ formulas. In particular, all $\Box^+$-formulas of Section 4 are $\mathsf{I}_{\forall}$ and thus they express safety properties.

(ii) *Runtime verification*. We recall that in runtime verification the task is to evaluate a property with respect to the current history (which is finite at each given instant) of a dynamic system, and to check whether this property is satisfied in all its possible future evolutions [Bauer *et al.*, 2010; Baader and Lippmann, 2014; De Giacomo *et al.*, 2014a]. Here we discuss the relationship between our semantic conditions and the *maxims* for runtime verification introduced by Bauer *et al.* 2010 which relate finite trace semantics to the infinite case. The authors suggest that every semantics to be used in runtime verification should satisfy the following requirements.

- *Impartiality*: never evaluate to true or false a formula on a finite trace, if one of its infinite extensions can possibly change its value.
- *Anticipation*: if a formula has the same truth value on every infinite extension of a finite trace, then it is equally evaluated also on that finite prefix.

Impartiality cannot be guaranteed for $T^f_{\mathcal{U}}\mathcal{QL}$: $\Box^+P(x)$ is an example of a formula violating this maxim. On the other hand, $\Diamond\top$ is a formula that violates anticipation. Impartiality, as formalised in [Bauer *et al.*, 2010], is captured by $\mathsf{F}_{\Rightarrow\forall}$ and $\mathsf{F}_{\Leftarrow\exists}$ properties. Instead, the formal version of anticipation corresponds to properties $\mathsf{F}_{\Leftarrow\forall}$ and $\mathsf{F}_{\Rightarrow\exists}$ in our setting. Therefore, any set of $T_{\mathcal{U}}\mathcal{QL}$ formulas satisfying both impartiality and anticipation should belong to the intersection of $\mathsf{F}_{\forall}$ and $\mathsf{F}_{\exists}$ formulas. Concerning the possibility to syntactically characterise these formulas, we have that, due to Theorems 12 and 13, impartiality and anticipation are not guaranteed to be preserved for $\Diamond^+$- or $\Box^+$-formulas.

Temporal Description Logics. We now consider temporal description logics. We define the temporal language $T_{\mathcal{U}}\mathcal{ALC}$ as a temporal extension of the description logic $\mathcal{ALC}$ [Gabbay *et al.*, 2003]. Let $\mathsf{N_C}, \mathsf{N_R} \subseteq \mathsf{N_P}$ be, respectively, sets of unary and binary predicates called *concept* and *role names*. A $T_{\mathcal{U}}\mathcal{ALC}$ *concept* is an expression of the form: $C, D ::= A \mid \neg C \mid C \sqcap D \mid \exists R.C \mid C\,\mathcal{U}\,D$, where $A \in \mathsf{N_C}$ and $R \in \mathsf{N_R}$. A $T_{\mathcal{U}}\mathcal{ALC}$ *axiom* is either a *concept inclusion (CI)* of the form $C \sqsubseteq D$, or an *assertion*, α, of the form $A(a)$ or $R(a,b)$, where C, D are $T_{\mathcal{U}}\mathcal{ALC}$ concepts, $A \in \mathsf{N_C}$, $R \in \mathsf{N_R}$, and $a, b \in \mathsf{N_I}$. $T_{\mathcal{U}}\mathcal{ALC}$ *formulas* have the form: $\varphi, \psi ::= \alpha \mid C \sqsubseteq D \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi\,\mathcal{U}\,\psi$. Since a $T_{\mathcal{U}}\mathcal{ALC}$ formula can be mapped into an equisatisfiable $T_{\mathcal{U}}\mathcal{QL}^2_{\boxed{1}}$ formula [Gabbay *et al.*, 2003] we can transfer the upper bounds of Theorems 4 and 5 to $T^f_{\mathcal{U}}\mathcal{ALC}$ and $T^k_{\mathcal{U}}\mathcal{ALC}$; the lower bounds can be obtained in a similar way as in the mentioned theorems. Moreover, from Theorems 6 and 7, we obtain immediately that $T^f_{\mathcal{U}}\mathcal{ALC}$ has both the bounded trace and domain properties.

We now consider the satisfiability problem in $T^k_{\mathcal{U}}\mathcal{ALC}$ *restricted to global CIs*, defined as the fragment of $T^k_{\mathcal{U}}\mathcal{ALC}$ in which formulas can only be of the form $\mathcal{T} \wedge \Box(\mathcal{T}) \wedge \phi$, where $\mathcal{T}$ is a conjunction of CIs and ϕ does not contain CIs. The ExpTime upper bound we provide has a rather challenging proof that uses a form of *type elimination* [Gabbay *et al.*, 2003; Lutz *et al.*, 2008; Gutiérrez-Basulto *et al.*, 2016], but in a setting where the number of time points is bounded by a natural number $k > 0$. The main challenge in solving this problem when the number of time points is arbitrarily large but finite is mainly due to the presence of *last* sub-formulas (i.e., formulas of the form $\neg\bigcirc\bot$) that can hold just in the last instant of the model. The complexity is tight since satisfiability in $\mathcal{ALC}$ is already ExpTime-hard [Gabbay *et al.*, 2003].

Theorem 17. *Satisfiability in $T^f_{\mathcal{U}}\mathcal{ALC}$ is* ExpSpace*-complete and in $T^k_{\mathcal{U}}\mathcal{ALC}$ is* NExpTime*-complete. Moreover, satisfiability in $T^k_{\mathcal{U}}\mathcal{ALC}$ restricted to global CIs is* ExpTime*-complete.*

6 Conclusion

We investigated first-order temporal logic over finite traces, studying satisfiability of its two-variable monodic fragment, $T^f_{\mathcal{U}}\mathcal{QL}^2_{\boxed{1}}$. While being ExpSpace-complete over arbitrary finite traces, it lowers down to NExpTime in case of $T^k_{\mathcal{U}}\mathcal{QL}^2_{\boxed{1}}$, interpreted over traces with at most k time points. Similar results have been shown for a temporal extension of the descrption logic $\mathcal{ALC}$, with $T^k_{\mathcal{U}}\mathcal{ALC}$ restricted to global CIs being ExpTime-complete. Moreover, in an effort to systematically clarify the correlations between finite vs. infinite reasoning we introduced various semantic conditions that allow to formally specify when it is possible to blur the distinction between finite and infinite traces. Grammars for $T_{\mathcal{U}}\mathcal{QL}$ formulas satisfying some of these conditions have been provided as well. In particular, we have shown that for $\Diamond^+$- and $\Box^+$-formulas, equivalence over finite and infinite traces coincide. Some notions used in planning (particularly, insensitivity to infiniteness [De Giacomo *et al.*, 2014b]) and verification have been lifted to the first-order setting, and related to our semantic conditions for blurring the distinction between reasoning over finite and infinite traces.

As future work, we plan to apply the semantic conditions to study the specific case where infinite extensions of finite traces are obtained by repeating the last instant forever [Bauer and Haslum, 2010], as well as to the analysis of monitoring functions for runtime verification [Bauer *et al.*, 2010; Baader and Lippmann, 2014; De Giacomo *et al.*, 2014a]. It would also be interesting to study the precise complexity of the satisfiability problem for $T^f_{\mathcal{U}}\mathcal{ALC}$ with just global CIs.

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A Proofs for Section 3

Lemma 1. *For every infinite trace $\mathfrak{I}$, $\mathfrak{I} \models \psi_f$ iff $\mathfrak{I} = \mathfrak{F} \cdot_E \mathfrak{I}'$, for some finite trace $\mathfrak{F}$ and some infinite trace $\mathfrak{I}'$.*

Proof. ψ_f is satisfied in $\mathfrak{I}$ iff there is $k > 0$ such that, for all $d \in \Delta$, it holds that: $d \notin E^{\mathcal{I}_j}$, for all $j \in [0, k)$; and $d \in E^{\mathcal{I}_i}$, for all $i \in [k, \infty)$. That is, $\mathfrak{I} = \mathfrak{F} \cdot_E \mathfrak{I}'$, for some finite trace $\mathfrak{F}$ and some infinite trace $\mathfrak{I}'$. ☐

Lemma 2. *Let $\mathfrak{F} \cdot_E \mathfrak{I}$ be an end extension of a finite trace $\mathfrak{F}$. For every $T_{\mathcal{U}}^f \mathcal{QL}$ formula φ and every assignment $\mathfrak{a}$, $\mathfrak{F} \models^{\mathfrak{a}} \varphi$ iff $\mathfrak{F} \cdot_E \mathfrak{I} \models^{\mathfrak{a}} \varphi^{\dagger}$.*

Proof. Let $\mathfrak{F} = (\Delta^{\mathfrak{F}}, (\mathcal{F}_n)_{n \in [0,l]})$ be a finite trace and let $\mathfrak{F} \cdot_E \mathfrak{I} = (\Delta^{\mathfrak{F} \cdot \mathfrak{I}}, (\mathcal{F} \cdot \mathcal{I}_n)_{n \in [0,\infty)})$ be an end extension of $\mathfrak{F}$. Firstly, we prove, by induction on subformulas ψ of φ, that for all $n \in [0, l]$ and all assignments $\mathfrak{a}$:

$$\mathfrak{F}, n \models^{\mathfrak{a}} \psi \text{ iff } \mathfrak{F} \cdot_E \mathfrak{I}, n \models^{\mathfrak{a}} \psi^{\dagger}.$$

If $\psi = P(\bar{\tau})$, the statement follows immediately from the definition of $\mathfrak{F} \cdot_E \mathfrak{I}$ and $\cdot^{\dagger}$. The proof of the inductive cases $\psi = \neg\chi$, $\psi = (\chi \wedge \zeta)$, and $\psi = \exists x \chi$ is straightforward. The remaining case is proved as follows.

$\psi = (\chi \mathcal{U} \zeta)$. We have that $\mathfrak{F}, n \models^{\mathfrak{a}} \chi \mathcal{U} \zeta$ iff there is $m \in (n, l]$ such that $\mathfrak{F}, m \models^{\mathfrak{a}} \zeta$ and for all $i \in (n, m)$, $\mathfrak{F}, i \models^{\mathfrak{a}} \chi$. By i.h. and definition of $\mathfrak{F} \cdot_E \mathfrak{I}$ this happens iff there is $m \in (n, l]$ s.t. $\mathfrak{F} \cdot_E \mathfrak{I}, m \models^{\mathfrak{a}} \zeta^{\dagger}$ and for all $i \in (n, m)$, $\mathfrak{F} \cdot_E \mathfrak{I}, i \models^{\mathfrak{a}} \chi^{\dagger}$. Since $E^{\mathcal{F} \cdot_E \mathcal{I}_j} = \emptyset$ for all $j \in [0, l]$, this means that $\mathfrak{F} \cdot_E \mathfrak{I}, n \models^{\mathfrak{a}} \chi^{\dagger} \, \mathcal{U} \, (\zeta^{\dagger} \wedge \forall x \neg E(x))$. That is, $\mathfrak{F} \cdot_E \mathfrak{I}, n \models^{\mathfrak{a}} (\chi \, \mathcal{U} \, \zeta)^{\dagger}$ (recall that $\forall x \neg E(x) = \psi_{f_1}$).

Thus, in particular, we have: $\mathfrak{F} \models \varphi$ iff $\mathfrak{F} \cdot_E \mathfrak{I} \models \varphi^{\dagger}$. ☐

Theorem 3. *Let φ be a $T_{\mathcal{U}}^f \mathcal{QL}$ formula. Then φ is satisfiable iff $\varphi^{\dagger} \wedge \psi_f$ is a satisfiable $T_{\mathcal{U}}^i \mathcal{QL}$ formula.*

Proof. If φ is satisfied in some finite trace $\mathfrak{F}$, then (by Lemmas 1 and 2) any end extension $\mathfrak{F} \cdot_E \mathfrak{I}$ satisfies $\varphi^{\dagger} \wedge \psi_f$. Conversely, suppose that $\varphi^{\dagger} \wedge \psi_f$ is satisfied in some infinite trace $\mathfrak{I}$. By Lemma 1, $\mathfrak{I} = \mathfrak{F} \cdot_E \mathfrak{I}'$, for some finite trace $\mathfrak{F}$ and some infinite trace $\mathfrak{I}'$. Since $\mathfrak{F} \cdot_E \mathfrak{I}' \models \varphi^{\dagger}$, by Lemma 2, we have that $\mathfrak{F} \models \varphi$. ☐

To show Theorem 5 we use standard definitions for *quasimodels* [Hodkinson *et al.*, 2000; Gabbay *et al.*, 2003], presented here for convenience of the reader. Let φ be a $T_{\mathcal{U}}^f \mathcal{QL}_{\boxed{1}}$ sentence. Let $\mathsf{N_I}(\varphi)$ be the set of individuals occurring in φ and let $\mathsf{sub}(\varphi)$ be the set of subformulas of φ. For every formula $\psi(y)$ of the form $\psi_1 \mathcal{U} \psi_2$ with one free variable y, we fix a *surrogate* $R_{\psi}(y)$; and for every sentence ψ of the form $\psi_1 \mathcal{U} \psi_2$ we fix a surrogate p_{ψ}, where R_{ψ} and p_{ψ} are symbols not occurring in φ. Given a $T_{\mathcal{U}}^f \mathcal{QL}_{\boxed{1}}$ formula φ, we denote by $\overline{\varphi}$ the result of replacing in φ all subformulas of the form $\psi_1 \mathcal{U} \psi_2$ which are not in the scope of any other occurrence of $\mathcal{U}$ by their surrogates. Thus, $\overline{\varphi}$ does not contain occurrences of temporal operators. Let x be a variable not occurring in φ. We write $\psi\{y\}$ for a sentence ψ or a formula $\psi(y)$ with (one) free variable y; and $\psi\{x/y\}$ for a sentence ψ or the result of replacing free occurrences of y in $\psi(y)$ by x. Let $\mathsf{sub}_x(\varphi)$ be the closure under single negation of all $\psi\{x/y\}$ with $\psi\{y\} \in \mathsf{sub}(\varphi)$. A *type* for φ is a subset t of $\{\overline{\psi} \mid \psi \in \mathsf{sub}_x(\varphi)\} \cup \mathsf{N_I}(\varphi)$ such that:

- $\overline{\psi_1} \wedge \overline{\psi_2} \in t$ iff $\overline{\psi_1} \in t$ and $\overline{\psi_2} \in t$, for every $\psi_1 \wedge \psi_2 \in \mathsf{sub}_x(\varphi)$;
- $\neg\overline{\psi} \in t$ iff $\overline{\psi} \notin t$, for every $\neg\psi \in \mathsf{sub}_x(\varphi)$; and
- t contains at most one element of $\mathsf{N_I}(\varphi)$.

We omit 'for φ' when there is no risk of confusion. Let $\mathsf{sub}_0(\overline{\varphi})$ be the set of all $\overline{\psi}$ with ψ a sentence in $\mathsf{sub}_x(\varphi)$. We say that the types t, t' *agree on* $\mathsf{sub}_0(\varphi)$ if $t \cap \mathsf{sub}_0(\overline{\varphi}) = t' \cap \mathsf{sub}_0(\overline{\varphi})$. Denote with $\mathsf{tp}(\varphi)$ the set of all types for φ. If $a \in t \cap \mathsf{N_I}(\varphi)$ then t 'describes' a named element. We write t^a to indicate this and call it a *named type*. A *state candidate* is a subset $\mathfrak{C}$ of $\mathsf{tp}(\varphi)$ with only types that agree on $\mathsf{sub}_0(\varphi)$, containing exactly one t^a for each $a \in \mathsf{N_I}(\varphi)$, and such that $\{t \mid t^a \in \mathfrak{C}\} \subseteq \mathfrak{C}$. Consider a classical first-order interpretation I with d in the domain Δ. Clearly, the set $t^I(d) = \{\overline{\psi} \mid \psi \in \mathsf{sub}_x(\varphi), I \models \overline{\psi}[d]\} \cup \{a \in \mathsf{N_I}(\varphi) \mid \text{if } a^I = d\}$ is a type for φ. An interpretation I *realizes* a state canditate $\mathfrak{C}$ if $\mathfrak{C} = \{t^I(d) \mid d \in \Delta\}$.

$\mathfrak{C}$ is realisable if there is a structure realizing it. A state candidate $\mathfrak{C}$ is realisable iff the sentence

$$\mathsf{real}_{\mathfrak{C}} = \bigwedge_{t \in \mathfrak{C}} \exists x \bigwedge_{\psi\{x\} \in t} \psi\{x\} \wedge \forall x \bigvee_{t \in \mathfrak{C}} \bigwedge_{\psi\{x\} \in t} \psi\{x\}$$

$$\wedge \bigwedge_{t^a \in \mathfrak{C}} \bigwedge_{\psi\{x\} \in t^a} \psi\{a/x\}$$

is true in some first-order interpretation [Gabbay *et al.*, 2003, Lemma 11.6]. For $T_{\mathcal{U}} \mathcal{QL}^2_{\boxed{1}}$ formulas, given a state candidate $\mathfrak{C}$, the formula $\mathsf{real}_{\mathfrak{C}}$ is satisfiable iff it is satisfiable by an interpretation of exponential size.

Lemma 18. *[Gabbay* et al.*, 2003, Theorem 11.31] Let $\mathfrak{C}$ be a state candidate for a $T_{\mathcal{U}} \mathcal{QL}^2_{\boxed{1}}$ formula φ. Then, the formula $\mathsf{real}_{\mathfrak{C}}$ is satisfiable iff it is satisfiable by an interpretation I of size exponential in the size of the input formula φ.*

Let us now turn to our main results for formulas interpreted over finite traces, i.e., formulas in $T_{\mathcal{U}}^f \mathcal{QL}^2_{\boxed{1}}$. A quasimodel for a $T_{\mathcal{U}}^f \mathcal{QL}^2_{\boxed{1}}$ sentence φ is a pair $(S, \mathfrak{R})$ where S is a finite sequence $S(0), \ldots, S(n)$ of realizable state candidates $S(i)$, and $\mathfrak{R}$ is a set of functions r, called *runs*, mapping each $i \in \{0, \ldots, n\}$ to a type in $S(i)$, satisfying the following conditions:

1. for every $\psi_1 \mathcal{U} \psi_2 \in \mathsf{sub}_x(\varphi)$ and every $i \in [0, n]$, we have $\overline{\psi_1 \mathcal{U} \psi_2} \in r(i)$ iff there is $j \in (i, n]$ such that $\overline{\psi_2} \in r(j)$ and $\overline{\psi_1} \in r(l)$ for all $l \in (i, j)$;
2. for every $a \in \mathsf{N_I}(\varphi)$, every $r \in \mathfrak{R}$ and every $i, j \in [0, n]$, we have $a \in r(i)$ iff $a \in r(j)$;
3. $\overline{\varphi} \in t$ for some $t \in S(0)$; and
4. for every $i \in [0, n]$ and every $t \in S(i)$ there is a run $r \in \mathfrak{R}$ such that $r(i) = t$.

Every quasimodel for φ describes an interpretation satisfying φ and, conversely, every such interpretation can be abstracted into a quasimodel for φ. We formalise this well-known notion in the following lemma that follows form an easy adapation of [Gabbay *et al.*, 2003, Lemma 11.22] to the case of finite traces.

Lemma 19. *Let φ be a $T^f_{\mathcal{U}}\mathcal{QL}^2_{\boxed{1}}$ formula. There is a finite trace satisfying φ with at most k time points iff there is a quasimodel for φ with a sequence of quasistates of length at most k.*

We now devise a non-deterministic algorithm to check satisfiability of a $T^k_{\mathcal{U}}\mathcal{QL}^2_{\boxed{1}}$ formula φ in NEXPTIME. It follows from the definition of types, that the number of distinct types for φ is exponential in $|\varphi|$. First we compute in exponential time w.r.t. $|\varphi|$ the set of all types for φ. We guess a sequence $I(0), \dots, I(n)$ of first-order interpretations (each of them of size exponential in $|\varphi|$ by Lemma 18); a sequence $S(0), \dots, S(n)$ of sets of types for φ of length $n \leq k$; and for each type at position i in this sequence we also guess a sequence of types of length n. Denote by $\mathfrak{R}$ the set of such sequences of types. We now check (a) whether each $S(i)$ is a realizable state candidate (by checking whether $I(i)$ satisfies $S(i)$); (b) whether each sequence in $\mathfrak{R}$ satisfies Conditions (1) and (2); and (c) whether φ is in a type in $S(0)$ (Condition 3). Condition 4 is satisfied by definition of $\mathfrak{R}$.

All these conditions can be checked in exponential time w.r.t. $|\varphi|$ and $|k|$. The algorithm returns 'satisfiable' iff all conditions are satisfied. Since the conditions exactly match the definition of a quasimodel for φ, their satisfaction implies that $(S, \mathfrak{R})$ is a quasimodel for φ.

Theorem 5. *Satisfiability in $T^k_{\mathcal{U}}\mathcal{QL}^2_{\boxed{1}}$ is* NEXPTIME-*complete.*

Proof. Hardness follows from Lemma 6.2 in [Baader *et al.*, 2012], proved for $\mathcal{ALC}$-LTL with rigid concepts, which can be expressed in $T^k_{\mathcal{U}}\mathcal{QL}^2_{\boxed{1}}$. The proof is by reduction from the bounded version of the domino problem. The same reduction applies for satisfiability in $T^k_{\mathcal{U}}\mathcal{QL}^2_{\boxed{1}}$ since a k-bounded solution for the problem can be mapped into a finite trace with at most k time points (assuming that k is given in binary the solution is exponential w.r.t. the size of the input).

For the upper bound, by Lemma 19, there is a finite trace with at most k time points iff there is a quasimodel with at most k quasistates. Then, we can decide satisfiability using the non-deterministic algorithm. ☐

Theorem 7. *Satisfiability of φ in $T^k_{\mathcal{U}}\mathcal{QL}^2_{\boxed{1}}$ implies the existence of a model with domain size exponential in k and φ.*

Proof. By Lemma 19 there is a model with at most k time points iff there is a quasimodel for φ with a sequence of quasistates (for φ) of length at most k. This means that runs have size at most k. Since each run is a sequence of types, the number of possible runs is bounded by $|\mathsf{tp}(\varphi)|^k$. The correspondence between models and quasimodels (Lemma 19) maps each run to a domain element. Thus, there is a model with domain size bounded by $|\mathsf{tp}(\varphi)|^k$. ☐

B Proofs for Section 4

Lemma 20. *Let φ be a $T_{\mathcal{U}}\mathcal{QL}$ formula, and $Q, Q' \in \{\exists, \forall\}$, with $Q \neq Q'$. Then, φ is $\mathsf{F}_{\Rightarrow Q}$ iff $\neg\varphi$ is $\mathsf{F}_{\Leftarrow Q'}$. Moreover, φ is $\mathsf{I}_{\Rightarrow Q}$ iff $\neg\varphi$ is $\mathsf{I}_{\Leftarrow Q'}$.*

Proof. Suppose that φ is $\mathsf{F}_{\Rightarrow Q}$, i.e.: $\forall \mathfrak{F}(\mathfrak{F} \models^{\mathfrak{a}} \varphi \Rightarrow Q\mathfrak{I} \in \mathit{Ext}(\mathfrak{F}).\mathfrak{I} \models^{\mathfrak{a}} \varphi)$, for all assignments $\mathfrak{a}$. This is equivalent to: $\forall \mathfrak{F}(Q'\mathfrak{I} \in \mathit{Ext}(\mathfrak{F}).\mathfrak{I} \models^{\mathfrak{a}} \neg\varphi \Rightarrow \mathfrak{F} \models^{\mathfrak{a}} \neg\varphi)$, for all assignments $\mathfrak{a}$. That is, $\neg\varphi$ is $\mathsf{F}_{\Leftarrow Q'}$. The remaining case is analogous. ☐

Lemma 21. *Let φ and ψ be $T_{\mathcal{U}}\mathcal{QL}$ formulas. If $\varphi \wedge \neg\psi$ and $\psi \wedge \neg\varphi$ are $\mathsf{F}_{\Rightarrow\exists}$, then $\varphi \equiv_{\mathfrak{I}} \psi$ implies $\varphi \equiv_{\mathfrak{F}} \psi$. Moreover, if $\varphi \wedge \neg\psi$ and $\psi \wedge \neg\varphi$ are $\mathsf{I}_{\Rightarrow\exists}$, then $\varphi \equiv_{\mathfrak{F}} \psi$ implies $\varphi \equiv_{\mathfrak{I}} \psi$.*

Proof. Suppose that $\varphi \wedge \neg\psi$ and $\psi \wedge \neg\varphi$ are $\mathsf{F}_{\Rightarrow\exists}$. We have $\varphi \equiv_{\mathfrak{I}} \psi$ iff $\varphi \wedge \neg\psi$ and $\psi \wedge \neg\varphi$ are not $T^i_{\mathcal{U}}\mathcal{QL}$ satisfiable. Assume $\varphi \equiv_{\mathfrak{I}} \psi$. By $\mathsf{F}_{\Rightarrow\exists}$, the previous step implies: $\varphi \wedge \neg\psi$ and $\psi \wedge \neg\varphi$ are not $T^f_{\mathcal{U}}\mathcal{QL}$ satisfiable, which is equivalent to $\varphi \equiv_{\mathfrak{F}} \psi$. Moreover, suppose that $\varphi \wedge \neg\psi$ and $\psi \wedge \neg\varphi$ are $\mathsf{I}_{\Rightarrow\exists}$. We have that $\varphi \equiv_{\mathfrak{F}} \psi$ iff $\varphi \wedge \neg\psi$ and $\psi \wedge \neg\varphi$ are not $T^f_{\mathcal{U}}\mathcal{QL}$ satisfiable If $\varphi \equiv_{\mathfrak{F}} \psi$, by $\mathsf{I}_{\Rightarrow\exists}$ we have $\varphi \wedge \neg\psi$ and $\psi \wedge \neg\varphi$ are not $T^i_{\mathcal{U}}\mathcal{QL}$ satisfiable. Thus, $\varphi \equiv_{\mathfrak{I}} \psi$. ☐

Lemma 22. *Let φ and ψ be $T_{\mathcal{U}}\mathcal{QL}$ formulas. If φ and ψ are $\mathsf{F}_{\exists}$ or $\mathsf{F}_{\forall}$, then $\varphi \wedge \neg\psi$ and $\psi \wedge \neg\varphi$ are $\mathsf{F}_{\Rightarrow\exists}$. Moreover, if φ and ψ are $\mathsf{I}_{\exists}$ or $\mathsf{I}_{\forall}$, then $\varphi \wedge \neg\psi$ and $\psi \wedge \neg\varphi$ are $\mathsf{I}_{\Rightarrow\exists}$.*

Proof. Suppose that φ and ψ are $\mathsf{F}_{\exists}$ (the case for $\mathsf{F}_{\forall}$ is similar), and let $\chi \in \{\varphi, \psi\}$. Since, in particular, χ is $\mathsf{F}_{\Leftarrow\exists}$, by Lemma 20 we have that $\neg\chi$ is also $\mathsf{F}_{\Rightarrow\forall}$, and thus $\mathsf{F}_{\Rightarrow\exists}$. Thus, we have that both χ and $\neg\chi$ are $\mathsf{F}_{\Rightarrow\exists}$. Suppose now, towards a contradiction, that $\varphi \wedge \neg\psi$ is not $\mathsf{F}_{\Rightarrow\exists}$. This means that there is a finite trace $\mathfrak{F}$ such that $\mathfrak{F} \models^{\mathfrak{a}} \varphi$ and $\mathfrak{F} \not\models^{\mathfrak{a}} \psi$, and for all $\mathfrak{I} \in \mathit{Ext}(\mathfrak{F})$: $\mathfrak{I} \not\models^{\mathfrak{a}} \varphi$ or $\mathfrak{I} \models^{\mathfrak{a}} \psi$. Given that φ is $\mathsf{F}_{\Rightarrow\exists}$, there is a $\mathfrak{I}' \in \mathit{Ext}(\mathfrak{F})$ such that $\mathfrak{I}' \models^{\mathfrak{a}} \varphi$. Also, since ψ is $\mathsf{F}_{\Leftarrow\exists}$, we have that for all $\mathfrak{I} \in \mathit{Ext}(\mathfrak{F})$, $\mathfrak{I} \not\models^{\mathfrak{a}} \psi$. Consider now an arbitrary $\mathfrak{I} \in \mathit{Ext}(\mathfrak{F})$: if $\mathfrak{I} \models^{\mathfrak{a}} \psi$, we have a contradiction from the fact that for all $\mathfrak{I} \in \mathit{Ext}(\mathfrak{F})$, $\mathfrak{I} \not\models^{\mathfrak{a}} \psi$. Then, for all $\mathfrak{I} \in \mathit{Ext}(\mathfrak{F})$, we have $\mathfrak{I} \not\models^{\mathfrak{a}} \varphi$, which contradicts the fact that there is a $\mathfrak{I}' \in \mathit{Ext}(\mathfrak{F})$ such that $\mathfrak{I}' \models^{\mathfrak{a}} \varphi$. This absurd shows that $\varphi \wedge \neg\psi$ is $\mathsf{F}_{\Rightarrow\exists}$ (the proof for $\psi \wedge \neg\varphi$ is analogous).

Moreover, suppose φ and ψ are $\mathsf{I}_{\exists}$ (the case for $\mathsf{I}_{\forall}$ is similar), and again let $\chi \in \{\varphi, \psi\}$. Since, in particular, χ is $\mathsf{I}_{\Leftarrow\exists}$, by Lemma 20 we have that $\neg\chi$ is also $\mathsf{I}_{\Rightarrow\forall}$, and thus $\mathsf{I}_{\Rightarrow\exists}$. Thus, we have that both χ and $\neg\chi$ are $\mathsf{I}_{\Rightarrow\exists}$. Suppose now, towards a contradiction, that $\varphi \wedge \neg\psi$ is not $\mathsf{I}_{\Rightarrow\exists}$. We have that there is an infinite trace $\mathfrak{I}$ and an assignment $\mathfrak{a}$ such that $\mathfrak{I} \models^{\mathfrak{a}} \varphi$ and $\mathfrak{I} \models^{\mathfrak{a}} \neg\psi$, and for all $\mathfrak{F} \in \mathit{Pre}(\mathfrak{I})$: $\mathfrak{F} \not\models^{\mathfrak{a}} \varphi$ or $\mathfrak{F} \models^{\mathfrak{a}} \psi$. Since φ is $\mathsf{I}_{\Rightarrow\exists}$, there is a $\mathfrak{F}' \in \mathit{Pre}(\mathfrak{I})$ such that $\mathfrak{F}' \models^{\mathfrak{a}} \varphi$. Moreover, since $\neg\psi$ is $\mathsf{I}_{\Rightarrow\forall}$, we have that for all $\mathfrak{F} \in \mathit{Pre}(\mathfrak{I})$, $\mathfrak{F} \models^{\mathfrak{a}} \neg\psi$. Given an arbitrary $\mathfrak{F} \in \mathit{Pre}(\mathfrak{I})$, if $\mathfrak{F} \models^{\mathfrak{a}} \psi$, we contradict the fact that for all $\mathfrak{F} \in \mathit{Pre}(\mathfrak{I})$, $\mathfrak{F} \not\models^{\mathfrak{a}} \psi$. Thus, for all $\mathfrak{F} \in \mathit{Pre}(\mathfrak{I})$, we have $\mathfrak{F} \not\models^{\mathfrak{a}} \varphi$, contradicting the fact that there is a $\mathfrak{F}' \in \mathit{Pre}(\mathfrak{I})$ such that $\mathfrak{F}' \models^{\mathfrak{a}} \varphi$. This absurd proves that $\varphi \wedge \neg\psi$ is $\mathsf{I}_{\Rightarrow\exists}$ (the proof for $\psi \wedge \neg\varphi$ is analogous). ☐

Theorem 9. *Given $T_{\mathcal{U}}\mathcal{QL}$ formulas φ and ψ, $\varphi \equiv_{\mathfrak{I}} \psi$ implies $\varphi \equiv_{\mathfrak{F}} \psi$ whenever both φ and ψ are (1) $\mathsf{F}_{\exists}$; or (2) $\mathsf{F}_{\forall}$; or (3) $\mathsf{F}_{\Rightarrow\exists}$ and $\mathsf{I}_{\Rightarrow\forall}$.*

Proof. Points (1) and (2) follow from Lemma 21 and Lemma 22. We provide also a direct proof. Assume φ and ψ to be $\mathsf{F}_\exists$ (the proof for $\mathsf{F}_\forall$ is analogous). Given an assignment $\mathfrak{a}$, let $\mathfrak{F}$ be a finite trace such that $\mathfrak{F} \models^\mathfrak{a} \varphi$. By $\mathsf{F}_\exists$, there is an infinite trace $\mathfrak{I} \in \mathit{Ext}(\mathfrak{F})$ such that $\mathfrak{I} \models^\mathfrak{a} \varphi$ Since $\varphi \equiv_\mathfrak{I} \psi$, we have that $\mathfrak{I} \models^\mathfrak{a} \psi$. By $\mathsf{F}_\exists$, $\mathfrak{F} \models^\mathfrak{a} \psi$. The converse direction can be obtained analogously, by swapping φ and ψ.

We now show Point (3). Given a finite trace $\mathfrak{F}$ and an assignment $\mathfrak{a}$, suppose that $\mathfrak{F} \models^\mathfrak{a} \varphi$. Since φ is $\mathsf{F}_{\Rightarrow\exists}$, we have that for some $\mathfrak{I} \in \mathit{Ext}(\mathfrak{F})$, $\mathfrak{I} \models^\mathfrak{a} \varphi$. By assumption, $\varphi \equiv_\mathfrak{I} \psi$, so $\mathfrak{I} \models^\mathfrak{a} \psi$. As ψ is $\mathsf{I}_{\Rightarrow\forall}$, $\mathfrak{I} \models^\mathfrak{a} \psi$ implies that, for all $\mathfrak{F}' \in \mathit{Pre}(\mathfrak{I})$, $\mathfrak{F}' \models^\mathfrak{a} \psi$. Thus, in particular, $\mathfrak{F} \models^\mathfrak{a} \psi$. The other direction can be obtained by swapping φ and ψ. ☐

Theorem 10. *Given $T_\mathcal{U}\mathcal{QL}$ formulas φ and ψ, $\varphi \equiv_\mathfrak{F} \psi$ implies $\varphi \equiv_\mathfrak{I} \psi$ whenever both φ and ψ are (1) $\mathsf{I}_\exists$; or (2) $\mathsf{I}_\forall$; or (3) $\mathsf{F}_{\Rightarrow\forall}$ and $\mathsf{I}_{\Rightarrow\exists}$.*

Proof. Points (1) and (2) follow again from Lemma 21 and Lemma 22. Here we give a direct proof as well. Suppose that φ and ψ are $\mathsf{I}_\exists$ (the proof for for $\mathsf{I}_\forall$ is analogous). Given an assignment $\mathfrak{a}$, let $\mathfrak{I}$ be an infinite trace such that $\mathfrak{I} \models^\mathfrak{a} \varphi$. By $\mathsf{I}_\exists$, there is $\mathfrak{F} \in \mathit{Pre}(\mathfrak{I})$ such that $\mathfrak{F} \models^\mathfrak{a} \varphi$. As $\varphi \equiv_\mathfrak{F} \psi$, this means that $\mathfrak{F} \models^\mathfrak{a} \psi$. Since ψ is $\mathsf{I}_\exists$, we have $\mathfrak{I} \models^\mathfrak{a} \psi$. The converse direction is obtained analogously, by swapping φ and ψ.

We now show Point (3). Let $\mathfrak{I}$ be an infinite trace and $\mathfrak{a}$ be an assignment such that $\mathfrak{I} \models^\mathfrak{a} \varphi$. As φ is $\mathsf{I}_{\Rightarrow\exists}$, there is $\mathfrak{F} \in \mathit{Pre}(\mathfrak{I})$ such that $\mathfrak{F} \models^\mathfrak{a} \varphi$. Given that $\varphi \equiv_\mathfrak{F} \psi$, $\mathfrak{F} \models^\mathfrak{a} \psi$. By $\mathsf{F}_{\Rightarrow\forall}$, for all $\mathfrak{I}' \in \mathit{Ext}(\mathfrak{F})$: $\mathfrak{I}' \models^\mathfrak{a} \psi$. Therefore, we have also $\mathfrak{I} \models^\mathfrak{a} \psi$. The converse direction is obtained in a similar way by swapping φ and ψ. ☐

Theorem 11. *All non-temporal $T_\mathcal{U}\mathcal{QL}$ formulas satisfy the finite/infinite trace properties $\mathsf{F}_\exists$, $\mathsf{F}_\forall$, $\mathsf{I}_\exists$, and $\mathsf{I}_\forall$.*

Proof. Clearly, assuming that φ has no temporal operators, for any finite/infinite trace $\mathfrak{M}$, $\mathfrak{M}$ satisfies φ iff any extension/prefix of $\mathfrak{M}$ satisfies φ. ☐

In the proofs of Theorems 12 and 13 we use the following notation. Given a finite trace $\mathfrak{F} = (\Delta, \mathcal{F}_{n\in[0,l]})$, we denote by $\mathfrak{F}^\omega$ the extension of $\mathfrak{F}$ with the infinite trace $(\mathfrak{F}^l)^\omega$, where $\mathfrak{F}^l$ is the finite trace with only the last instant of $\mathfrak{F}$. Also, given and assignment $\mathfrak{a}$ and an element d of a domain Δ (of a finite or infinite trace), we write $\mathfrak{a}_d$ for the result of modifying $\mathfrak{a}$ so that x maps to d. The next lemma is useful to prove Theorem 12.

Lemma 23. *Let $\mathfrak{F}$ be a finite trace and let $\mathfrak{F}'$ be a prefix of $\mathfrak{F}$. For all assignments $\mathfrak{a}$, and all $\Diamond$-formulas φ, if $\mathfrak{F}' \models^\mathfrak{a} \varphi$ then $\mathfrak{F} \models^\mathfrak{a} \varphi$.*

Proof. The proof is by induction. In the base case we have all non-temporal $\Diamond$-formulas. Assume that the lemma holds for a $\Diamond$-formula φ. Let $\mathfrak{F}$ be a finite trace, let $\mathfrak{F}'$ be a prefix of $\mathfrak{F}$ and let $\mathfrak{a}$ be an assignment. If $\mathfrak{F}' \models^\mathfrak{a} \Diamond\varphi$ then there is $n > 0$ such that $\mathfrak{F}', n \models^\mathfrak{a} \varphi$. As $\mathfrak{F}'$ is a prefix of $\mathfrak{F}$, we have that $\mathfrak{F}, n \models^\mathfrak{a} \varphi$. This means that there is $n > 0$ such that $\mathfrak{F}, n \models^\mathfrak{a} \varphi$. Thus, $\mathfrak{F} \models^\mathfrak{a} \Diamond\varphi$. Other cases can be proved by straightforward applications of the inductive hypothesis. ☐

Theorem 12. *All $\Diamond^+$-formulas are $\mathsf{F}_\forall$ and $\mathsf{I}_\exists$. Moreover, all $\Diamond^+\forall$-formulas are $\mathsf{F}_\forall$ and all $\Diamond$-formulas are $\mathsf{I}_\exists$.*

Proof. We first show that all $\Diamond^+\forall$-formulas are $\mathsf{F}_\forall$. In Claim 1, we show that all $\Diamond^+\forall$-formulas are $\mathsf{F}_{\Rightarrow\forall}$ (in fact for the $\Rightarrow$ we can also allow $\Diamond\varphi$ in the grammar). Then, in Claim 2, we show that all $\Diamond^+\forall$-formulas are $\mathsf{F}_{\Leftarrow\forall}$.

Claim 1. *For all finite traces $\mathfrak{F}$, all assignments $\mathfrak{a}$, and all $\Diamond^+\forall$-formulas φ, if $\mathfrak{F} \models^\mathfrak{a} \varphi$ then, for all $\mathfrak{I} \in \mathit{Ext}(\mathfrak{F})$, $\mathfrak{I} \models^\mathfrak{a} \varphi$.*

Proof of Claim 1 The proof is by induction. In the base case we have all non-temporal $\Diamond^+\forall$-formulas (Theorem 11). Assume that the claim holds for φ and ψ, and there is a finite trace $\mathfrak{F}$ and an assignment $\mathfrak{a}$ such that $\mathfrak{F} \models^\mathfrak{a} \varphi'$. We now argue that, for all $\mathfrak{I} \in \mathit{Ext}(\mathfrak{F})$, $\mathfrak{I} \models^\mathfrak{a} \varphi'$, where φ' is as follows.

- For $\varphi' = \Diamond\varphi$: by assumption $\mathfrak{F} \models^\mathfrak{a} \Diamond\varphi$. This means that there is $n \in (0, l]$ such that $\mathfrak{F}, n \models^\mathfrak{a} \varphi$. In other words, $\mathfrak{F}^n \models^\mathfrak{a} \varphi$, where $\mathfrak{F}^n$ is the suffix of $\mathfrak{F}$ starting from time point n. By the inductive hypothesis, if $\mathfrak{F}^n \models^\mathfrak{a} \varphi$ then, for all $\mathfrak{I} \in \mathit{Ext}(\mathfrak{F}^n)$, $\mathfrak{I} \models^\mathfrak{a} \varphi$. This implies that for all $\mathfrak{I} \in \mathit{Ext}(\mathfrak{F})$, there is $n \in (0, \infty)$ such that $\mathfrak{I}, n \models^\mathfrak{a} \varphi$. So, $\mathfrak{I} \models^\mathfrak{a} \Diamond\varphi$ for all $\mathfrak{I} \in \mathit{Ext}(\mathfrak{F})$.
- For $\varphi' = \forall x\varphi$: by assumption $\mathfrak{F} \models^\mathfrak{a} \forall x\varphi$. This means that for all $d \in \Delta$, $\mathfrak{F} \models^{\mathfrak{a}_d} \varphi[d]$, where $\mathfrak{a}_d$ extends $\mathfrak{a}$ by mapping x to d. By applying the inductive hypothesis on every $\varphi[d]$ with $d \in \Delta$, we have that, for all $d \in \Delta$ and all $\mathfrak{I} \in \mathit{Ext}(\mathfrak{F})$, $\mathfrak{I} \models^{\mathfrak{a}_d} \varphi[d]$. By semantics of $\forall$, for all $\mathfrak{I} \in \mathit{Ext}(\mathfrak{F})$, $\mathfrak{I} \models^\mathfrak{a} \forall x\varphi$.
- For $\varphi' = \exists x\varphi$: by assumption $\mathfrak{F} \models^\mathfrak{a} \exists x\varphi$. This means that there is $d \in \Delta$ such that $\mathfrak{F} \models^{\mathfrak{a}_d} \varphi[d]$, where $\mathfrak{a}_d$ extends $\mathfrak{a}$ by mapping x to d. By applying the inductive hypothesis, we have that, for all $\mathfrak{I} \in \mathit{Ext}(\mathfrak{F})$, $\mathfrak{I} \models^{\mathfrak{a}_d} \varphi[d]$. By semantics of $\exists$, for all $\mathfrak{I} \in \mathit{Ext}(\mathfrak{F})$, $\mathfrak{I} \models^\mathfrak{a} \exists x\varphi$.
- The remaining cases can be proved in a straightforward way using the inductive hypothesis.

We observe that problematic formulas for this claim are, e.g., $\Diamond \mathit{last}$. This formula holds in any finite trace $\mathfrak{F}$ but it does not on any extension of $\mathfrak{F}$. Claim 1 relies on the fact that the grammar rule for $\Diamond$-formulas does not allow *last*. The boolean case $\varphi \vee \psi$ also relies on the fact that the claim is for $\Diamond$-formulas, since, e.g., the formula $\Diamond\Box^+P(x) \vee \Diamond\Box^+Q(x)$ holds on any finite trace (with more than one time point) where $P(x) \vee Q(x)$ is satisfied in the last time point but there are extensions which do not satisfy $\Diamond\Box^+P(x) \vee \Diamond\Box^+Q(x)$.

Claim 2. *For all finite traces $\mathfrak{F}$, all assignments $\mathfrak{a}$, and all $\Diamond^+\forall$-formulas φ, if, for all $\mathfrak{I} \in \mathit{Ext}(\mathfrak{F})$, $\mathfrak{I} \models^\mathfrak{a} \varphi$ then $\mathfrak{F} \models^\mathfrak{a} \varphi$.*

Proof of Claim 2 If, for all $\mathfrak{I} \in \mathit{Ext}(\mathfrak{F})$, $\mathfrak{I} \models^\mathfrak{a} \varphi$ then, in particular, $\mathfrak{F}^\omega \models^\mathfrak{a} \varphi$. We show that $\mathfrak{F}^\omega \models^\mathfrak{a} \varphi$ implies $\mathfrak{F} \models^\mathfrak{a} \varphi$. The proof is by induction. In the base case we have all non-temporal $\Diamond^+\forall$-formulas. Assume that the lemma holds for a $\Diamond^+\forall$-formula φ.

- If $\mathfrak{F}^\omega \models^\mathfrak{a} \Diamond^+\varphi$ then there is $n \geq 0$ such that $\mathfrak{F}^\omega, n \models^\mathfrak{a} \varphi$. By definition of $\mathfrak{F}^\omega$, there is a suffix $\mathfrak{F}'$ of $\mathfrak{F}$ such that $\mathfrak{F}'^\omega$ corresponds to the suffix of $\mathfrak{F}^\omega$ starting at n. By inductive hypothesis, $\mathfrak{F}' \models \varphi$. As $\mathfrak{F}'$ is a suffix of $\mathfrak{F}$, we have that $\mathfrak{F} \models^\mathfrak{a} \Diamond^+\varphi$.

- If $\mathfrak{F}^\omega \models^\mathfrak{a} \forall x\varphi$. This means that, for all $d \in \Delta$, $\mathfrak{F}^\omega \models^\mathfrak{a} \varphi[d]$. By the inductive hypothesis, for all $d \in \Delta$, $\mathfrak{F} \models^\mathfrak{a} \varphi[d]$. So, $\mathfrak{F} \models^\mathfrak{a} \forall x\varphi$.
- If $\mathfrak{F}^\omega \models^\mathfrak{a} \exists x\varphi$. This means that there is $d \in \Delta$ such that $\mathfrak{F}^\omega \models^\mathfrak{a} \varphi[d]$. By the inductive hypothesis, $\mathfrak{F} \models^\mathfrak{a} \varphi[d]$. This means that $\mathfrak{F} \models^\mathfrak{a} \exists x\varphi$.
- The remaining cases can be proved in a straightforward way using the inductive hypothesis.

We point out that this claim does not hold for $\Diamond$-formulas (only for $\Diamond^+$-formulas). To see this consider the formula $\Diamond\top$, which holds in any infinite trace but not on a finite trace with only one time point.

We now show that all $\Diamond$-formulas are $\mathsf{I}_\exists$. In Claim 3, we show that all $\Diamond$-formulas are $\mathsf{I}_{\Rightarrow\exists}$. Then, in Claim 4, we show that all $\Diamond$-formulas are $\mathsf{I}_{\Leftarrow\exists}$.

Claim 3. *For all infinite traces $\mathfrak{I}$, all assignments $\mathfrak{a}$, and all $\Diamond$-formulas φ, if $\mathfrak{I} \models^\mathfrak{a} \varphi$ then there is $\mathfrak{F} \in Pre(\mathfrak{I})$ such that $\mathfrak{F} \models^\mathfrak{a} \varphi$.*

Proof of Claim 3 The proof is by induction as in Claim 1. In the base case we have all non-temporal $\Diamond$-formulas (Theorem 11). Assume that the claim holds for φ and ψ.

- For formulas of the form $\Diamond\varphi$ the argument is as follows. Let $\mathfrak{I}$ be an infinite trace. By assumption $\mathfrak{I} \models^\mathfrak{a} \Diamond\varphi$. This means that there is $n \in (0,\infty)$ such that $\mathfrak{I}, n \models^\mathfrak{a} \varphi$. In other words, $\mathfrak{I}^n \models^\mathfrak{a} \varphi$, where $\mathfrak{I}^n$ is the suffix of $\mathfrak{I}$ starting from time point n. By the inductive hypothesis, if $\mathfrak{I}^n \models^\mathfrak{a} \varphi$ then there is $\mathfrak{F} \in Pre(\mathfrak{I}^n)$ such that $\mathfrak{F} \models^\mathfrak{a} \Diamond\varphi$. Then, there is $\mathfrak{F}' \in Pre(\mathfrak{I})$ such that $\mathfrak{F}' \models^\mathfrak{a} \Diamond\varphi$.
- For formulas of the form $\exists x\varphi$ the argument is as follows. Let $\mathfrak{I} = (\Delta, \mathcal{I}_{n\in[0,\infty)})$ be an infinite trace. By assumption $\mathfrak{I} \models^\mathfrak{a} \exists x\varphi$. This means that there is $d \in \Delta$ such that $\mathfrak{I} \models^\mathfrak{a} \varphi[d]$. By the inductive hypothesis, if $\mathfrak{I} \models^\mathfrak{a} \varphi[d]$ then there is $\mathfrak{F} \in Pre(\mathfrak{I})$ such that $\mathfrak{F} \models^\mathfrak{a} \varphi[d]$. So, $\mathfrak{F} \models^\mathfrak{a} \exists x\varphi$.
- For formulas of the form $\varphi \wedge \psi$ the argument is as follows. Let $\mathfrak{I}$ be an infinite trace. By assumption $\mathfrak{I} \models^\mathfrak{a} \varphi \wedge \psi$. This means that $\mathfrak{I} \models^\mathfrak{a} \varphi$ and $\mathfrak{I} \models^\mathfrak{a} \psi$. By the inductive hypothesis, there are $\mathfrak{F}, \mathfrak{F}' \in Pre(\mathfrak{I})$ such that $\mathfrak{F} \models^\mathfrak{a} \varphi$ and $\mathfrak{F}' \models^\mathfrak{a} \psi$. By definition of $\mathfrak{F}$ and $\mathfrak{F}'$, either $\mathfrak{F}'$ is a prefix of $\mathfrak{F}$ or vice versa. Assume w.l.o.g. that $\mathfrak{F}'$ is a prefix of $\mathfrak{F}$. By Lemma 23, if $\mathfrak{F}' \models^\mathfrak{a} \psi$ then $\mathfrak{F} \models^\mathfrak{a} \psi$. Then, $\mathfrak{F} \models^\mathfrak{a} \varphi$ and $\mathfrak{F} \models^\mathfrak{a} \psi$, and so, $\mathfrak{F} \models^\mathfrak{a} \varphi \wedge \psi$.
- The remaining cases are a straightforward application of the inductive hypothesis.

Claim 4. *For all infinite traces $\mathfrak{I}$, all assignments $\mathfrak{a}$, and all $\Diamond$-formulas φ, if there is $\mathfrak{F} \in Pre(\mathfrak{I})$ such that $\mathfrak{F} \models^\mathfrak{a} \varphi$ then $\mathfrak{I} \models^\mathfrak{a} \varphi$.*

Proof of Claim 4 The proof is by induction as in Claim 1. In the base case we have all non-temporal $\Diamond$-formulas (Theorem 11). Assume that the claim holds for φ and ψ.

- For formulas of the form $\Diamond\varphi$ the argument is as follows. Let $\mathfrak{I}$ be an infinite trace. By assumption there is $\mathfrak{F} \in Pre(\mathfrak{I})$ such that $\mathfrak{F} \models^\mathfrak{a} \Diamond\varphi$. This means that there is $n \in (0,l]$ such that $\mathfrak{F}, n \models^\mathfrak{a} \varphi$. In other words, $\mathfrak{F}^n \models^\mathfrak{a} \varphi$, where $\mathfrak{F}^n$ is the suffix of $\mathfrak{F}$ starting from time point n. By the inductive hypothesis, if $\mathfrak{F}^n \models^\mathfrak{a} \varphi$ then $\mathfrak{I}^n \models^\mathfrak{a} \varphi$, where $\mathfrak{I}^n$ is the suffix of $\mathfrak{I}$ starting from time point n. This implies that there is $n \in (0,\infty)$ such that $\mathfrak{I}, n \models^\mathfrak{a} \varphi$. So, $\mathfrak{I} \models^\mathfrak{a} \Diamond\varphi$.
- For formulas of the form $\exists x\varphi$ the argument is as follows. Let $\mathfrak{I} = (\Delta, \mathcal{I}_{n\in[0,\infty)})$ be an infinite trace. By assumption there is $\mathfrak{F} \in Pre(\mathfrak{I})$ such that $\mathfrak{F} \models^\mathfrak{a} \exists x\varphi$. This means that there is $d \in \Delta$ such that $\mathfrak{F} \models^\mathfrak{a} \varphi[d]$. By the inductive hypothesis, if $\mathfrak{F} \models^\mathfrak{a} \varphi[d]$ then $\mathfrak{I} \models^\mathfrak{a} \varphi[d]$. So, $\mathfrak{I} \models^\mathfrak{a} \exists x\varphi$.
- The remaining cases can be proved by a straightforward application of the inductive hypothesis.

$\square$

The next lemmas are useful to prove Theorem 13.

Lemma 24. *Let $\mathfrak{F}$ be a finite trace and let $\mathfrak{F}'$ be a prefix of $\mathfrak{F}$. For all assignments $\mathfrak{a}$, and all $\Box$-formulas φ, if $\mathfrak{F}' \not\models^\mathfrak{a} \varphi$ then $\mathfrak{F} \not\models^\mathfrak{a} \varphi$.*

Proof. The proof is by induction. In the base case we have all non-temporal $\Box$-formulas. Assume that the lemma holds for a $\Box$-formula φ. Let $\mathfrak{F}$ be a finite trace, let $\mathfrak{F}'$ be a prefix of $\mathfrak{F}$ and let $\mathfrak{a}$ be an assignment. If $\mathfrak{F}' \not\models^\mathfrak{a} \Box\varphi$ then there is $n > 0$ such that $\mathfrak{F}', n \not\models^\mathfrak{a} \varphi$. As $\mathfrak{F}'$ is a prefix of $\mathfrak{F}$, we have that $\mathfrak{F}, n \not\models^\mathfrak{a} \varphi$. This means that there is $n > 0$ such that $\mathfrak{F}, n \not\models^\mathfrak{a} \varphi$. Thus, $\mathfrak{F} \not\models^\mathfrak{a} \Box\varphi$. Other cases can be proved by straightforward applications of the inductive hypothesis. $\square$

Lemma 25. *For all finite traces $\mathfrak{F}$, all infinite traces $\mathfrak{I} \in Ext(\mathfrak{F})$, all assignments $\mathfrak{a}$, and all $\Box^+\exists$-formulas φ, if $\mathfrak{F} \models^\mathfrak{a} \Box^+\varphi$ then $\mathfrak{F}^\omega \models^\mathfrak{a} \Box^+\varphi$.*

Proof. The proof is by induction. In the base case we have all non-temporal $\Box^+\exists$-formulas. Assume that the lemma holds for φ and ψ. We now argue that if $\mathfrak{F} \models^\mathfrak{a} \Box^+\Box^+\varphi$ then $\mathfrak{F}^\omega \models^\mathfrak{a} \Box^+\Box^+\varphi$, by the inductive hypothesis and the semantics of $\Box^+$. The remaining cases can be proved in a straightforward way using the inductive hypothesis. $\square$

Recall that, given and assignment $\mathfrak{a}$ and an element d of a domain Δ (of a finite or infinite trace), we write $\mathfrak{a}_d$ for the result of modifying $\mathfrak{a}$ so that x maps to d.

Theorem 13. *All $\Box^+$-formulas are $\mathsf{F}_\exists$ and $\mathsf{I}_\forall$. Moreover, all $\Box^+\exists$-formulas are $\mathsf{F}_\exists$ and all $\Box$-formulas are $\mathsf{I}_\forall$.*

Proof. We first show that all $\Box^+\exists$-formulas are $\mathsf{F}_\exists$. In Claim 1 we show that all $\Box^+\exists$-formulas are $\mathsf{F}_{\Rightarrow\exists}$. Then, in Claim 2 we show that all $\Box^+\exists$-formulas are $\mathsf{F}_{\Leftarrow\exists}$.

Claim 1. *For all finite traces $\mathfrak{F}$, all assignments $\mathfrak{a}$, and all $\Box^+\exists$-formulas φ, if $\mathfrak{F} \models^\mathfrak{a} \varphi$ then there is $\mathfrak{I} \in Ext(\mathfrak{F})$ such that $\mathfrak{I} \models^\mathfrak{a} \varphi$.*

Proof of Claim 1 We show that $\mathfrak{F} \models^\mathfrak{a} \varphi$ implies $\mathfrak{F}^\omega \models^\mathfrak{a} \varphi$. The proof is by induction. In the base case we have all non-temporal $\Box^+\exists$-formulas (Theorem 11). Assume that the claim holds for φ.

- The case for $\Box^+\varphi$ is by Lemma 25.

- If $\mathfrak{F} \models^{\mathfrak{a}} \forall x\varphi$. This means that, for all $d \in \Delta$, $\mathfrak{F} \models^{\mathfrak{a}} \varphi[d]$. By the inductive hypothesis, for all $d \in \Delta$, $\mathfrak{F}^{\omega} \models^{\mathfrak{a}} \varphi[d]$. So, $\mathfrak{F}^{\omega} \models^{\mathfrak{a}} \forall x\varphi$.
- If $\mathfrak{F} \models^{\mathfrak{a}} \exists x\varphi$. This means that there is $d \in \Delta$ such that $\mathfrak{F} \models^{\mathfrak{a}} \varphi[d]$. By the inductive hypothesis, $\mathfrak{F}^{\omega} \models^{\mathfrak{a}} \varphi[d]$. This means that $\mathfrak{F}^{\omega} \models^{\mathfrak{a}} \exists x\varphi$.
- The remaining cases can be proved in a straightforward way using the inductive hypothesis.

We observe that the problematic operator for this claim is $\Box$, e.g., $\Box\bot$ holds in a finite trace $\mathfrak{F}$ with only one time point but it does not on any extension, in particular, not on $\mathfrak{F}^{\omega}$. This claim relies on the fact that the grammar rule for $\Box^{+}$-formulas does not allow the $\Box$ operator.

Claim 2. *For all finite traces $\mathfrak{F}$, all assignments $\mathfrak{a}$, and all $\Box^{+}\exists$-formulas φ, if there exists $\mathfrak{I} \in Ext(\mathfrak{F})$ such that $\mathfrak{I} \models^{\mathfrak{a}} \varphi$ then we have that $\mathfrak{F} \models^{\mathfrak{a}} \varphi$.*

Proof of Claim 2 The proof is by induction, as in Claim 1. In the base case we have all non-temporal $\Box^{+}\exists$-formulas (Theorem 11). Assume that the claim holds for φ and ψ.

- For formulas of the form $\Box^{+}\varphi$ the argument is as follows. Let $\mathfrak{F} = (\Delta, \mathcal{F}_{n\in[0,l]})$ be a finite trace and let $\mathfrak{I}$ be an element of $Ext(\mathfrak{F})$. Assume $\mathfrak{I} \models^{\mathfrak{a}} \Box^{+}\varphi$. If $\mathfrak{I} \models^{\mathfrak{a}} \Box^{+}\varphi$ then, for all $n \in [0,\infty)$, $\mathfrak{I}, n \models^{\mathfrak{a}} \varphi$. By applying the inductive hypothesis on every point in $[0,l]$, we have that, for all $n \in [0,l]$, $\mathfrak{F}, n \models^{\mathfrak{a}} \varphi$. So, $\mathfrak{F} \models^{\mathfrak{a}} \Box^{+}\varphi$.
- The case for formulas of the form $\forall x\varphi$ is as follows. Let $\mathfrak{F} = (\Delta, \mathcal{F}_{n\in[0,l]})$ be a finite trace. Assume that there is $\mathfrak{I} \in Ext(\mathfrak{F})$ such that $\mathfrak{I} \models^{\mathfrak{a}} \forall x\varphi$. By semantics of $\forall$, we have that, for all $d \in \Delta$, $\mathfrak{I} \models^{\mathfrak{a}_d} \varphi[d]$. By applying the inductive hypothesis on every $\varphi[d]$ with $d \in \Delta$ we have that $\mathfrak{F} \models^{\mathfrak{a}_d} \varphi[d]$, for all $d \in \Delta$. So, $\mathfrak{F} \models^{\mathfrak{a}} \forall x\varphi$.
- The case for formulas of the form $\exists x\varphi$ is as follows. Let $\mathfrak{F} = (\Delta, \mathcal{F}_{n\in[0,l]})$ be a finite trace. Assume that there is $\mathfrak{I} \in Ext(\mathfrak{F})$ such that $\mathfrak{I} \models^{\mathfrak{a}} \exists x\varphi$. By semantics of $\exists$, there is $d \in \Delta$ such that $\mathfrak{I} \models^{\mathfrak{a}_d} \varphi[d]$. By applying the inductive hypothesis, we have that $\mathfrak{F} \models^{\mathfrak{a}_d} \varphi[d]$, for some $d \in \Delta$. So, $\mathfrak{F} \models^{\mathfrak{a}} \exists x\varphi$.
- The remaining cases are a straightforward application of the inductive hypothesis.

We now show that all $\Box$-formulas are $\mathsf{I}_{\forall}$. In Claim 3, we show that all $\Box$-formulas are $\mathsf{I}_{\Rightarrow\forall}$. Then, in Claim 4, we show that all $\Box$-formulas are $\mathsf{I}_{\Leftarrow\forall}$.

Claim 3. *For all infinite traces $\mathfrak{I}$, all assignments $\mathfrak{a}$, and all $\Box$-formulas φ, if $\mathfrak{I} \models^{\mathfrak{a}} \varphi$ then, for all $\mathfrak{F} \in Pre(\mathfrak{I})$, we have that $\mathfrak{F} \models^{\mathfrak{a}} \varphi$.*

Proof of Claim 3 The proof is by induction as in Claim 1. In the base case we have all non-temporal $\Box$-formulas (Theorem 11). Assume that the claim holds for φ and ψ.

- For formulas of the form $\Box\varphi$ the argument is as follows. Let $\mathfrak{I}$ be an infinite trace such that $\mathfrak{I} \models^{\mathfrak{a}} \Box\varphi$ and let $\mathfrak{F} = (\Delta, \mathcal{F}_{n\in[0,l]})$ be an arbitrary element of $Pre(\mathfrak{I})$. If $\mathfrak{I} \models^{\mathfrak{a}} \Box\varphi$ then, for all $n \in (0,\infty)$, $\mathfrak{I}, n \models^{\mathfrak{a}} \varphi$. By applying the inductive hypothesis on every point in $(0,l]$, we have that, for all $n \in (0,l]$, $\mathfrak{F}, n \models^{\mathfrak{a}} \varphi$. This means that $\mathfrak{F} \models^{\mathfrak{a}} \Box\varphi$. Since $\mathfrak{F}$ was an arbitrary element of $Pre(\mathfrak{I})$, the argument holds for all $\mathfrak{F} \in Pre(\mathfrak{I})$.
- The case for formulas of the form $\forall x\varphi$ is as follows. Let $\mathfrak{I} = (\Delta, \mathcal{I}_{n\in[0,\infty)})$ be an infinite trace and let $\mathfrak{F}$ be an arbitrary element of $Pre(\mathfrak{I})$. Assume $\mathfrak{I} \models^{\mathfrak{a}} \forall x\varphi$. By semantics of $\forall$, we have that, for all $d \in \Delta$, $\mathfrak{I} \models^{\mathfrak{a}_d} \varphi[d]$. By applying the inductive hypothesis on every $\varphi[d]$ with $d \in \Delta$ we have that $\mathfrak{F} \models^{\mathfrak{a}_d} \varphi[d]$, for all $d \in \Delta$. By semantics of $\forall$, $\mathfrak{F} \models^{\mathfrak{a}} \forall x\varphi$. As $\mathfrak{F}$ was an arbitrary element of $Pre(\mathfrak{I})$, the argument holds for all $\mathfrak{F} \in Pre(\mathfrak{I})$.
- The remaining cases can be proved by a straightforward application of the inductive hypothesis.

Claim 4. *For all infinite traces $\mathfrak{I}$, all assignments $\mathfrak{a}$, and all $\Box$-formulas φ, if for all $\mathfrak{F} \in Pre(\mathfrak{I})$, $\mathfrak{F} \models^{\mathfrak{a}} \varphi$ then $\mathfrak{I} \models^{\mathfrak{a}} \varphi$.*

Proof of Claim 4 The proof is by induction as in Claim 1. In the base case we have all non-temporal $\Box$-formulas (Theorem 11). Assume that the claim holds for φ and ψ.

- The case for formulas of the form $\Box\varphi$ is by the semantics of $\Box$ (since $\mathfrak{I} \not\models^{\mathfrak{a}} \Box\varphi$ implies that there is $\mathfrak{F} \in Pre(\mathfrak{I})$ such that $\mathfrak{F} \not\models^{\mathfrak{a}} \Box\varphi$ for finite traces with more than one time point, which contradicts the assumption).
- The case for formulas of the form $\forall x\varphi$ is as follows. Let $\mathfrak{I} = (\Delta, \mathcal{I}_{n\in[0,\infty)})$ be an infinite trace. Assume $\mathfrak{F} \models^{\mathfrak{a}} \forall x\varphi$, for all $\mathfrak{F} \in Pre(\mathfrak{I})$. By semantics of $\forall$, for all $\mathfrak{F} \in Pre(\mathfrak{I})$ and all $d \in \Delta$, $\mathfrak{F} \models^{\mathfrak{a}_d} \varphi[d]$. By applying the inductive hypothesis on every $\varphi[d]$ with $d \in \Delta$, we obtain $\mathfrak{I} \models^{\mathfrak{a}_d} \varphi[d]$, for all $d \in \Delta$. So, $\mathfrak{I} \models^{\mathfrak{a}} \forall x\varphi$.
- The case for formulas of the form $\varphi \vee \psi$ is as follows. Assume that, for all $\mathfrak{F} \in Pre(\mathfrak{I})$, $\mathfrak{F} \models^{\mathfrak{a}} \varphi \vee \psi$ and suppose to the contrary that $\mathfrak{I} \not\models^{\mathfrak{a}} \varphi \vee \psi$. Then, there are $\mathfrak{F}, \mathfrak{F}' \in Pre(\mathfrak{I})$ such that $\mathfrak{F} \not\models^{\mathfrak{a}} \varphi$ and $\mathfrak{F}' \not\models^{\mathfrak{a}} \psi$ (otherwise we contradict the inductive hypothesis). By definition of $\mathfrak{F}$ and $\mathfrak{F}'$, either $\mathfrak{F}'$ is a prefix of $\mathfrak{F}$ or vice versa. Assume w.l.o.g. that $\mathfrak{F}'$ is a prefix of $\mathfrak{F}$. By Lemma 24, $\mathfrak{F} \not\models^{\mathfrak{a}} \psi$. Then, $\mathfrak{F} \not\models^{\mathfrak{a}} \varphi \vee \psi$, which contradicts the assumption that, for all $\mathfrak{F}'' \in Pre(\mathfrak{I})$, $\mathfrak{F}'' \models^{\mathfrak{a}} \varphi \vee \psi$.
- The remaining cases can be proved by a straightforward application of the inductive hypothesis.

□

C Proofs for Section 5

C.1 Planning

For the proofs of the next theorems, we will use the following definitions. Let Σ be a finite subset of $\mathsf{N_P}$. Assume w.l.o.g. that the $T_{\mathcal{U}}^{f}\mathcal{QL}$ formulas we mention in this subsection have predicates in Σ, and that Σ contains the end of time predicate E. Given an infinite trace $\mathfrak{I}$, the Σ*-reduct of* $\mathfrak{I}$ is the infinite trace $\mathfrak{I}_{|\Sigma}$ coinciding with $\mathfrak{I}$ on Σ and such that $X^{\mathcal{I}_{n|\Sigma}} = \emptyset$, for $X \notin \Sigma$ and $n \in [0,\infty)$.

Lemma 26. *For every infinite trace $\mathfrak{I}$, $\mathfrak{I} \models \chi_f$ iff $\mathfrak{I}_{|\Sigma} = \mathfrak{F} \cdot_E \mathfrak{E}$, for some finite trace $\mathfrak{F}$.*

Proof. ($\Leftarrow$) If $\mathfrak{I}_{|\Sigma} = \mathfrak{F} \cdot_E \mathfrak{E}$, by Lemma 1, $\mathfrak{F} \cdot_E \mathfrak{E} \models \psi_f$. Moreover, where l is the last time point of $\mathfrak{F}$, we have by definition: for all $n \in [0, l]$, $E^{\mathcal{F} \cdot_E \mathcal{E}_n} = \emptyset$; for all $n \in [l+1, \infty)$, $E^{\mathcal{F} \cdot_E \mathcal{E}_n} = \Delta$, and for every $P \in \mathsf{N_P} \setminus \{E\}$, $P^{\mathcal{F} \cdot_E \mathcal{E}_n} = \emptyset$. Thus, for all $n \in (0, \infty)$, for all objects d and all tuples of objects $\bar{d}$ in Δ, we have: if $d \in E^{\mathcal{F} \cdot_E \mathcal{E}_n}$, then $(d, \bar{d}) \notin P$, for all $P \in \Sigma \setminus \{E\}$. Therefore, $\mathfrak{F} \cdot_E \mathfrak{E} \models \chi_{f_1}$, and hence $\mathfrak{F} \cdot_E \mathfrak{E} \models \chi_f$.

($\Rightarrow$) Suppose $\mathfrak{I} \models \chi_f$. Since in particular it satisfies ψ_f, by Lemma 1, we have that $\mathfrak{I} = \mathfrak{F} \cdot_E \mathfrak{I}'$, for some finite trace $\mathfrak{F} = (\Delta, (\mathcal{F}_n)_{n \in [0,l]})$ and some infinite trace $\mathfrak{I}'$. Thus:

$$E^{\mathcal{F} \cdot \mathcal{I}'_n} = \begin{cases} \emptyset, & \text{for all } n \in [0, l] \\ \Delta, & \text{for all } n \in [l+1, \infty) \end{cases}$$

Since $\mathfrak{I} \models \chi_{f_1}$, for all $n \in [l+1, \infty)$, for all objects d and all tuples of objects $\bar{d}$ in Δ, we have that: if $d \in E^{\mathcal{F} \cdot_E \mathcal{E}_n} = \Delta$, then $(d, \bar{d}) \notin P$, for all $P \in \Sigma \setminus \{E\}$. This is equivalent to $P^{\mathcal{I}_n} = \emptyset$, for all $P \in \Sigma \setminus \{E\}$. Therefore, we have that $\mathfrak{I}_{|\Sigma} = \mathfrak{F} \cdot_E \mathfrak{E}$. ☐

Lemma 27. *Let φ be a $T_{\mathcal{U}}\mathcal{QL}$ formula, $\mathfrak{F}$ a finite trace, and $\mathfrak{a}$ an assignment. We have that: $\mathfrak{F} \models^{\mathfrak{a}} \varphi$ iff $\mathfrak{F} \cdot_E \mathfrak{E} \models^{\mathfrak{a}} \varphi^{\dagger}$.*

Proof. By definition of $\mathfrak{F} \cdot_E \mathfrak{E}$ and as a consequence of Lemma 2. ☐

Theorem 14. *A $T_{\mathcal{U}}\mathcal{QL}$ formula φ is insensitive to infiniteness iff the $T^i_{\mathcal{U}}\mathcal{QL}$ logical implication $\chi_f \models \varphi \leftrightarrow \varphi^{\dagger}$ holds.*

Proof. ($\Rightarrow$) Assume that φ is insensitive. We want to prove that, for every infinite trace $\mathfrak{I}$ and all assignments $\mathfrak{a}$, if $\mathfrak{I} \models^{\mathfrak{a}} \chi_f$, then $\mathfrak{I} \models^{\mathfrak{a}} \varphi \leftrightarrow \varphi^{\dagger}$. Suppose $\mathfrak{I} \models^{\mathfrak{a}} \chi_f$. By Lemma 26, $\mathfrak{I}_{|\Sigma} = \mathfrak{F} \cdot_E \mathfrak{E}$, for a finite trace $\mathfrak{F}$. Moreover, thanks to Lemma 27, $\mathfrak{F} \models^{\mathfrak{a}} \varphi$ if and only if $\mathfrak{F} \cdot_E \mathfrak{E} \models^{\mathfrak{a}} \varphi^{\dagger}$. Since φ is by hypothesis insensitive, for every finite trace $\mathfrak{F}$ and all assignments $\mathfrak{a}$, $\mathfrak{F} \models^{\mathfrak{a}} \varphi$ if and only if $\mathfrak{F} \cdot_E \mathfrak{E} \models^{\mathfrak{a}} \varphi$. Thus, $\mathfrak{F} \cdot_E \mathfrak{E} \models^{\mathfrak{a}} \varphi$ if and only if $\mathfrak{F} \cdot_E \mathfrak{E} \models^{\mathfrak{a}} \varphi^{\dagger}$. That is, $\mathfrak{F} \cdot_E \mathfrak{E} \models^{\mathfrak{a}} \varphi \leftrightarrow \varphi^{\dagger}$, and therefore $\mathfrak{I} \models \varphi \leftrightarrow \varphi^{\dagger}$ (since all the predicates occurring in $\varphi, \varphi^{\dagger}$ are in Σ).

($\Leftarrow$) Assume that $\chi_f \models \varphi \leftrightarrow \varphi^{\dagger}$. By Lemma 26, for every infinite trace $\mathfrak{I}$ and every assignment $\mathfrak{a}$, $\mathfrak{I} \models^{\mathfrak{a}} \chi_f$ means that $\mathfrak{I}_{|\Sigma} = \mathfrak{F} \cdot_E \mathfrak{E}$, for a finite trace $\mathfrak{F}$. Given our assumption, this implies $\mathfrak{F} \cdot_E \mathfrak{E} \models^{\mathfrak{a}} \varphi \leftrightarrow \varphi^{\dagger}$, that is $\mathfrak{F} \cdot_E \mathfrak{E} \models^{\mathfrak{a}} \varphi$ if and only if $\mathfrak{F} \cdot_E \mathfrak{E} \models^{\mathfrak{a}} \varphi^{\dagger}$, for all assignments $\mathfrak{a}$. By Lemma 27, $\mathfrak{F} \cdot_E \mathfrak{E} \models^{\mathfrak{a}} \varphi^{\dagger}$ if and only if $\mathfrak{F} \models^{\mathfrak{a}} \varphi$. In conclusion, we obtain that, for all assignments $\mathfrak{a}$, $\mathfrak{F} \cdot_E \mathfrak{E} \models^{\mathfrak{a}} \varphi$ if and only if $\mathfrak{F} \models^{\mathfrak{a}} \varphi$, meaning that φ is insensitive. ☐

Theorem 15. *Let φ, ψ be insensitive $T_{\mathcal{U}}\mathcal{QL}$ formulas. Then $\neg\varphi$, $\exists x\varphi$, and $\varphi \wedge \psi$ are insensitive.*

Proof. Let $\mathfrak{F}$ be a finite trace and $\mathfrak{a}$ be an assignment. For $\neg\varphi$, we have that $\mathfrak{F} \models^{\mathfrak{a}} \neg\varphi$ iff $\mathfrak{F} \not\models^{\mathfrak{a}} \varphi$. Since φ is insensitive by hypothesis, this means that $\mathfrak{F} \cdot_E \mathfrak{E} \not\models^{\mathfrak{a}} \varphi$. Therefore, $\neg\varphi$ is insensitive as well. For $\exists x\varphi$, we have that $\mathfrak{F} \models^{\mathfrak{a}} \exists x\varphi$ iff $\mathfrak{F} \models^{\mathfrak{a}'} \varphi[d]$, for some $d \in \Delta$. Given that φ is insensitive, this is equivalent to $\mathfrak{F} \cdot_E \mathfrak{E} \models^{\mathfrak{a}'} \varphi[d]$, for some $d \in \Delta$. That is, $\mathfrak{F} \cdot_E \mathfrak{E} \models^{\mathfrak{a}} \exists x\varphi$, and so $\exists x\varphi$ is insensitive. For $\varphi \wedge \psi$, we have that $\mathfrak{F} \models^{\mathfrak{a}} \varphi \wedge \psi$ is equivalent to $\mathfrak{F} \models^{\mathfrak{a}} \varphi$ and $\mathfrak{F} \models^{\mathfrak{a}} \psi$. Since both φ and ψ are assumed to be insensitive, the previous step is equivalent to: $\mathfrak{F} \cdot_E \mathfrak{E} \models^{\mathfrak{a}} \varphi$ and $\mathfrak{F} \cdot_E \mathfrak{E} \models^{\mathfrak{a}} \psi$, i.e., $\mathfrak{F} \cdot_E \mathfrak{E} \models^{\mathfrak{a}} \varphi \wedge \psi$. ☐

Theorem 16. *Let φ and ψ be insensitive $T_{\mathcal{U}}\mathcal{QL}$ formulas. Then $\varphi \equiv_{\mathfrak{I}} \psi$ implies $\varphi \equiv_{\mathfrak{F}} \psi$.*

Proof. The result can be seen as a consequence of Theorem 14. We provide also a direct proof. Given a finite trace $\mathfrak{F}$ and an assignment $\mathfrak{a}$, if $\mathfrak{F} \models^{\mathfrak{a}} \varphi$ then, as φ is insensitive, $\mathfrak{F} \cdot_E \mathfrak{E} \models^{\mathfrak{a}} \varphi$. By assumption, $\varphi \equiv_{\mathfrak{I}} \psi$, so $\mathfrak{F} \cdot_E \mathfrak{E} \models^{\mathfrak{a}} \psi$. As ψ is insensitive, $\mathfrak{F} \cdot_E \mathfrak{E} \models^{\mathfrak{a}} \psi$ implies $\mathfrak{F} \models^{\mathfrak{a}} \psi$. The converse direction is obtained by swapping φ and ψ. ☐

C.2 Temporal Description Logics

To prove Theorem 17 we use *quasimodels* [Gabbay *et al.*, 2003], which have been used to prove the satisfiability of various temporal DLs. Our definitions here are similar to those in Section A, now adapted to temporal $\mathcal{ALC}$. Our upper bound is obtained by a type elimination procedure.

Theorem 17. *Satisfiability in $T^f_{\mathcal{U}}\mathcal{ALC}$ is* ExpSpace*-complete and in $T^k_{\mathcal{U}}\mathcal{ALC}$ is* NExpTime*-complete. Moreover, satisfiability in $T^k_{\mathcal{U}}\mathcal{ALC}$ restricted to global CIs is* ExpTime*-complete.*

Proof. We show that satisfiability in $T^k_{\mathcal{U}}\mathcal{ALC}$ restricted to global CIs is in ExpTime. Let φ be a $T^k_{\mathcal{U}}\mathcal{ALC}$ formula restricted to global CIs. Assume w.l.o.g. that φ does not contain abbreviations (i.e., it only contains the logical connectives $\neg$, $\sqcap$, $\wedge$, the existential quantifier $\exists$, and the temporal operator $\mathcal{U}$). Let $\mathsf{N_I}(\varphi)$ be the set of individuals occurring in φ. Following the notation provided by Baader *et al.* 2017, denote by $\mathsf{cl^f}(\varphi)$ the closure under single negation of the set of all formulas occurring in φ. Similarly, we denote by $\mathsf{cl^c}(\varphi)$ the closure under single negation of the set of all concepts union the concepts $A_a, \exists R.A_a$, for any $a \in \mathsf{N_I}(\varphi)$ and R a role occurring in φ, where A_a is fresh. A *concept type for* φ is any subset t of $\mathsf{cl^c}(\varphi) \cup \mathsf{N_I}(\varphi)$ such that:

T1 $\neg C \in t$ iff $C \notin t$, for all $\neg C \in \mathsf{cl^c}(\varphi)$;

T2 $C \sqcap D \in t$ iff $C, D \in t$, for all $C \sqcap D \in \mathsf{cl^c}(\varphi)$; and

T3 t contains at most one individual name in $\mathsf{N_I}(\varphi)$.

Similarly, we define *formula types* $t \subseteq \mathsf{cl^f}(\varphi)$ for φ with the conditions:

T1′ $\neg\phi \in t$ iff $\phi \notin t$, for all $\neg\phi \in \mathsf{cl^f}(\varphi)$; and

T2′ $\phi \wedge \psi \in t$ iff $\phi, \psi \in t$, for all $\phi \wedge \psi \in \mathsf{cl^f}(\varphi)$.

We omit 'for φ' when there is no risk of confusion. A concept type describes one domain element at a single time point, while a formula type expresses constraints on all domain elements. If $a \in t \cap \mathsf{N_I}(\varphi)$, then t describes a named element. We write t_a to indicate this and call it a *named type*.

The next notion captures how sets of types need to be constrained so that the DL dimension is respected. We say that a pair of concept types (t, t') is *R-compatible* if $\{\neg F \mid \neg\exists R.F \in t\} \subseteq t'$. A *quasistate* for φ is a set S of concept or formula types for φ such that:

Q1 S contains exactly one formula type t_S;

Q2 S contains exactly one named type t_a for each $a \in \mathsf{N_I}(\varphi)$;

Q3 for all $C \sqsubseteq D \in \mathsf{cl}^{\mathsf{f}}(\varphi)$, we have $C \sqsubseteq D \in t_S$ iff $C \in t$ implies $D \in t$ for all concept types $t \in S$;

Q4 for all $A(a) \in \mathsf{cl}^{\mathsf{f}}(\varphi)$, we have $A(a) \in t_S$ iff $A \in t_a$

Q5 $t \in S$ and $\exists R.D \in t$ implies there is $t' \in S$ such that $D \in t'$ and (t, t') is R-compatible;

Q6 for all $R(a, b) \in \mathsf{cl}^{\mathsf{f}}(\varphi)$, we have $R(a, b) \in t_S$ iff (t_a, t_b) is R-compatible.

A *(concept/formula) run segment* for φ is a finite sequence $\sigma = \sigma(0) \dots \sigma(n)$ composed exclusively of concept or formula types, respectively, such that:

R1 for all $a \in \mathsf{N_I}(\varphi)$ and all $i \in (0, n]$, we have $a \in \sigma(0)$ iff $a \in \sigma(i)$;

R2 for all $\alpha \mathcal{U} \beta \in \mathsf{cl}^*(\varphi)$ and all $i \in [0, n]$, we have $\alpha \mathcal{U} \beta \in \sigma(i)$ iff there is $j \in (i, n]$ such that $\beta \in \sigma(j)$ and $\alpha \in \sigma(m)$ for all $m \in (i, j)$,

where cl^* is either cl^{c} or cl^{f} (as appropriate), and **R1** does not apply to formula run segments. Intuitively, a concept run segment describes the temporal dimension of a single domain element, whereas a formula run segment describes constraints on the whole DL interpretation.

Finally, a *quasimodel* for φ is a pair $(S, \mathfrak{R})$, with S a finite sequence of quasistates $S(0)S(1) \dots S(n)$ and $\mathfrak{R}$ a non-empty set of run segments such that:

M1 $\varphi \in t_S$ where t_S is the formula type in $S(0)$;

M2 for every $\sigma \in \mathfrak{R}$ and every $i \in [0, n]$, $\sigma(i) \in S(i)$; and, conversely, for every $t \in S(i)$, there is $\sigma \in \mathfrak{R}$ with $\sigma(i) = t$.

By **M2** and the definition of a quasistate for φ, $\mathfrak{R}$ always contains exactly one formula run segment and one named run segment for each $a \in \mathsf{N_I}(\varphi)$.

Every quasimodel for φ describes an interpretation satisfying φ and, conversely, every such interpretation can be abstracted into a quasimodel for φ. We formalise this notion for finite traces with the following claim.

Claim 1. *There is a finite trace satisfying φ with at most k time points iff there is a quasimodel for φ with a sequence of quasistates of length at most k.*

Assume w.l.o.g. that the $T^k_{\mathcal{U}}\mathcal{ALC}$ formula φ restricted to global CIs, which is equivalent to a formula of the form $\mathcal{T} \wedge \Box(\mathcal{T}) \wedge \phi$ where ϕ does not contain inclusions and $\mathcal{T}$ is a conjunction of inclusions, has $\mathcal{T}$ equivalent to $\top \sqsubseteq C_{\mathcal{T}}$ (and $C_{\mathcal{T}}$ is of polynomial size w.r.t. the size of $\mathcal{T}$). We compute in exponential time w.r.t. the size of φ the set $\mathsf{tp}(\varphi)$ of all formula types for ϕ and the set of all concept types for φ satisfying the following condition (in addition to **T1**-**T3**):

T4 $C_{\mathcal{T}} \in t$.

There is a quasimodel for ϕ with all concept types satisfying this last condition iff there is a quasimodel for φ. We formalise this statement with the following straightforward claim.

Claim 2. *There is a quasimodel for ϕ with all concept types satisfying **T4** iff there is a quasimodel for φ.*

We say that a pair (t, t') of (concept/formula) types is *$\mathcal{U}$-compatible* if:

- $\alpha \mathcal{U} \beta \in t$ iff either $\beta \in t'$ or $\{\alpha, \alpha \mathcal{U} \beta\} \subseteq t'$, for all $\alpha \mathcal{U} \beta \in \mathsf{cl}^*(\varphi)$,

where cl^* is either cl^{c} or cl^{f} (as appropriate).

Our type elimination algorithm iterates over the values in $[1, k-1]$ to determine in exponential time in $|k|$, with k given in binary, the length of the sequence of quasistates of a quasimodel for φ, if one exists. For each $l \in [1, k-1]$, the l-th iteration starts with sets:

$$S_0, \dots, S_{l-1}, S_l$$

and each S_i is initially set to $\mathsf{tp}(\varphi)$. We start by exhaustively eliminating concept types t from some S_i, with $i \in [0, l]$, if t violates one of the following conditions:

E1 for all $\exists R.D \in t$, there is $t' \in S_i$ such that $D \in t'$ and (t, t') is R-compatible;

E2 if $i > 0$, there is $t' \in S_{i-1}$ such that (t', t) is $\mathcal{U}$-compatible;

E3 if $i < l$, there is $t' \in S_{i+1}$ such that (t, t') is $\mathcal{U}$-compatible;

E4 if $i = l$ then there is no $C \mathcal{U} D \in t$.

For each $a \in \mathsf{N_I}(\varphi)$, if t is a named type t_a then, in **E2** and **E3**, we further require that the mentioned types in a $\mathcal{U}$-compatible pair contain a. This phase of the algorithm stops when no further concept types can be eliminated. Next, for each formula type t, we say that a function f_t, mapping each $a \in \mathsf{N_I}(\varphi)$ to a named type containing a, is *consistent with* t if: (i) for all $A(a) \in \mathsf{cl}^{\mathsf{f}}(\varphi)$, $A(a) \in t$ iff $A \in f_t(a)$; and (ii) for all $R(a, b) \in \mathsf{cl}^{\mathsf{f}}(\varphi)$, $R(a, b) \in t$ iff $(f_t(a), f_t(b))$ is R-compatible. We are going to use these functions to construct our quasimodel as follows. We first add to each S_i all f_t consistent with each $t \in S_i$ such that the image of f_t is contained in S_i. We then exhaustively eliminate such functions f_t from some S_i, with $i \in [0, l]$, if f_t violates one of the following conditions:

E1′ if $i < l$, there is $f_{t'} \in S_{i+1}$ such that (t, t') is $\mathcal{U}$-compatible and, for all $a \in \mathsf{N_I}(\varphi)$, $(f_t(a), f_{t'}(a))$ is $\mathcal{U}$-compatible;

E2′ if $i = l$ then there is no $\psi \mathcal{U} \psi' \in t$.

It remains to ensure that each S_i contains exactly one formula type t_i and one named type t_a for each $a \in \mathsf{N_I}(\phi)$ (and no functions f_t). For this choose any formula type function f_{t_0} in S_0 such that $\phi \in t_0$ (if one exists) and remove formula types $t'_0 \neq t_0$ from S_0. Then, for each $i \in [1, l]$, select a formula type function $f_{t_i} \in S_i$ such that (t_{i-1}, t_i) is $\mathcal{U}$-compatible and for all $a \in \mathsf{N_I}(\varphi)$, $(f_{t_{i-1}}(a), f_{t_i}(a))$ is $\mathcal{U}$-compatible, removing formula types $t'_i \neq t_i$ from S_i, where f_{t_i} is the selected function. The existence of such f_{t_i} is ensured by **E1′**. For each selected function f_{t_i} and each $a \in \mathsf{N_I}(\varphi)$, with $i \in [1, l]$, we remove from S_i all named types t_a containing a such that $t_a \neq f_{t_i}(a)$. We now have that each S_i contains exactly one formula type t_i and one named type t_a for each $a \in \mathsf{N_I}(\phi)$. Finally, we proceed removing all functions f_t. We have thus constructed a sequence of quasistates. Until concepts/formulas $\alpha \mathcal{U} \beta$ are satisfied thanks to the $\mathcal{U}$-compatibility conditions and the fact that there are no expressions of the form $\alpha \mathcal{U} \beta$ in concept/formula types in the last quasistate.

This last step does not affect Conditions **E1**-**E4** (in particular **E1**) for the remaining concept types since for each named type there is an unnamed (concept) type which is the result of removing the individual name from it, and if the named type was not removed during type elimination then the corresponding unnamed type was also not removed. If the algorithm succeeds on these steps with a surviving concept type $t \in S_0$ and a formula type t_{S_0} in S_0 such that $\phi \in t_{S_0}$ then it returns 'satisfiable'. Otherwise, it increments l or returns 'unsatisfiable' if $l = k - 1$ (i.e., there are no further iterations).

Claim 3. *The type elimination algorithm returns 'satisfiable' iff there is a quasimodel for* φ.

Proof of Claim 3. For ($\Rightarrow$), let $S^* = S_0^*, \ldots, S_l^*$ be the result of the type elimination procedure. Define $(S^*, \mathfrak{R})$ with $\mathfrak{R}$ as the set of sequences σ of (concept/formula) types such that, for all $i \in [0, l]$:

1. $\sigma(i) \in S_i^*$, and for every $t \in S_i^*$, there is $\sigma \in \mathfrak{R}$ with $\sigma(i) = t$;
2. for all $a \in \mathsf{N_I}(\varphi)$, we have $a \in \sigma(0)$ iff $a \in \sigma(i)$;
3. for all $\alpha\,\mathcal{U}\,\beta \in \mathsf{cl}^*(\varphi)$, we have $\alpha\,\mathcal{U}\,\beta \in \sigma(i)$ iff there is $j \in (i, l]$ such that $\beta \in \sigma(j)$ and $\alpha \in \sigma(n)$ for all $n \in (i, j)$.

where cl^* is either $\mathsf{cl^c}$ or $\mathsf{cl^f}$ (as appropriate). We now argue that $(S^*, \mathfrak{R})$ is a quasimodel for ϕ. We first argue that S^* is a sequence of quasistates for ϕ. **E1** ensures Condition **Q5**, while Condition **Q3** is satisfied since ϕ does not have concept inclusions. For Conditions **Q4** and **Q6**, we have the fact that named types are taken from functions consistent with the formula types. The last step of our algorithm consists in eliminating formula and named types so that we satisfy Conditions **Q1** and **Q2**. Thus, S^* is a sequence of quasistates for ϕ. Concerning the construction of $\mathfrak{R}$, Point 2 can be enforced thanks to our selection procedure for named types, while Point 3 is a consequence of

- Conditions **E2**, **E3** and **E4**, for concept types; and
- Conditions **E1**$'$ and **E2**$'$, for formula types, together with the selection procedure.

Points 2 and 3 coincide with Conditions **R1** and **R2**, so $\mathfrak{R}$ is a set of run segments for ϕ. Finally, when the algorithm returns 'satisfiable' then Condition **M1** holds, while Point 1 ensures that Condition **M2** holds. Thus, $(S^*, \mathfrak{R})$ is a quasimodel for ϕ. Since it satisfies Condition **T4**, by Claim 2, there is a quasimodel for φ.

For the other direction ($\Leftarrow$), assume there is a quasimodel for φ. By Claim 2, this implies the existence of a quasimodel $(S', \mathfrak{R})$ for ϕ satisfying Condition **T4**. Assume S' is of the form $S_0' \ldots S_{l-1}' S_l'$, for some $l \in [1, k-1]$. Let $S_0^*, \ldots, S_l^*$ be the result of the type elimination at the l-th iteration. Since $(S', \mathfrak{R})$ is a quasimodel, each concept type satisfies **E1**. Moreover, Conditions **E2**-**E4** are consequences of the existence of run segments through each type (by **M2**). Then, for all unnamed (concept) types t, if $t \in S_i'$ then $t \in S_i^*$, $i \in [0, l]$. If t is a formula type or a named type then $t \in S_i'$ does not necessarily imply that $t \in S_i^*$, $i \in [0, l]$. However, the existence of such types implies that the algorithm should find a sequence of functions f_t satisfying **E1**$'$ and **E2**$'$ which is then used to select formula and named types satisfying the quasimodel conditions. In particular, the selection procedure will select a function f_{t_0} associated with a formula type $t_0 \in S_0^*$ containing ϕ. So there is a surviving formula type in S_0^* containing ϕ. If the formula contains individuals, then the named types in the image of the selected function are in S_0^* and the algorithm returns 'satisfiable'. Otherwise, since $(S', \mathfrak{R})$ is a quasimodel, there is an unnamed concept type $t \in S_0'$ which is also in S_0^* by definition of our type elimination procedure for unnamed concept types.

This finishes the proof of Claim 3.

We now argue that our type elimination algorithm runs in exponential time. Since there are polynomially many individuals (w.r.t. the size of ϕ) occurring in ϕ, the number of functions f_t consistent with a formula type is exponential. As the number of (concept/formula) types is exponential the total number of functions and types to consider is exponential. In every step some concept type or function is eliminated (by **E1**-**E4** or by **E1**$'$-**E2**$'$, respectively). Conditions **E1**-**E4** and **E1**$'$-**E2**$'$ can clearly be checked in exponential time. Also, the selection procedure of functions for each S_i, which determine the formula and named types in the result of the algorithm, can also be checked in exponential time since we can pick any function in S_{i+1} satisfying the $\mathcal{U}$-compatibility relation, which is a local condition. As this can also be implemented in exponential time, we finish our proof. $\square$

We leave satisfiability in $T_{\mathcal{U}}^{f}\mathcal{ALC}$ *restricted to global CIs*, analogously defined as the fragment of $T_{\mathcal{U}}^{f}\mathcal{ALC}$ with only formulas of the form $\mathcal{T} \wedge \Box(\mathcal{T}) \wedge \phi$, as an open problem. It is known that the complexity of the satisfiability problem in this fragment over infinite traces is ExpTime-complete [Lutz *et al.*, 2008; Baader *et al.*, 2017]. Though, the end of time formula ψ_f is not expressible in this fragment. We cannot use the same strategy of defining a translation for the semantics based on infinite traces, as we did in Section 'Satisfiability over Finite Traces' The upper bound in [Lutz *et al.*, 2008] is based on type elimination. The main difficulty of devising a type elimination procedure is that the number of time points is not fixed and the argument in [Lutz *et al.*, 2008] showing that there is a quasimodel iff there is a quasimodel $(S, \mathfrak{R})$ such that $S(i+1) \subseteq S(i)$, for all $i \geq 0$, is not applicable to finite traces. A type with a concept equivalent to $\neg\bigcirc\top$ can only be in the last quasistate of the quasimodel. So it is not clear whether one can show that if there is a quasimodel then there is a quasimodel with an exponential sequence of quasistates, as in Theorem 17.