Temporalizing description logics

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1 Introduction

Traditional first order predicate logic is known to be designed for representing and manipulating *static* knowledge (e.g. mathematical theories). So are many of its applications. Knowledge representation systems based on concept description logics are not exceptions.

In the framework of a description logic, one can represent an application domain in terms of concepts, roles, and object names. Concepts are understood as classes of objects, roles as binary relations between objects, and object names denote certain objects in the domain. The expressive power of the description logic depends on the concept and role constructors available in its language. Typical examples are conjunction, negation and restricted quantification of concepts, and composition, union, inversion, and reflexive transitive closure of roles. In general, description logics can be characterized as variable-free fragments of first order logic, sometimes augmented with fixpoint-operators (see de Giacomo and Lenzerini, 1994). Unlike first order logic, description logics are often decidable and, moreover, they are effectively implementable (see e.g. Brachman and Schmolze, 1985, Borgida et al., 1989, Baader and Hollunder, 1991). Recently description logics have found numerous applications, in particular, to information systems (Catarci and Lenzerini 1993), databases (Borgida 1995), software engineering (Wright et al. 1993). They have also been advocated as a unifying framework for different types of databases and knowledge representation formalisms (Bergamaschi and Sartori 1992).

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To capture various *dynamic* features of application domains in computer science and artificial intelligence (such as program executions, information flows, temporal databases, multi-agent distributive systems, etc.), first order logic is usually extended by explicit program, temporal, epistemic or some other kind of "modal" operators. However, this often results in logics of even a higher degree of undecidability, for instance, recursively non-enumerable (see e.g. Gabbay et al., 1994, Kröger, 1990, Szalas and Holenderski, 1988), which is the main reason why mostly only the propositional fragment of temporal, dynamic and other logics of this sort has been studied and used in practice.

On the other hand, having such a natural, well motivated and established knowledge representation formalism as description logics, it would be strange not to try to extend it by adding, say, a temporal dimension so that the underlying description logic would represent knowledge about states of a process while the temporal component describe the behaviour of the process in time, i.e., the resulting sequence of states.

The main aim of this paper is to show that by combining rather expressive decidable description logics and point-based temporal propositional logics we can obtain decidable hybrids. In a sense our results can be regarded as an optimal compromise between expressive power and decidability: even harmless looking extensions of the constructed systems lead to undecidable logics.

We deal with three types of underlying description logics. First we consider the logic $\mathcal{CIQ}$ developed and investigated by de Giacomo and Lenzerini (1996) and de Giacomo (1995). It has the usual concept constructors including number restrictions and an extensive set of role constructors: union, chaining, transitive reflexive closure, inversion, and test. (Note that because of the transitive reflexive closure constructor this logic is not a fragment of first order logic.) We allow not only TBox-reasoning but also object names and assertions of the form $a : C$ (object a is in concept C), aRb (objects a and b are in relation R). Two other description logics are $\mathcal{CIO}$ and $\mathcal{CNO}$ introduced by de Giacomo (1995). In their languages one can form concepts $\{a\}$ for all object names a, which are interpreted as singletons and correspond to names or nominals known in the modal logic literature (see e.g. Blackburn, 1993). In these cases to obtain decidability either the constructor of inverse roles or number restrictions have to be omitted.

In the temporal dimension, we consider the operators "Since" and "Until" over natural and integers numbers, and the operators "sometime in the future" and "sometime in the past" over arbitrary strict linear orders and rational numbers. The pure temporal part of our logics is also well known and investigated; see e.g. (Gabbay et al. 1994).

In the variety of possible ways of combining the formalisms of description and temporal logics we follow that one which was first proposed by Baader and Laux (1995) who integrated polymodal $\mathbf{K}$ with the description logic $\mathcal{ALC}$ by applying modal operators to both concepts and formulas. In our case, we also allow applications of the temporal operators to concepts and formulas. This way seems to be an optimal choice, for, as was shown by Baader and Ohlbach (1995), modal operators applicable to roles can ruin decidability.

Our attempt to combine description and temporal logics is not the first one. Some ways of introducing a temporal dimension in description logics have already been investigated in the literature. Schmiedel (1990) proposed a very expressive temporal description logic based on intervals as introduced by Halpern and Shoham (1991); however, it turned out to be undecidable. Devanbu and Litman (1991), Weida and Litman (1992, 1994), Artale and Franconi (1994) continued this work by weakening Schmiedel's logic (they integrated constraint networks and fragments of Allen's interval calculus into description logics). Schild (1993) introduced a decidable point-based temporal description logic in which temporal operators can be applied only to concepts. On the other hand, a number of approaches to combining modal and temporal logics have been proposed. Finger and Gabbay (1992) studied temporal modal logics in which (speaking in terms of description logics) temporal operators are applied only to formulas. Both Schild's and Finger–Gabbay's constructions are covered by our approach. Fagin et al. (1995) considered a logic for modelling the behaviour of parallel processes on the basis of epistemic and temporal operators. Their system for one agent who does not forget, does not learn and knows time is a fragment of our logics based on natural numbers. Reynolds (1996) interpreted this system on arbitrary strict linear orders. This is also covered by our formalisms.

The paper is organized in the following way. Having defined (in Sections 2 and 3) the syntax and semantics of the temporal description logic $\mathcal{CIQ}_{\mathcal{US}}$, we introduce and investigate (in Section 4) our main tool for establishing decidability, the notion of a quasimodel. Unlike standard models, worlds in quasimodels are always finite; however, modulo a given formula, every model can be represented as a suitable quasimodel. In (Wolter and Zakharyaschev 1998) we used the notion of a quasimodel for proving the decidability of other combinations of modal and description logics. In Sections 5 and 6 we establish the decidability of the satisfiability problem for various temporal description logics based on $\mathcal{CIQ}$, and Section 7 extends the obtained results to temporal logics based on $\mathcal{CIO}$ and $\mathcal{CIN}$. The paper closes with a discussion of open problems.

2 Basic description logic

The underlying concept description logic we deal with in the first part of the paper was introduced by de Giacomo and Lenzerini (1996) and de Giacomo (1995) under the name $\mathcal{CIQ}$.

Definition 1 (language). The *language* of $\mathcal{CIQ}$ is based upon a list of *concept names* $C_0, C_1, \ldots$, a list of *role names* $R_0, R_1, \ldots$, and a list of *object names* $a_0, a_1, \ldots$. Starting from these we can form compound roles, concepts, and formulas using the following constructors. First, by a *basic role* we mean any role name R_i as well as its "inversion" R_i^-. Now, if R, S are roles, B is a basic role, C, D are concepts (for the basis of our inductive definition we assume basic roles to be roles and concept names to be concepts), and $n < \omega$, then

$$R \vee S,\ R \circ S,\ R^*,\ R^-,\ C?$$

are *roles* and

$$\top,\ C \wedge D,\ \neg C,\ \exists R.C,\ \exists_{\geq n} B.C$$

are *concepts*. *Atomic formulas* are expressions of the form

$$\top,\ C = D,\ a : C,\ aRb,$$

where C and D are concepts, R is a role name and a, b are object names. If φ and ψ are formulas then so are $\varphi \wedge \psi$ and $\neg\varphi$.

The connectives (or operations) $\rightarrow$ and $\vee$ are defined in the standard way:

$$E_1 \rightarrow E_2 = \neg(E_1 \wedge \neg E_2),\ E_1 \vee E_2 = \neg(\neg E_1 \wedge \neg E_2),$$

where expressions E_1, E_2 are either concepts or formulas.

The intended meaning of the introduced constructors will be clear from Definition 3 below.

Definition 2 (model). A $\mathcal{CIQ}$*-model* is a structure of the form

$$I = \left\langle \Delta, R_0^I, \dots, C_0^I, \dots, a_0^I, \dots \right\rangle,$$

where Δ is a non-empty set, the *domain* of the model, R_i^I ($i = 0, \dots$) are binary relations on Δ (interpreting the role names), C_i^I subsets of Δ (interpreting the concept names), and a_i^I are objects in Δ (interpreting the object names).

Definition 3 (satisfaction). For a $\mathcal{CIQ}$-model I, the *value* C^I of a concept C, the *value* R^I of a role R, and the *truth-relation* $\models$ are defined inductively in the following way:

1. $\top^I = \Delta$ and $C^I = C_i^I$, for $C = C_i$;
2. $(C \wedge D)^I = C^I \cap D^I$;
3. $(\neg C)^I = \Delta^I - C^I$;
4. $x \in (\exists R.C)^I$ iff $\exists y \in C^I\ xR^I y$;
5. $x \in (\exists_{\geq n} R.C)^I$ iff $\left|\{y \in C^I : xR^I y\}\right| \geq n$;
6. $(R \vee S)^I = R^I \cup S^I$;
7. $(R \circ S)^I = R^I \circ S^I$ (the composition of R^I and S^I);
8. $(R^*)^I = (R^I)^*$ (the transitive and reflexive closure of R^I);
9. $(R^-)^I = (R^I)^{-1}$ (the inversion of R^I);
10. $(C?)^I = \{\langle x, x\rangle : x \in C^I\}$;
11. $I \models \top$;
12. $I \models C = D$ iff $C^I = D^I$;

13. $I \models a : C$ iff $a^I \in C^I$;

14. $I \models aRb$ iff $a^I R^I b^I$;

15. $I \models \varphi \wedge \psi$ iff $I \models \varphi$ and $I \models \psi$;

16. $I \models \neg\varphi$ iff $I \not\models \varphi$.

(Here and below $|X|$ is the cardinality of X.) A formula φ is called *satisfiable* if there is a $\mathcal{CIQ}$-model I such that $I \models \varphi$.

As was shown by de Giacomo and Lenzerini (1996), the satisfiability problem for $\mathcal{CIQ}$ is decidable; however, it becomes undecidable for the extended language in which one can construct concepts of the form $\exists_{\geq n} R.C$ for all (not only basic) roles R; see (de Giacomo and Lenzerini 1996), where the reader can find also some examples illustrating the expressive power of $\mathcal{CIQ}$. Another important fact observed by de Giacomo and Lenzerini (1996) is that $\mathcal{CIQ}$ does not have the finite model property: there exists a formula satisfiable in an infinite model but not in finite ones.

3 Temporal description logic

We now add to the static language $\mathcal{CIQ}$ a temporal dimension.

Definition 4 (language). Let $\mathcal{CIQ}_{\mathcal{US}}$ be the extension of $\mathcal{CIQ}$ with the binary temporal operators $\mathcal{U}$ (Until) and $\mathcal{S}$ (Since) which may be applied to concepts and formulas, i.e., if C, D are concepts and φ, ψ formulas then $C\mathcal{U}D$, $C\mathcal{S}D$ are concepts and $\varphi\mathcal{U}\psi$, $\varphi\mathcal{S}\psi$ formulas. $\mathcal{CIQ}_{\mathcal{U}}$ is the extension of $\mathcal{CIQ}$ with only $\mathcal{U}$. And by $\mathcal{CIQ}_{\diamond}$ we denote the extension of $\mathcal{CIQ}$ with the operators $\diamond^+$ (sometime in the future) and $\diamond^-$ (sometime in the past) defined by

$$\diamond^+ E = \top\mathcal{U}E, \ \diamond^- E = \top\mathcal{S}E,$$

where E is either a concept or a formula.

Below we define models and other semantic notions only for the full language $\mathcal{CIQ}_{\mathcal{US}}$; they are easily relativized to its fragments $\mathcal{CIQ}_{\mathcal{U}}$ and $\mathcal{CIQ}_{\diamond}$.

Definition 5 (model). A $\mathcal{CIQ}_{\mathcal{US}}$*-model* with *domain* Δ is a pair

$$\mathfrak{M} = \langle \langle W, < \rangle, I \rangle$$

in which $\langle W, < \rangle$ is a strict linear order[1] and I a function associating with each $w \in W$ a $\mathcal{CIQ}$-model

$$I(w) = \left\langle \Delta, R_0^{I(w)}, \ldots, C_0^{I(w)}, \ldots, a_0^{I(w)}, \ldots \right\rangle$$

such that $a_i^{I(u)} = a_i^{I(v)}$ for any $u, v \in W$. Without loss of generality we may (and often will) identify the objects $a_i^{I(w)}$ with the object names a_i.

[1] I.e., $<$ is an irreflexive transitive relation on W such that $u < v$ or $v < u$ for all $u \neq v$.

It is worth emphasizing that all our models satisfy the *constant domain assumption*; as was shown in (Wolter and Zakharyaschev 1998), the cases of expanding and varying domains are reducible to that of constant domains at least as far as the decidability of the satisfiability problem is concerned.

Definition 6 (satisfaction). Given a $\mathcal{CIQ}_{\mathcal{US}}$-model $\mathfrak{M} = \langle\langle W, <\rangle, I\rangle$ and a "world" w in it, the *values* $C^{I(w)}$ and $R^{I(w)}$ of a concept C and a role R in w, and the *truth-relation* $(\mathfrak{M}, w) \models \varphi$ (or simply $w \models \varphi$, if $\mathfrak{M}$ is understood) are computed inductively according to the rules of Definition 3 and the following clauses:

1. $x \in (C\mathcal{U}D)^{I(w)}$ iff there is $v > w$ such that $x \in D^{I(v)}$ and $x \in C^{I(u)}$ for every u in the interval $(w, v) = \{u \in W : w < u < v\}$;

2. $x \in (C\mathcal{S}D)^{I(w)}$ iff there is $v < w$ such that $x \in D^{I(v)}$ and $x \in C^{I(u)}$ for every $u \in (v, w)$;

3. $w \models \psi\mathcal{U}\chi$ iff there is $v > w$ such that $v \models \chi$ and $u \models \psi$ for every $u \in (w, v)$;

4. $w \models \psi\mathcal{S}\chi$ iff there is $v < w$ such that $v \models \chi$ and $u \models \psi$ for every $u \in (v, w)$.

A formula φ is *satisfiable* in the frame $\langle W, <\rangle$ if there is a $\mathcal{CIQ}_{\mathcal{US}}$-model based on $\langle W, <\rangle$ and a world w in it such that $w \models \varphi$.

In this paper, our concern is only the satisfiability problem in various frames. Other standard reasoning tasks, say subsumption or instantiation, are known to be reducible to it. The entailment problems in both local and global formulations can also be reduced to the satisfiability problem:

- (*local consequence*) $\Gamma \models \varphi$, for a finite set of formulas Γ, iff for every model $\mathfrak{M} = \langle\langle W, <\rangle, I\rangle$ in a given class and every $w \in W$, we have $w \models \varphi$ whenever $w \models \Gamma$; it is easily seen that $\Gamma \models \varphi$ iff $\bigwedge \Gamma \wedge \neg\varphi$ is not satisfiable in the class;

- (*global consequence*) $\Gamma \models^* \varphi$ iff, for every $\mathfrak{M}$ in a given class, we have $w \models \varphi$ for all $w \in W$ whenever $w \models \Gamma$ for all $w \in W$; in this case $\Gamma \models^* \varphi$ iff
$$\Gamma \cup \{\Box^+\psi : \psi \in \Gamma\} \cup \{\Box^-\psi : \psi \in \Gamma\} \models \varphi.$$

In the semantics introduced above object names are interpreted globally, whereas role and concept names are interpreted locally (in the AI literature locally interpreted terms are known as fluents). We can easily simulate global concepts with the help of the equation

$$C = \Box^+ C \wedge \Box^- C.$$

A concept C satisfying this equation in each world of a model is a global concept in the sense that $C^{I(w)}$ does not depend on w. On the other hand, global role names cannot be simulated by means of local ones, and this restriction is essential for the satisfiability problem to be decidable. Indeed, let us assume that role names are interpreted globally, i.e, $R^{I(v)} = R^{I(w)}$ for all $v, w \in W$. Then the resulting description logic would contain as fragments products of modal logics (see (Gabbay and Shehtman 1998) and (Marx and Venema 1997)) interpreted in structures of the form

$$\langle W, < \rangle \times \langle \Delta, R \rangle .$$

As was shown by Spaan (1993) and Marx (1997), the global consequence problem for products of this form is mostly undecidable. From their results it follows immediately, for example, that the satisfiability problem for $\mathcal{ALC}_{\mathcal{U}}$ in the frame $\langle \mathbb{N}, < \rangle$ with the global interpretation of role names is undecidable. Thus, to ensure decidability we are forced either to interpret role names locally or to omit some boolean operators and universal role quantification. The latter way was taken by Artale and Franconi (1994) who considered interval-based temporal description logics. In this paper we deal with only the former choice.

Our main aim is to prove the following

Theorem 7. *There are algorithms that are capable of deciding whether*

1. *a given $\mathcal{CIQ}_{\mathcal{US}}$-formula is satisfiable in $\langle \mathbb{Z}, < \rangle$ and in $\langle \mathbb{N}, < \rangle$ ($\mathbb{Z}$ and $\mathbb{N}$ are the sets of all integer and natural numbers, respectively) and whether*
2. *a given $\mathcal{CIQ}_{\diamond}$-formula is satisfiable in some (strictly linearly ordered) frame as well as in $\langle \mathbb{Q}, < \rangle$ ($\mathbb{Q}$ is the set of all rational numbers).*

As in (Wolter and Zakharyaschev 1998), our first step is to represent $\mathcal{CIQ}_{\mathcal{US}}$-models in the form of quasimodels, sequences of certain finite structures called quasiworlds.

4 Quasimodels

Fix a $\mathcal{CIQ}_{\mathcal{US}}$-formula φ. Let $ob\varphi$ be the set of all object names in φ. And by $con\varphi$ and $sub\varphi$ we denote the closure under negation of, respectively, the set of all concepts in φ and the set of all subformulas in φ. Without loss of generality we may identify E and $\neg\neg E$, for every concept or formula E; so both $con\varphi$ and $sub\varphi$ are finite.

Definition 8 (types). A *concept type* t for φ is a subset of $con\varphi$ such that

- $C \wedge D \in t$ iff $C, D \in t$, for every $C \wedge D \in con\varphi$;
- $\neg C \in t$ iff $C \notin t$, for every $C \in con\varphi$.

By a *named concept type* for φ we mean the pair $\langle a, t \rangle$ in which $a \in ob\varphi$ and t is a concept type for φ. We will denote $\langle a, t \rangle$ by t_a and write $C \in t_a$ instead of $C \in t$, for t in $\langle a, t \rangle$. A *formula type* Φ for φ is a subset of $sub\varphi$ such that

- $\psi \wedge \chi \in \Phi$ iff $\psi, \chi \in \Phi$, for every $\psi \wedge \chi \in sub\varphi$;
- $\neg\psi \in \Phi$ iff $\psi \notin \Phi$, for every $\psi \in sub\varphi$.

Definition 9 (quasiworld candidate). Let T be a set of concept types for φ, T^o a set containing one named concept type t_a for every $a \in ob\varphi$, and let Φ be a formula type for φ. The triple $\langle T, T^o, \Phi\rangle$ is called a *quasiworld candidate* for φ if the following holds:

- $t \in T$ for every $\langle a, t\rangle \in T^o$;
- $(a : C) \in \Phi$ iff $C \in t_a$, for every $(a : C) \in sub\varphi$ and every $t_a \in T^o$;
- $(C = D) \in \Phi$ iff each $t \in T$ contains or does not contain simultaneously both C and D, for every $(C = D) \in sub\varphi$.

It should be clear that for every quasiworld candidate $\langle T, T^o, \Phi\rangle$ for φ we have

$$|T| \leq 2^{|con\varphi|},\ |T^o| = |ob\varphi|\,,\ |\Phi| \leq 2^{|sub\varphi|}.$$

Also, it is not hard to see that, given a triple $\langle T, T^o, \Phi\rangle$ as described in the first sentence of Definition 9, one can effectively decide whether it is a quasiworld candidate for φ or not.

Definition 10 (extended $\mathcal{CIQ}$-model). By an *extended* $\mathcal{CIQ}$*-model* for φ we mean a $\mathcal{CIQ}$-model

$$I = \left\langle \Delta, R_0^I, \ldots, C_0^I, \ldots, (C\mathcal{U}D)^I, \ldots, (C'\mathcal{S}D')^I, \ldots, a_0^I, \ldots \right\rangle \tag{1}$$

in which all concepts of the form $C\mathcal{U}D$ and $C'\mathcal{S}D'$ occurring in φ are regarded as concept names. For every $x \in \Delta$ we put

$$t^I(x) = \{C \in con\varphi : x \in C^I\},\ \ [x]^I = \{y \in \Delta : t^I(x) = t^I(y)\}.$$

Clearly, $t^I(x)$ is a concept type.

Definition 11 (quasiworld). Say that an extended $\mathcal{CIQ}$-model I for φ of the form (1) *realizes* a quasiworld candidate $\mathfrak{w} = \langle T, T^o, \Phi\rangle$ for φ if the following conditions hold:

1. $T = \{t^I(x) : x \in \Delta\}$;
2. for every $a \in ob\varphi$, $t_a = \left\langle a, t^I(a)\right\rangle$;
3. for every $aRb \in sub\varphi$, $a^I R^I b^I$ iff $aRb \in \Phi$.

A realizable quasiworld candidate $\mathfrak{w}$ for φ is called a *quasiworld* for φ. Instead of $\psi \in \Phi$ we will often write $\mathfrak{w} \models \psi$ and say that ψ is *true* in $\mathfrak{w}$.

Lemma 12. *Given a quasiworld candidate for φ, one can effectively recognize whether it is quasiworld for φ.*

Proof It is easy to see that a quasiworld candidate $\langle T, T^o, \Phi\rangle$ for φ is realizable iff the conjunction of the formulas

$$\bigvee\{\bigwedge t : t \in T\} = \top,\ a : \bigwedge t_a \text{ for } t_a \in T^o,$$
$$aRb \text{ for } aRb \in \Phi, \text{ and } \neg(aRb) \text{ for } \neg(aRb) \in \Phi$$

($\bigwedge t$ is the conjunction of all concepts in t) is satisfiable in an extended $\mathcal{CIQ}$-model for φ. It remains to recall that, according to (de Giacomo and Lenzerini 1996), the satisfiability problem for $\mathcal{CIQ}$ is decidable. ❑

Observe that the number of distinct quasiworlds for φ does not exceed

$$\sharp(\varphi) = 2^{2^{|con\varphi|}} \cdot |ob\varphi| \cdot 2^{|con\varphi|} \cdot 2^{|sub\varphi|}.$$

Fix a strictly linearly ordered frame $\mathfrak{F} = \langle W, <\rangle$ and consider a sequence

$$Q = \langle \mathfrak{w}_w : w \in W\rangle \tag{2}$$

of quasiworlds $\mathfrak{w}_w = \langle T_w, T^o_w, \Phi_w\rangle$ for φ. We will call it an $\mathfrak{F}$-*sequence* for φ. Concept types in T_w will be denoted by t_w, named concept types in T^o_w by t^w_a, $a \in ob\varphi$. $Q(w)$ is another name for $\mathfrak{w}_w$. More generally, for any sequence s of some elements indexed by worlds $w \in W$, $s(w)$ will denote the member of s indexed by w.

Definition 13 (run). A *run* in Q is a sequence $r = \langle r(w) : w \in W\rangle$ such that

(a) $r(w) \in T_w$ for every $w \in W$;

(b) for every concept $C\mathcal{U}D \in con\varphi$ and every $w \in W$, $C\mathcal{U}D \in r(w)$ iff there exists $u > w$ such that $D \in r(u)$ and $C \in r(v)$, for all $v \in (w, u)$;

(c) for every concept $C\mathcal{S}D \in con\varphi$ and every $w \in W$, $C\mathcal{S}D \in r(w)$ iff there exists $u < w$ such that $D \in r(u)$ and $C \in r(v)$, for all $v \in (u, w)$.

Definition 14 (quasimodel). An $\mathfrak{F}$-sequence Q for φ of the form (2) is called a *quasimodel for* φ *based on* $\mathfrak{F}$ if the following conditions hold:

(d) for every $a \in ob\varphi$, the sequence $r_a = \langle t^w_a : w \in W\rangle$ is a run in Q;

(e) for every $w \in W$ and every $t \in T_w$, there is a run r in Q such that $r(w) = t$;

(f) for every $w \in W$ and every $\psi\mathcal{U}\chi \in sub\varphi$, we have $Q(w) \models \psi\mathcal{U}\chi$ iff there exists $u > w$ such that $Q(u) \models \chi$ and $Q(v) \models \psi$ for all $v \in (w, u)$;

(g) for every $w \in W$ and every $\psi\mathcal{S}\chi \in sub\varphi$, we have $Q(w) \models \psi\mathcal{S}\chi$ iff there exists $u < w$ such that $Q(u) \models \chi$ and $Q(v) \models \psi$ for all $v \in (u, w)$.

A formula $\psi \in sub\varphi$ is said to be *satisfied* in Q if $Q(w) \models \psi$ for some $w \in W$.

Theorem 15. *A formula φ is satisfiable in a $\mathcal{CIQ}_{\mathcal{US}}$-model based on $\langle W, <\rangle$ iff it is satisfiable in a quasimodel for φ based on $\langle W, <\rangle$.*

Proof ($\Rightarrow$) Suppose that φ is satisfied in a $\mathcal{CIQ}_{\mathcal{US}}$-model $\langle\langle W,<\rangle, I\rangle$ with domain Δ. For every $w \in W$, we define $\mathfrak{w}_w = \langle T_w, T_w^o, \Phi_w\rangle$ by taking

$$T_w = \{t^{I(w)}(x) : x \in \Delta\},$$
$$T_w^o = \{t_a^w = \left\langle a, t^{I(w)}(a)\right\rangle : a \in ob\varphi\},$$
$$\Phi_w = \{\psi \in sub\varphi : w \models \psi\}.$$

It is not hard to see that $\mathfrak{w}_w$ is a quasiworld for φ (realized in $I(w)$ extended by the concepts $C\mathcal{U}D$ and $C'\mathcal{S}D'$ in φ) and $Q = \langle \mathfrak{w}_w : w \in W\rangle$ is a quasimodel on $\langle W,<\rangle$ satisfying φ (the sequence $\left\langle t^{I(w)}(x) : w \in W\right\rangle$ is a run through $t^{I(u)}(x)$, for every $u \in W$ and every $x \in \Delta$).

($\Leftarrow$) To show the converse we require the following lemma.

Lemma 16. *There is a cardinal $\kappa \geq \aleph_0$ such that, for any cardinal $\kappa' \geq \kappa$, every quasiworld $\mathfrak{w}$ for φ is realized in an extended $\mathcal{CIQ}$-model J in which $\left|[x]^J\right| = \kappa'$ for all x in the domain of J.*

Proof For each quasiworld $\mathfrak{u}$ for φ fix an extended $\mathcal{CIQ}$-model $I_\mathfrak{u}$ realizing $\mathfrak{u}$. Let $\Delta_\mathfrak{u}$ be the domain of $I_\mathfrak{u}$. Then we define κ to be the supremum of $\aleph_0$ and $\left|[x]^{I_\mathfrak{u}}\right|$, for all quasiworlds $\mathfrak{u}$ for φ and all $x \in \Delta_\mathfrak{u}$. We show that κ satisfies the required conditions.

Suppose $\mathfrak{w}$ is a quasiworld for φ and $\kappa' \geq \kappa$. Take an extended $\mathcal{CIQ}$-model

$$I = \left\langle \Delta, R_0^I, \dots, C_0^I, \dots, (C\mathcal{U}D)^I, \dots, (C'\mathcal{S}D')^I, \dots, a_0^I, \dots\right\rangle$$

realizing $\mathfrak{w}$ and such that $\left|[x]^I\right| \leq \kappa$ for every $x \in \Delta$. Now we define

$$J = \left\langle \Delta', R_0^J, \dots, C_0^J, \dots, (C\mathcal{U}D)^J, \dots, (C'\mathcal{S}D')^J, \dots, a_0^J, \dots\right\rangle$$

to be the disjoint union of κ' copies of I; more precisely, we put

$$\Delta' = \{\langle x, \xi\rangle : x \in \Delta, \xi < \kappa'\},$$
$$R_i^J = \{\langle\langle x,\xi\rangle, \langle y,\xi\rangle\rangle : \langle x, y\rangle \in R_i^I, \xi < \kappa'\},$$
$$C_i^J = \{\langle x, \xi\rangle : x \in C_i^I, \xi < \kappa'\},$$
$$a_i^J = \left\langle a_i^I, 0\right\rangle.$$

Clearly, $\left|[x]^J\right| = \kappa'$ for every $x \in \Delta'$, and one can readily check by induction that J realizes $\mathfrak{w}$. ❑

Let us now return to the proof of our theorem. Suppose φ is satisfied in a quasimodel $Q = \langle \mathfrak{w}_w : w \in W\rangle$ with $\mathfrak{w}_w = \langle T_w, T_w^o, \Phi_w\rangle$. Assume also that κ' is a cardinal exceeding the cardinality of the set Ω of all runs in Q and the cardinal κ supplied by Lemma 16 as well. Let

$$\Delta = \{\langle r, \xi\rangle : r \in \Omega, \xi < \kappa'\}.$$

Notice that $|\{\langle r, \xi\rangle \in \Omega : r(w) = t\}| = \kappa'$, for every $w \in W$ and every $t \in T_w$. By Lemma 16, for every $w \in W$ there exists an extended $\mathcal{CIQ}$-model

$$I(w) = \left\langle \Delta, R_0^{I(w)}, \dots, C_0^{I(w)}, \dots, (C\mathcal{U}D)^{I(w)}, \dots, (C'\mathcal{S}D')^{I(w)}, \dots, a_0^{I(w)}, \dots \right\rangle$$

such that

- $a^{I(w)} = \langle r_a, 0\rangle$, for each $a \in ob\varphi$;
- $t^{I(w)}(\langle r, \xi\rangle) = r(w)$, for every $r \in \Omega$ and every $\xi < \kappa'$.

For $w \in W$ let

$$J(w) = \left\langle \Delta, R_0^{I(w)}, \dots, C_0^{I(w)}, \dots, a_0^{I(w)}, \dots \right\rangle.$$

Consider the $\mathcal{CIQ}_{\mathcal{US}}$-model $\mathfrak{M} = \langle\langle W, <\rangle, J\rangle$ and show by induction on the construction of $\psi \in sub\varphi$ that

$$\mathfrak{w}_w \models \psi \text{ iff } (\mathfrak{M}, w) \models \psi. \tag{3}$$

Observe first that for every $C \in con\varphi$, we have $C^{I(w)} = C^{J(w)}$. This is also proved by induction the only non-trivial step in which is to show

$$(C\mathcal{U}D)^{I(w)} = (C\mathcal{U}D)^{J(w)},\ (C\mathcal{S}D)^{I(w)} = (C\mathcal{S}D)^{J(w)}$$

assuming that $C^{I(u)} = C^{J(u)}$ and $D^{I(u)} = D^{J(u)}$ for all $u \in W$.

Suppose $\langle r, \xi\rangle \in \Delta$. By the definition of $I(w)$, $\langle r, \xi\rangle \in (C\mathcal{U}D)^{I(w)}$ iff $r(w) \in C\mathcal{U}D$. By (b) of Definition 13, this means that there is $u > w$ such that $D \in r(u)$ and $C \in r(v)$ for all $v \in (w, u)$, which is equivalent to $\langle r, \xi\rangle \in D^{I(u)}$ and $\langle r, \xi\rangle \in C^{I(v)}$ for $v \in (w, u)$, and so, by IH, $\langle r, \xi\rangle \in (C\mathcal{U}D)^{J(w)}$. The concept $C\mathcal{S}D$ is treated analogously.

By the definition of a quasiworld, it follows that (3) holds for atomic ψ. The induction step for $\psi = \chi_1 \wedge \chi_2$ and $\psi = \neg\chi_1$ is trivial, and the cases $\psi = \chi_1 \mathcal{U} \chi_2$, $\psi = \chi_1 \mathcal{S} \chi_2$ follow from (f) and (g) in Definition 14.

Thus $\mathfrak{M}$ satisfies φ. ❑

5 Satisfiability problem for $\mathcal{CIQ}_{\mathcal{U}}$ and $\mathcal{CIQ}_{\mathcal{US}}$

In this section we prove the first claim of Theorem 7. To make the idea of the proof more transparent, we develop a satisfiability checking algorithm for $\mathcal{CIQ}_{\mathcal{U}}$-formulas in the frame $\langle \mathbb{N}, <\rangle$.

Fix a $\mathcal{CIQ}_{\mathcal{U}}$-formula φ. Unless otherwise indicated, we will assume in this section that all quasimodels are based on $\langle \mathbb{N}, <\rangle$.

Given a sequence $s = s(0), s(1), \dots$ and $i \geq 0$, we denote by $s^{\leq i}$ and $s^{>i}$ the head $s(0), \dots, s(i)$ and the tail $s(i+1), s(i+2), \dots$ of s, respectively; $s_1 * s_2$ is the concatenation of sequences s_1 and s_2; $|s|$ denotes the length of s and

$$s^* = s * s * s * \dots$$

Lemma 17. *Let $Q = Q(0), Q(1), \ldots$ be a quasimodel for φ and $Q(n) = Q(m)$ for some $n < m$. Then $Q_{nm} = Q^{\leq n} * Q^{>m}$ is also a quasimodel for φ.*

Proof It suffices to observe that if r_1 and r_2 are runs in Q with $r_1(n) = r_2(m)$ then $r_1^{\leq n} * r_2^{>m}$ is a run in Q_{nm}. ❑

Definition 18. If a subsequence of a quasimodel Q for φ is a quasimodel for φ itself then we call it a *subquasimodel* of Q.

For example, Q_{nm} in Lemma 17 is a subquasimodel of Q.

Lemma 19. *Every quasimodel Q for φ contains a subquasimodel $Q' = Q_1 * Q_2$ such that $|Q_1| \leq \sharp(\varphi)$ and each quasiworld in Q_2 occurs in this sequence infinitely many times.*

Proof Let n be the maximal number such that $Q(n) \neq Q(m)$ for all $m > n$. If $n = 0$ then we take $Q' = Q = Q_2$ (Q_1 is empty). Otherwise we apply Lemma 17 to the quasimodel $Q = Q^{\leq n} * Q^{>n}$ deleting from its head $Q^{\leq n}$ all repeating quasiworlds, which gives us a subquasimodel $Q' = Q_1 * Q^{>n}$ satisfying the required properties. ❑

Definition 20. Suppose that $Q = \langle \mathfrak{w}_i : i \in \mathbb{N} \rangle$ is a sequence of quasiworlds $\mathfrak{w}_i = \langle T_i, T_i^o, \Phi_i \rangle$ for φ and r is a sequence of elements from T_i, $i \in \mathbb{N}$, such that $r(i) \in T_i$. Say that r *realizes* a concept $C\mathcal{U}D \in r(n)$ *in m steps* if there is $l \leq m$ such that $D \in r(n+l)$ and $C \in r(n+k)$ for all $k \in (0, l)$. A formula $\psi\mathcal{U}\chi \in \Phi_n$ is *realized in m steps* if there is $l \leq m$ such that $\chi \in \Phi_{n+l}$ and $\psi \in \Phi_{n+k}$ for all $k \in (0, l)$.

Lemma 21. *Let $Q = Q_1 * Q_2$ be a quasimodel for φ (with quasiworlds of the form $\langle T_i, T_i^o, \Phi_i \rangle$ for $i \in \mathbb{N}$) satisfying the requirements of Lemma 19, let $n = |Q_1| + 1$ and $\flat(\varphi) = 2^{|con\varphi|} + |ob\varphi|$. Then Q contains a subquasimodel of the form $Q_1 * Q_0 * Q_2^{>l}$, for some $l \geq 0$, such that*

(i) *$|Q_0| \leq \flat^2(\varphi) \cdot |con\varphi| \cdot \sharp(\varphi) + |sub\varphi| \cdot \sharp(\varphi) + \sharp(\varphi)$;*

(ii) *for every $t \in T_n$ there is a run r through t realizing all concepts of the form $C\mathcal{U}D \in r(n)$ in $|Q_0|$ steps (for $t_a \in T_n^o$ the run r_a realizes all concepts $C\mathcal{U}D \in r_a(n)$ in $|Q_0|$ steps);*

(iii) *every formula $\psi\mathcal{U}\chi \in \Phi_n$ is realized in $|Q_0|$ steps;*

(iv) *$Q_0(1) = Q_2^{>l}(1)$.*

Proof Suppose $t \in T_n$, $C\mathcal{U}D \in t$ and r is a run in Q through t, i.e., $r(n) = t$. Then there exists $m > 0$ such that $D \in r(n+m)$ and $C \in r(n+k)$ for all $k \in (0, m)$. Assume now that $0 < i < j < m$, $r(n+i) = r(n+j)$ and $Q(n+i) = Q(n+j)$. In view of Lemma 17, $Q_1 * Q_2^{\leq i} * Q_2^{>j}$ is a subquasimodel of Q and $r^{\leq n+i} * r^{>n+j}$ is a run through t. It follows that we can construct a subquasimodel $Q_1 * Q_2^{\leq 1} * Q_3$ of Q and a run r_1 in it which comes through t and realizes $C\mathcal{U}D$ in $m_1 \leq \flat(\varphi) \cdot \sharp(\varphi)$ steps.

Then we consider another concept $C'\mathcal{U}D' \in t$ and assume that it is realized in $m_2 > m_1$ steps in r_1. Using Lemma 17 once again (and deleting repeating quasiworlds in the interval $Q_3(m_1), \ldots, Q_3(m_2)$) we select a subquasimodel $Q_1 * Q_2^{\leq 1} * Q_3^{\leq m_1} * Q_4$ of Q and a run r_2 through t which realizes both $C\mathcal{U}D$ and $C'\mathcal{U}D'$ in $2 \cdot \flat(\varphi) \cdot \sharp(\varphi)$ steps.

Having analyzed all distinct concepts of the form $C\mathcal{U}D \in t$ we obtain a subquasimodel $Q_1 * Q_2^{\leq 1} * Q'$ of Q and a run r' through t which realizes all those concepts in $m' \leq |con\varphi| \cdot \flat(\varphi) \cdot \sharp(\varphi)$ steps.

After that we consider in the same manner another concept type $t' \in T_n$. However this time we can delete quasiworlds only after $Q'(m')$, and so to realize in some run through t' the concepts $C\mathcal{U}D \in t'$ we need $\leq 2 \cdot |con\varphi| \cdot \flat(\varphi) \cdot \sharp(\varphi)$ steps. And so on. Since $|T_n| + |T_n^o| \leq \flat(\varphi)$, to satisfy (ii) at most $|con\varphi| \cdot \flat^2(\varphi) \cdot \sharp(\varphi)$ quasiworlds are required.

The formulas $\psi\mathcal{U}\chi \in sub\varphi$ that are true in $Q_2(1)$ are treated analogously. This may give us $\leq |sub\varphi| \cdot \sharp(\varphi)$ more quasiworlds. And $\leq \sharp(\varphi)$ quasiworlds may be required to comply with (iv). ❑

Definition 22 (suitable pair). A pair t, t' of concept types for φ is called *suitable* if for every $C\mathcal{U}D \in con\varphi$,

$$C\mathcal{U}D \in t \text{ iff either } D \in t' \text{ or } C \in t' \text{ and } C\mathcal{U}D \in t'.$$

Lemma 23. *Suppose Q_1 and Q_2 are finite sequences of quasiworlds for φ of length l_1 and l_2, respectively, and let*

$$Q = Q_1 * Q_2^*$$

with $Q(n) = \langle T_n, T_n^o, \Phi_n \rangle$. Then Q is a quasimodel for φ whenever the following conditions hold:

1. *for every $i \leq l_1 + l_2$ and every $t' \in T_{i+1}$, there is $t \in T_i$ such that the pair t, t' is suitable;*
2. *for every $i \leq l_1 + 1$ and every $t_i \in T_i$, all concepts of the form $C\mathcal{U}D \in t_i$ are realized in $l_1 + l_2 - i$ steps in some sequence $t_i, t_{i+1}, \ldots, t_{l_1+l_2}$ in which $t_{i+j} \in T_{i+j}$ and every pair of adjacent elements is suitable (for $t_a^i \in T_i^o$ one can take the sequence $t_a^i, t_a^{i+1}, \ldots, t_a^{l_1+l_2}$, where $t_a^j \in T_j^o$);*
3. *for every $i \leq l_1 + l_2$, and every formula $\psi\mathcal{U}\chi \in sub\varphi$,*

   $$Q(i) \models \psi\mathcal{U}\chi \text{ iff either } Q(i+1) \models \chi \text{ or } Q(i+1) \models \psi \text{ and } Q(i+1) \models \psi\mathcal{U}\chi;$$

4. *for every $i \leq l_1 + 1$, all formulas of the form $\psi\mathcal{U}\chi \in \Phi_i$ are realized in $l_1 + l_2 - i$ steps.*

Proof Condition (d) follows from 2. To construct a run through $t_m \in T_m$, we first take concept types $t_i \in T_i$, for $i < m$, such that every pair of adjacent elements in the sequence $t_1, \ldots, t_m$ is suitable—this can be done by 1. Then using condition 2 we select a sequence $t_m, \ldots, t_{m+n}$, for some $n \leq l_1 + l_2$, such that every pair of adjacent elements in it is suitable and all concepts of the form $C\mathcal{U}D \in t_m$ are realized in it in n steps. After that we select such a sequence starting from t_{m+n} and so on. It is readily seen that the resulting sequence is a run in Q. This establishes (e). And condition (f) follows from 3 and 4. ❑

As a consequence of the two preceding lemmas we immediately obtain

Theorem 24. *A $\mathcal{CIQ}_{\mathcal{U}}$-formula φ is satisfiable in $\langle \mathbb{N}, < \rangle$ iff there are two sequences Q_1 and Q_2 of quasiworlds for φ such that $Q_1 * Q_2^*$ satisfies conditions 1–4 of Lemma 23, all quasiworlds in Q_1 are distinct (and so $|Q_1| \leq \sharp(\varphi)$),*

$$|Q_2| \leq \flat^2(\varphi) \cdot |con\varphi| \cdot \sharp(\varphi) + |sub(\varphi)| \cdot \sharp(\varphi) + \sharp(\varphi),$$

and $Q(1) \models \varphi$.

Proof By Theorem 15 and Lemmas 19, 21, φ is satisfiable in $\langle \mathbb{N}, < \rangle$ iff φ is true in the first quasiworld of a quasimodel of the form $Q_1 * Q_0 * Q_2^{>l}$ described in Lemma 21. It remains to observe that $Q_1 * Q_0^*$ satisfies the conditions of Lemma 23. ❑

This provides us with an algorithm which is capable of deciding, given an arbitrary $\mathcal{CIQ}_{\mathcal{U}}$-formula, whether it is satisfiable in $\langle \mathbb{N}, < \rangle$. In a similar manner one can construct a satisfiability checking algorithm for $\mathcal{CIQ}_{\mathcal{US}}$-formulas in the frame $\langle \mathbb{Z}, < \rangle$. We leave this to the reader, since no new ideas are required.

6 Satisfiability problem for $\mathcal{CIQ}_{\diamond}$

The aim of this section is to prove the second claim of Theorem 7. Now our frames are strict linear orders. For $\mathcal{CIQ}_{\diamond}$ Definition 6 becomes somewhat simpler: its items 1–4 should be replaced by the following:

1. $x \in (\diamond^+ C)^{I(w)}$ iff there is $v > w$ such that $x \in C^{I(v)}$;
2. $x \in (\diamond^- C)^{I(w)}$ iff there is $v < w$ such that $x \in C^{I(v)}$;
3. $w \models \diamond^+ \psi$ iff there is $v > w$ such that $v \models \psi$;
4. $w \models \diamond^- \psi$ iff there is $v < w$ such that $v \models \psi$.

Fix an arbitrary $\mathcal{CIQ}_{\diamond}$-formula φ.

Definition 25 (suitable triple). Let $\mathfrak{u} = \langle T_{\mathfrak{u}}, T_{\mathfrak{u}}^o, \Phi_{\mathfrak{u}} \rangle$ and $\mathfrak{v} = \langle T_{\mathfrak{v}}, T_{\mathfrak{v}}^o, \Phi_{\mathfrak{v}} \rangle$ be quasiworlds for φ and $\sigma \subseteq T_{\mathfrak{u}} \times T_{\mathfrak{v}}$. The triple $\langle \mathfrak{u}, \mathfrak{v}, \sigma \rangle$ is called *suitable* if it satisfies the conditions:

- $\forall t \in T_{\mathfrak{u}} \exists t' \in T_{\mathfrak{v}}\ t \sigma t'$;
- $\forall t' \in T_{\mathfrak{v}} \exists t \in T_{\mathfrak{u}}\ t \sigma t'$;

- $\forall a \in ob\varphi\ t_a^{\mathfrak{u}} \sigma t_a^{\mathfrak{v}}$;
- $\forall \Diamond^+ C \in con\varphi \forall t \in T_{\mathfrak{u}} \forall t' \in T_{\mathfrak{v}}\ (t\sigma t'\ \&\ \Diamond^+ C \notin t \Rightarrow C \notin t'\ \&\ \Diamond^+ C \notin t')$;
- $\forall \Diamond^- C \in con\varphi \forall t T_{\mathfrak{u}} \forall t' \in T_{\mathfrak{v}}\ (t\sigma t'\ \&\ \Diamond^- C \notin t' \Rightarrow C \notin t\ \&\ \Diamond^- C \notin t)$;
- $\forall \Diamond^+ \psi \in sub\varphi\ (\mathfrak{u} \not\models \Diamond^+ \psi \Rightarrow \mathfrak{v} \not\models \psi\ \&\ \mathfrak{v} \not\models \Diamond^+ \psi)$;
- $\forall \Diamond^- \psi \in sub\varphi\ (\mathfrak{v} \not\models \Diamond^- \psi \Rightarrow \mathfrak{u} \not\models \psi\ \&\ \mathfrak{u} \not\models \Diamond^- \psi)$.

The relation σ is called a *connection* between $\mathfrak{u}$ and $\mathfrak{v}$. Note that the same pair of quasiworlds may have several different connections.

It is easily checked that if $\langle \mathfrak{u}, \mathfrak{v}, \tau \rangle$ and $\langle \mathfrak{v}, \mathfrak{w}, \rho \rangle$ are suitable triples then $\langle \mathfrak{u}, \mathfrak{w}, \tau \circ \rho \rangle$ is a suitable triple as well.

Definition 26 (satisfying set). Say that a set $\mathcal{S}$ of suitable triples for φ is a *satisfying set* for φ if the following conditions hold:

(S1) there is a triple in $\mathcal{S}$ which contains a quasiworld $\mathfrak{w}$ such that $\mathfrak{w} \models \varphi$;

(S2) if $\langle \mathfrak{u}, \mathfrak{v}, \sigma \rangle \in \mathcal{S}$ and $\mathfrak{v} \models \Diamond^+ \psi$, then there is $\langle \mathfrak{v}, \mathfrak{w}, \tau \rangle \in \mathcal{S}$ such that $\mathfrak{w} \models \psi$;

(S3) if $\langle \mathfrak{u}, \mathfrak{v}, \sigma \rangle \in \mathcal{S}$ and $\mathfrak{u} \models \Diamond^- \psi$, then there is $\langle \mathfrak{w}, \mathfrak{u}, \tau \rangle \in \mathcal{S}$ such that $\mathfrak{w} \models \psi$;

(S4) if $\langle \mathfrak{u}, \mathfrak{v}, \sigma \rangle \in \mathcal{S}$ and $\Diamond^+ C \in t \in T_{\mathfrak{v}}$, then there are $\langle \mathfrak{v}, \mathfrak{w}, \tau \rangle \in \mathcal{S}$ and $t' \in T_{\mathfrak{w}}$ such that $C \in t'$ and $t\tau t'$ (if $t = t_a$, for $a \in ob\varphi$, then one can take $t' = t'_a$);

(S5) if $\langle \mathfrak{u}, \mathfrak{v}, \sigma \rangle \in \mathcal{S}$ and $\Diamond^- C \in t \in T_{\mathfrak{u}}$, then there are $\langle \mathfrak{w}, \mathfrak{u}, \tau \rangle \in \mathcal{S}$ and $t' \in T_{\mathfrak{w}}$ such that $C \in t'$ and $t'\tau t$ (if $t = t_a$, for $a \in ob\varphi$, then one can take $t' = t'_a$);

(S6) if $\langle \mathfrak{u}, \mathfrak{v}, \sigma \rangle \in \mathcal{S}$, $\mathfrak{u} \models \Diamond^+ \psi$, $\mathfrak{v} \not\models \psi$ and $\mathfrak{v} \not\models \Diamond^+ \psi$, then there is a quasiworld $\mathfrak{w} \models \psi$ such that $\langle \mathfrak{u}, \mathfrak{w}, \tau \rangle \in \mathcal{S}$, $\langle \mathfrak{w}, \mathfrak{v}, \rho \rangle \in \mathcal{S}$, for some τ, ρ, and $\tau \circ \rho = \sigma$;

(S7) if $\langle \mathfrak{u}, \mathfrak{v}, \sigma \rangle \in \mathcal{S}$, $\mathfrak{v} \models \Diamond^- \psi$, $\mathfrak{u} \not\models \psi$ and $\mathfrak{u} \not\models \Diamond^- \psi$, then there is a quasiworld $\mathfrak{w} \models \psi$ such that $\langle \mathfrak{u}, \mathfrak{w}, \tau \rangle \in \mathcal{S}$, $\langle \mathfrak{w}, \mathfrak{v}, \rho \rangle \in \mathcal{S}$, for some τ, ρ, and $\tau \circ \rho = \sigma$;

(S8) if $\langle \mathfrak{u}, \mathfrak{v}, \sigma \rangle \in \mathcal{S}$, $\Diamond^+ C \in t \in T_{\mathfrak{u}}$, $t\sigma t'$, $C \notin t'$ and $\Diamond^+ C \notin t'$, then there are $\mathfrak{w}$ and $t'' \in T_{\mathfrak{w}}$ such that $C \in t''$, $\langle \mathfrak{u}, \mathfrak{w}, \tau \rangle \in \mathcal{S}$, $\langle \mathfrak{w}, \mathfrak{v}, \rho \rangle \in \mathcal{S}$, for some τ and ρ, $t\tau t'' \rho t'$, and $\tau \circ \rho = \sigma$ (if $t = t_a$, $t' = t'_a$ then one can take $t'' = t''_a$);

(S9) if $\langle \mathfrak{u}, \mathfrak{v}, \sigma \rangle \in \mathcal{S}$, $\Diamond^- C \in t \in T_{\mathfrak{v}}$, $t'\sigma t$, $C \notin t'$ and $\Diamond^- C \notin t'$, then there are $\mathfrak{w}$ and $t'' \in T_{\mathfrak{w}}$ such that $C \in t''$, $\langle \mathfrak{u}, \mathfrak{w}, \tau \rangle \in \mathcal{S}$, $\langle \mathfrak{w}, \mathfrak{v}, \rho \rangle \in \mathcal{S}$, for some τ and ρ, $t'\tau t'' \rho t$, and $\tau \circ \rho = \sigma$ (if $t = t_a$, $t' = t'_a$ then one can take $t'' = t''_a$).

The crucial step in constructing a satisfiability checking algorithm for $\mathcal{CIQ}_\Diamond$-formulas in strict linear orders is the following

Theorem 27. *A $\mathcal{CIQ}_\Diamond$-formula φ is satisfiable in a strict linear order with ≥ 2 elements iff there exists a satisfying set for φ.*

Since the number of distinct quasiworlds for any formula φ does not exceed $\sharp(\varphi)$, and every quasiworld contains at most $\flat(\varphi)$ concept types, one can effectively check whether there exists a satisfying set for φ (e.g., simply by looking through all sets of suitable triples for φ). It follows that Theorem 27 is enough to show the decidability of the satisfiability problem for $\mathcal{CIQ}_\diamond$-formulas in strict linear orders. (Of course, in order to obtain the decidability, it remains to observe that it is decidable whether a formula is decidable in a strict linear order with one element.) So we focus on the proof of this theorem. One direction is easy.

Proof ($\Rightarrow$) Suppose φ is satisfied in a $\mathcal{CIQ}_\diamond$-model $\mathfrak{M} = \langle\langle W, <\rangle, I\rangle$ with a least two elements. Define a set $\mathcal{S}$ by putting in it all triples $\langle\mathfrak{u}, \mathfrak{v}, \sigma\rangle$ for which there are worlds $u, v \in W$ such that $u < v$, $\mathfrak{u} = \mathfrak{w}_u$, $\mathfrak{v} = \mathfrak{w}_v$ (see the proof of Theorem 15), and $t\sigma t'$ iff there is x in the domain of $\mathfrak{M}$ such that $t = t^{I(u)}(x)$ and $t' = t^{I(v)}(x)$. It is readily seen that $\mathcal{S}$ is a satisfying set for φ. ❑

To prove the converse we require a number of definitions. Fix a satisfying set $\mathcal{S}$ for φ. We are going to construct a quasimodel satisfying φ by taking the limit of an inductively defined sequence of finite weak quasimodels over $\mathcal{S}$.

Definition 28 (weak quasimodel). By a *weak quasimodel* over $\mathcal{S}$ we mean a finite sequence

$$q = \langle\mathfrak{w}_1, \dots, \mathfrak{w}_n\rangle$$

of quasiworlds for φ such that $\langle\mathfrak{w}_i, \mathfrak{w}_{i+1}, \sigma_{ii+1}\rangle \in \mathcal{S}$ for some connection σ_{ii+1} and every $i \in (0, n)$. Instead of $\mathfrak{w}_i$ we also write $q(i) = \langle T_i, T_i^o, \Phi_i\rangle$. A sequence of the form

$$r = \langle t_1, \dots, t_n\rangle$$

such that $t_i \in T_i$ and $t_i\sigma_{ii+1}t_{i+1}$ will be called a *run* in q. As before, the run $\langle t_a^1, \dots, t_a^n\rangle$, for $t_a^i \in T_i^o$, is denoted by r_a.

It should be clear that for every $t \in T_i$, $i \in \{1, \dots, n\}$, there is a run r in q such that $r(i) = t$. It is not hard to check also that if $1 \le i < j \le n$, then $\langle\mathfrak{w}_i, \mathfrak{w}_j, \sigma_{ij}\rangle$ is a suitable triple, where

$$\sigma_{ij} = \sigma_{ii+1} \circ \sigma_{i+1i+2} \circ \dots \circ \sigma_{j-1j}.$$

Definition 29 (defect). A *defect* in a weak quasimodel $q = \langle q(1), \dots, q(n)\rangle$ over $\mathcal{S}$ is

- a pair $d = \langle i, \psi\rangle$ such that $1 \le i \le n$, $\psi = \diamond^+\chi \in sub\varphi$ (or $\psi = \diamond^-\chi \in sub\varphi$), $q(i) \models \psi$ and $q(j) \not\models \chi$ for any $j \in (i, n+1)$ (respectively, $j \in (0, i)$)

and

- a triple $d = \langle i, r, C\rangle$ such that $1 \le i \le n$, r is a run in q, $C = \diamond^+D \in con\varphi$ (or $C = \diamond^-D \in con\varphi$), $C \in r(i)$ and $D \notin r(j)$ for any $j \in (i, n+1)$ (respectively, $j \in (0, i)$).

Suppose d is a defect in a weak quasimodel $q = \langle q(1), \ldots, q(n)\rangle$ over $\mathcal{S}$. We construct a new weak quasimodel q^d which "cures" d. In accordance with the definition above, consider two cases.

Case 1: $d = \langle i, \psi\rangle$, for $\psi = \Diamond^+\chi$ (or $\psi = \Diamond^-\chi$). Let $j \geq i$ be the maximal (respectively, let $j \leq i$ be the minimal) number for which $\langle j, \psi\rangle$ is a defect in q. If $j = n$ ($j = 1$) then, by conditions (S2) and (S3), there is a quasiworld $\mathfrak{w} \models \chi$ such that $\langle \mathfrak{w}_n, \mathfrak{w}, \sigma\rangle \in \mathcal{S}$ ($\langle \mathfrak{w}, \mathfrak{w}_1, \sigma\rangle \in \mathcal{S}$), for some connection σ. Put

$$q^d = \langle q(1), \ldots, q(n), \mathfrak{w}\rangle \ \ (\text{or } q^d = \langle \mathfrak{w}, q(1), \ldots, q(n)\rangle).$$

When $j \neq n$ ($j \neq 1$) we select, according to (S6) and (S7), a quasiworld $\mathfrak{w} \models \chi$ such that $\langle q(j), \mathfrak{w}, \tau\rangle \in \mathcal{S}$, $\langle \mathfrak{w}, q(j+1), \rho\rangle \in \mathcal{S}$ (respectively, $\langle q(j-1), \mathfrak{w}, \tau\rangle \in \mathcal{S}$, $\langle \mathfrak{w}, q(j), \rho\rangle \in \mathcal{S}$) and $\tau \circ \rho = \sigma_{jj+1}$ ($\tau \circ \rho = \sigma_{j-1j}$). Then we insert $\mathfrak{w}$ right after (before) $q(j)$ in q thus obtaining

$$q^d = \langle q(1), \ldots, q(j), \mathfrak{w}, q(j+1), \ldots, q(n)\rangle\,,$$
$$(\text{or } q^d = \langle q(1), \ldots, q(j-1), \mathfrak{w}, q(j), \ldots, q(n)\rangle).$$

Case 2: $d = \langle i, r, \Diamond^+ C\rangle$. Again let $j \geq i$ be the maximal number for which $\langle j, r, \Diamond^+ C\rangle$ is a defect in q. If $j = n$ then, by (S4), there exist a quasiworld $\mathfrak{w}$ and a type $t \in T_{\mathfrak{w}}$ such that $\langle \mathfrak{w}_n, \mathfrak{w}, \sigma\rangle \in \mathcal{S}$, for some σ, $C \in t$, and $r(n)\sigma t$. In this case we put

$$q^d = \langle q(1), \ldots, q(n), \mathfrak{w}\rangle\,.$$

When $j \neq n$ we use (S8) to select a quasiworld $\mathfrak{w}$ and a type $t \in T_{\mathfrak{w}}$ such that $\langle q(j), \mathfrak{w}, \tau\rangle, \langle \mathfrak{w}, q(j+1), \rho\rangle \in \mathcal{S}$, $C \in t$, and $r(j)\tau t \rho r(j+1)$ and $\tau \circ \rho = \sigma_{jj+1}$. This yields us a weak quasimodel

$$q^d = \langle q(1), \ldots, q(j), \mathfrak{w}, q(j+1), \ldots, q(n)\rangle\,,$$

"curing" d. The case of $d = \langle i, r, \Diamond^- C\rangle$ is considered analogously.

We are in a position now to complete the proof of Theorem 27.

Proof Suppose $\mathcal{S}$ is a satisfying set for φ and $\mathfrak{O} = \langle W, <\rangle$ a dense strict linear order without endpoints. We construct by induction a sequence of weak quasimodels q_i over $\mathcal{S}$ and a sequence of subframes $\mathfrak{O}_i = \langle W_i, <_i\rangle$ of $\mathfrak{O}$, for $i = 0, 1, \ldots$.

Step 0. Take a triple $\langle \mathfrak{u}, \mathfrak{v}, \sigma\rangle \in \mathcal{S}$ such that $\mathfrak{u} \models \varphi$ or $\mathfrak{v} \models \varphi$ (it exists by (S1)) and let $w_1 < w_2$ in $\mathfrak{O}$. Then we put

$$q_0 = \langle \mathfrak{w}_{w_1}, \mathfrak{w}_{w_2}\rangle\,,\ \mathfrak{O}_0 = \langle W_0, <_0\rangle\,,$$

where $\mathfrak{w}_{w_1} = \mathfrak{u}$, $\mathfrak{w}_{w_2} = \mathfrak{v}$, $W_0 = \{w_1, w_2\}$ and $w_1 <_0 w_2$.

Step $i+1$. Suppose we have already constructed a weak quasimodel

$$q_i = \langle \mathfrak{w}_{w_1}, \ldots, \mathfrak{w}_{w_n}\rangle \tag{4}$$

and a subframe $\mathfrak{O}_i = \langle W_i, <_i\rangle$ of $\mathfrak{O}$ such that

$$W_i = \{w_1, \ldots, w_n\},\ w_1 <_i \cdots <_i w_n.$$

If the set D_i of all defects in q_i is empty then we are done: q_i is clearly a quasimodel based on $\mathfrak{O}_i$ and satisfying φ. Otherwise we take some $d \in D_i$, construct the weak quasimodel

$$q_i^d = \left\langle \mathfrak{w}_{w_1}, \ldots, \mathfrak{w}_{w_j}, \mathfrak{w}_w, \mathfrak{w}_{w_{j+1}}, \ldots, \mathfrak{w}_{w_n} \right\rangle, \tag{5}$$

for $j \in \{1, \ldots, n\}$, select some $w \in W$ such that $w_j < w < w_{j+1}$ ($w_n < w$, if $j = n$, and $w < w_1$, if $j = 1$) and define $\mathfrak{O}_i^d$ to be the subframe of $\mathfrak{O}$ containing $\mathfrak{O}_i$ and w.

Define a set D_i^d of defects in q_i^d in the following way. Suppose d' is a defect in D_i different from d. If $d' = \langle k, \psi \rangle$ then we put $d' = \langle k, \psi \rangle$ in D_i^d when $k \leq j$ and d' is a defect in q_i^d; when $k > j$, we put there $d' = \langle k+1, \psi \rangle$. And if $d' = \langle k, r, D \rangle$ then we fix a run r' in q_i^d extending r and put $d' = \langle k, r', D \rangle$ in D_i^d when $k \leq j$ and d' is a defect in q_i^d; when $k > j$, we put there $d' = \langle k+1, r', D \rangle$. Clearly, $\left| D_i^d \right| \leq |D_i| - 1$. If $D_i^d \neq \emptyset$ then we take a defect $d' \in D_i^d$, construct $q_i^{dd'}$, $\mathfrak{O}_i^{dd'}$, and so on. When all defects in D_i are cured, we obtain a weak quasimodel

$$q_{i+1} = \langle \mathfrak{w}_{w_1}, \ldots, \mathfrak{w}_{w_m} \rangle$$

and a subframe $\mathfrak{O}_{i+1} = \langle W_{i+1}, <_{i+1} \rangle$ of $\mathfrak{O}$ such that $W_{i+1} = \{w_1, \ldots, w_m\}$ and $w_1 <_{i+1} \cdots <_{i+1} w_m$.

Step ω. Finally, put

$$W_\omega = \bigcup_{i<\omega} W_i, \; <_\omega = \bigcup_{i<\omega} <_i, \; \mathfrak{O}_\omega = \langle W_\omega, <_\omega \rangle, \; Q = \langle \mathfrak{w}_w : w \in W_\omega \rangle.$$

We show now that Q is a quasimodel based on $\mathfrak{O}_\omega$ and satisfying φ.

Let $u \in W_\omega$, $\mathfrak{w}_u = \langle T_u, T_u^o, \Phi_u \rangle$ and $t' \in T_u$. We are going to construct a run in Q through t'. Note first that $\mathfrak{w}_u$ belongs to a weak quasimodel q_i of the form (4), for some $i < \omega$, and there is a run r in q_i coming through t'. Define an extension of r for each act of expanding q_i.

Suppose that we are "curing" a defect d in q_i and obtain q_i^d. If $d = \langle j, \psi \rangle$ or $d = \langle j, r_1, D \rangle$, for $r_1 \neq r$, then we take any run r' in q_i^d containing r and declare it to be the *extension* of r in q_i^d. And if $d = \langle j, r, \Diamond^+ C \rangle$ and q_i^d is of the form (5) (so that $t' = r(k)$ for some $k \leq j$) then we define the *extension* of r in q_i^d to be the run

$$r(1), \ldots, r(j), t, r(j+1), \ldots, r(n),$$

where $t \in C$ is the concept type in T_w selected in Case 2 above. For $d = \langle j, r, \Diamond^- C \rangle$ the extension of r in q_i^d is defined in a symmetrical way. Now, if r' is the extension of r in q' and r'' the extension of r' in q'' then r'' is the *extension* of r in q''. Finally, we define the *extension* of r in Q as the limit r_ω of the sequence of the extensions of r in q_{i+1}, q_{i+2}, etc.; more precisely, r_ω comes through $t \in T_w$, $w \in W_\omega$, iff the extension of r in some q_j, $j > i$, comes through t. (If the original r is r_a for some $a \in ob\varphi$, then we can always define r_ω so that it comes through all t_a^w, $w \in W_\omega$.)

The constructed extension r_ω is a run in Q coming through t'. Indeed, suppose $\Diamond^+ C \in r_\omega(w)$ for some $\Diamond^+ C \in con\varphi$ and some $w \in W_\omega$. Then the extension r' of r in q_j, for some $j \geq i$, comes through $r_\omega(w)$, say $r_\omega(w) = r'(k)$. If $\langle k, r', \Diamond^+ C\rangle$ is not a defect in r' then there is $m > k$ such that $C \in r'(m)$ and so $C \in r_\omega(v)$ for some $v >_\omega w$. And if $\langle k, r', \Diamond^+ C\rangle$ is a defect then it is cured in some extension of r', and again we must have $v >_\omega w$ with $C \in r_\omega(v)$. Conversely, assume that there is $v >_\omega w$ and $C \in r_\omega(v)$, for some $\Diamond^+ C \in con\varphi$. Consider the extension r' of r in some q_j containing both $\mathfrak{w}_w$ and $\mathfrak{w}_v$. Let $r'(k) = r_\omega(w)$ and $r'(m) = r_\omega(v)$, $k < m$. Since r' is a run in q_j and by the definition of a suitable triple, we must have $\Diamond^+ C \in r'(k) = r_\omega(w)$. The case of $\Diamond^- C$ is considered analogously.

Thus, r_ω is a run in Q through $t' \in T_u$. It is readily seen also that, for every $\Diamond^+\psi \in sub\varphi$ ($\Diamond^-\psi \in sub\varphi$), $Q(u) \models \Diamond^+\psi$ (respectively, $Q(u) \models \Diamond^-\psi$) iff $Q(v) \models \psi$ for some $v >_\omega u$ ($v <_\omega u$). So Q is a quasimodel based on $\mathfrak{O}_\omega$ and satisfying φ. ❑

This shows that the satisfiability problem for $\mathcal{CIQ}_\Diamond$-formulas in strict linear orders is decidable. To see that it is decidable also in $\langle \mathbb{Q}, < \rangle$ we require one more definition.

Definition 30 ($\mathbb{Q}$-satisfying set). Say that a satisfying set $\mathcal{S}$ for a formula φ is $\mathbb{Q}$-*satisfying* if for every $\langle \mathfrak{u}, \mathfrak{v}, \sigma\rangle \in \mathcal{S}$ there exist $\langle \mathfrak{u}', \mathfrak{u}, \tau'\rangle \in \mathcal{S}$, $\langle \mathfrak{v}, \mathfrak{v}', \rho'\rangle \in \mathcal{S}$, and $\langle \mathfrak{u}, \mathfrak{w}, \tau\rangle \in \mathcal{S}$, $\langle \mathfrak{w}, \mathfrak{v}, \rho\rangle \in \mathcal{S}$ such that $\tau \circ \rho = \sigma$.

Theorem 31. *A $\mathcal{CIQ}_\Diamond$-formula φ is satisfiable in $\langle \mathbb{Q}, < \rangle$ iff there exists a $\mathbb{Q}$-satisfying set for φ.*

Proof ($\Rightarrow$) is established in the same way as in the proof of Theorem 27.

($\Leftarrow$) Suppose $\mathcal{S}$ is a $\mathbb{Q}$-satisfying set for φ and $\mathfrak{O} = \langle \mathbb{Q}, < \rangle$. We define a sequence of weak quasimodels q_i over $\mathcal{S}$ almost in the same way as in the proof of Theorem 27. The only difference is that now, having cured all defects at step $i+1$ and constructed a weak quasimodel

$$q'_{i+1} = \langle \mathfrak{w}_{w_1}, \dots, \mathfrak{w}_{w_m} \rangle,$$

we define q_{i+1} to be a weak quasimodel

$$q_{i+1} = \left\langle \mathfrak{w}_{u_1}, \mathfrak{w}_{w_1}, \mathfrak{w}_{u_2}, \mathfrak{w}_{w_2}, \dots, \mathfrak{w}_{u_m}, \mathfrak{w}_{w_m}, \mathfrak{w}_{u_{m+1}} \right\rangle$$

in which $\langle \mathfrak{w}_{u_i}, \mathfrak{w}_{w_i}, \sigma_i\rangle \in \mathcal{S}$ and $\left\langle \mathfrak{w}_{w_m}, \mathfrak{w}_{u_{m+1}}, \sigma_{m+1} \right\rangle \in \mathcal{S}$, for some σ_i and σ_{m+1}, $i = 1, \dots, m$, such that $u_1 < w_1 < u_2 < w_2 < \dots < u_m < w_m < u_{n+1}$. As a result we construct a quasimodel satisfying φ and based on a subframe of $\mathfrak{O}$ which is isomorphic to $\mathfrak{O}$. ❑

7 Other temporal description logics

The methods of proving decidability developed above work actually for an arbitrary decidable description logic which is *closed under the disjoint union construction of Lemma 16.* Most description logics are of this sort. Of other logics especially interesting are those which allow the construction of the concept $\{a\}$ from every object name a. Such concepts can be understood as what is know in the modal logic literature as *nominals* or *names* (see e.g. Blackburn, 1993). Using this constructor one can form then the concept $\exists R.\{a\}$. The formula $\top = \exists R.\{a\}$ is true in a model iff xRa for all objects x in its domain. It follows that logics with this constructor cannot be closed under the formation of disjoint unions.

In this section we briefly explain how to modify our proofs in order to cope with the nominal constructor. We will be considering two rather expressive decidable description logics, namely, $\mathcal{CNO}$ and $\mathcal{CIO}$, first introduced by de Giacomo (1995).

Let $\mathcal{CI}$ and $\mathcal{CN}$ be the languages resulting from $\mathcal{CIQ}$ by omitting the constructors of qualified number restrictions $\exists_{\geq n}$ and of forming inversions of roles, respectively. Now, $\mathcal{CIO}$ and $\mathcal{CNO}$ are the extensions of, respectively, $\mathcal{CI}$ and $\mathcal{CN}$ by the following concept constructor:

- $\{a\}$ is a concept whenever a is an object name.

The concept $\{a\}$ is interpreted in a model I in a straightforward manner:

- $\{a\}^I = \{a^I\}$.

Temporal description logics $\mathcal{CIO}_{\mathcal{SU}}$ and $\mathcal{CNO}_{\mathcal{SU}}$ and their semantics are defined in the obvious way (we still assume that object names are rigid designators). Having concepts of the form $\{a\}$, there is no need to define as atomic formulas $a : C$ and aRb: they are equivalent to $\{a\} \to C = \top$ and $\{a\} \to \exists R.\{b\} = \top$, respectively. Now we have:

Theorem 32. *There are algorithms that are capable of deciding whether*

1. *a given* $\mathcal{CIO}_{\mathcal{US}}$*- or* $\mathcal{CNO}_{\mathcal{US}}$*-formula is satisfiable in* $\langle \mathbb{Z}, < \rangle$ *and in* $\langle \mathbb{N}, < \rangle$*, and whether*

2. *a given* $\mathcal{CIO}_{\diamond}$*- or* $\mathcal{CNO}_{\diamond}$*-formula is satisfiable in a strict linear order as well as in* $\langle \mathbb{Q}, < \rangle$*.*

We will point out the most important modifications in the proof of Theorem 7. By $ob\varphi$ we will denote the set of object names a such that $\{a\} \in con\varphi$.

First we should change the definition of a quasiworld candidate: in the present context it is a pair $\langle T, \Phi \rangle$ such that the third condition of Definition 9 holds and

- for every $a \in ob\varphi$ there exists precisely one $t \in T$ for which $\{a\} \in t$.

Note that in the definition of a quasiworld candidate we omit the set T^o; its role is now played by the types t containing concepts of the form $\{a\}$. We denote the type t containing $\{a\}$ by t_a and define T^o to be the set of all types of the form t_a. The notion of an extended model remains the same. An extended model I realizes a quasiworld candidate iff the first condition of Definition 11 holds and

- for every $a \in ob\varphi$, $t^I(a) = t_a$.

De Giacomo (1995) proves the decidability of the satisfiability problem for both $\mathcal{CIO}$ and $\mathcal{CNO}$. So one can effectively recognize whether a quasiworld candidate is a quasiworld.

The definition of a run also requires a modification. In Definition 13 we allowed runs r in which $r(u) = t_a^u$ and $r(v) \neq t_a^v$ for some $u \neq v$; now such runs should be forbidden (in accordance with the condition that $x \in \{a\}^{I(u)}$ iff $x \in \{a\}^{I(w)}$). More precisely, a run r still has to satisfy all the conditions of Definition 13 and also the following one

- if $r(u) = t_a^v$ then $r(v) = t_a^v$, for all $u, v \in W$.

The definition of a quasimodel should be clear now. The only important thing which remains to be modified is the proof of Theorem 15. Basically this reduces to the proof of an analogue of Lemma 16. Of course, we cannot claim now that $|[x]^J| = \kappa'$ for any x in the domain of J. We reformulate this lemma in the following way.

Lemma 33. *Let Q be a quasimodel for φ based on $\langle W, < \rangle$. There is a cardinal $\kappa \geq \aleph_0$ such that, for any cardinal $\kappa' \geq \kappa$, every $\mathcal{CIO}$-quasiworld ($\mathcal{CNO}$-quasiworld) $Q(w) = \mathfrak{w}$ is realized in an extended $\mathcal{CIO}$-model ($\mathcal{CNO}$-model) J in which $\left|[x]^J\right| = \kappa'$ for all x in the domain of J different from any a^J, $a \in ob\varphi$.*

Proof The lemma is trivial if $T_w^o = T_w$, for some $w \in W$, since in this case in any quasimodel realizing $Q(w)$ every x in the domain coincides with some $a \in ob\varphi$. (Note that in this case $T_{w'}^o = T_{w'}$, for any $w' \in W$.)

So suppose this is not the case. First we consider $\mathcal{CIO}$ and $\mathcal{CNO}$ simultaneously.

For each quasiworld $Q(w) = \mathfrak{u}$ fix an extended model $I_\mathfrak{u}$ realizing $\mathfrak{u}$. Let $\Delta_\mathfrak{u}$ be the domain of $I_\mathfrak{u}$. Then we define κ to be the supremum of $\aleph_0$ and $\left|[x]^{I_\mathfrak{u}}\right|$ for all quasiworlds $Q(w) = \mathfrak{u}$ and all $x \in \Delta_\mathfrak{u}$ with $x \neq a^{I_\mathfrak{u}}$ for any $a \in ob\varphi$. We show that κ satisfies the required conditions.

Suppose $Q(w) = \mathfrak{w}$ for some $w \in W$ and $\kappa' \geq \kappa$. Take an extended model

$$I = \left\langle \Delta, R_0^I, \dots, C_0^I, \dots, (C\mathcal{U}D)^I, \dots, (C'\mathcal{S}D')^I, \dots, a_0^I, \dots \right\rangle$$

realizing $\mathfrak{w}$ and such that $\left|[x]^I\right| \leq \kappa$ for every $x \in \Delta$, $x \neq a^I$ for any $a \in ob\varphi$. Let $N = \{a^I : a \in ob\varphi\}$ and

$$J = \left\langle \Delta', R_0^J, \dots, C_0^J, \dots, (C\mathcal{U}D)^J, \dots, (C'\mathcal{S}D')^J, \dots, a_0^J, \dots \right\rangle,$$

where

$$\Delta' = N \cup \{\langle x, \xi \rangle : x \in \Delta - N, \xi < \kappa'\},$$

$$C_i^J = \{\langle x, \xi \rangle : x \in (\Delta - N) \cap C_i^I, \xi < \kappa'\} \cup (C_i^J \cap N)$$

$$a_i^J = a_i^I.$$

The definition of R_i^J depends on whether we deal with $\mathcal{CIO}$ or $\mathcal{CNO}$. In both cases we have for all $\xi < \kappa'$, $x, y \in \Delta - N$, $a, b \in ob\varphi$:

- $\langle x, \xi \rangle \, R_i^J \, \langle y, \xi \rangle$ iff $x R_i^I y$,
- $a^I R_i^J b^I$ iff $a^I R_i^I b^I$, and
- $\langle x, \xi \rangle \, R_i^J a^I$ iff $x R_i^I a^I$.

In the case of $\mathcal{CIQ}$—because of the inverse constructor—we put for all $\xi < \kappa'$, $x \in \Delta - N$:

- $a^I R_i^J \langle x, \xi \rangle$ iff $a^I R_i^I x$.

It is readily checked that J satisfies the required conditions for $\mathcal{CIO}$. However, for $\mathcal{CNO}$ this may be not the case, since a^I may have more R_i successors now. In case of $\mathcal{CNO}$ we put instead:

- $a^I R_i^J \langle x, \xi \rangle$ iff $a^I R_i^I x$ and $\xi = 0$.

With this definition the resulting model is as required for $\mathcal{CNO}$. ❑

The remaining modifications required for the decidability proof are straightforward.

8 Open problems

This paper introduces temporal description logics as an expressive and *decidable* alternative to temporal predicate logics. We have proved the decidability of the satisfiability problem for $\mathcal{CIQ}_{\mathcal{US}}$-formulas in $\langle \mathbb{N}, < \rangle$ and $\langle \mathbb{Z}, < \rangle$, and of $\mathcal{CIQ}_{\diamond}$ in strict linear orders and $\langle \mathbb{Q}, < \rangle$. It would also be of interest to find solutions to the following problems:

- Is the satisfiability problem for $\mathcal{CIQ}_{\mathcal{US}}$-formulas in strict linear orders and $\langle \mathbb{Q}, < \rangle$ decidable ?
- Is the satisfiability problem for $\mathcal{CIQ}_{\diamond}$-formulas in $\langle \mathbb{R}, < \rangle$ decidable?
- What is the complexity of the satisfiability problems considered in this paper?

In the temporal extensions of $\mathcal{CIO}$ and $\mathcal{CNO}$ we assumed that object names (and so concepts of type $\{a\}$) are *rigid* designators: their extensions are defined globally and do not depend on the particular world. By allowing object names to be interpreted locally we obtain more expressive languages.

- Is the satisfiability problem for the resulting language decidable?

As was already noted, none of the underlying description languages considered here has the finite model property (fmp). And even if we take as the basis of our temporal logics a description logic with the fmp (say $\mathcal{ALC}$), it does not follow that the resulting temporal description logic having models with finite domains coincides with the logic whose models may have arbitrary domains. (see Wolter and Zakharyaschev, 1998). This observation leads to the following problem:

- Are the temporal description logics considered in this paper decidable when the domains of models are assumed to be finite?

References

Artale and Franconi 1994. A. Artale and E. Franconi. A computational account for a description logic of time and action. In *Proceedings of the fourth Conference on Principles of Knowledge Representation and Reasoning*, pages 3–14, Montreal, Canada, 1994. Morgan Kaufman.

Baader and Hollunder 1991. F. Baader and B. Hollunder. A terminological knowledge representation system with complete inference algorithms. In *Proceedings of the workshop on Processing Declarative Knowledge, PDK-91*, pages 67–86. Lecture Notes in Artificial Intelligence, No. 567. Springer Verlag, 1991.

Baader and Laux 1995. F. Baader and A. Laux. Terminological logics with modal operators. In *Proceedings of the 14th International Joint Conference on Artificial Intelligence*, pages 808–814, Montreal, Canada, 1995. Morgan Kaufman.

Baader and Ohlbach 1995. F. Baader and H.J. Ohlbach. A multi-dimensional terminological knowledge representation language. *Journal of Applied Non-Classical Logic*, 5:153–197, 1995.

Bergamaschi and Sartori 1992. S. Bergamaschi and C. Sartori. On taxonimic reasoning in conceptual design. *ACM Trans. on Database Systems*, 17:385–422, 1992.

Blackburn 1993. P. Blackburn. Nominal tense logic. *Notre Dame Journal of Formal Logic*, 34:56–83, 1993.

Borgida *et al.* 1989. A. Borgida, R.J. Brachman, D.L McGuiness, and L. Alperin Resnick. CLASSIC: A structural data model for objects. In *Proceedings of the ACM SIGMOD International Conference on Management of Data*, pages 59–67. Portland, Oreg., 1989.

Borgida 1995. A. Borgida. Description logics in data management. *IEEE Trans. on Knowledge and Data Engineering*, 7:671–682, 1995.

Brachman and Schmolze 1985. R.J. Brachman and J.G. Schmolze. An overview of the KL-ONE knowledge representation system. *Cognitive Science*, 9:171–216, 1985.

Catarci and Lenzerini 1993. T. Catarci and M. Lenzerini. Representing and using interschema knowledge in cooperative information systems. *J. of Intelligent and Cooperative Information Systems*, 2:375–398, 1993.

de Giacomo and Lenzerini 1994. G. de Giacomo and M. Lenzerini. Boosting the correspondence between description logics and propositional dynamic logics. In *Proceedings of the 12th Nat. Conf. on Artificial Intelligence (AAAI-94)*, pages 205–212. AAAI Press/The MIT Press, 1994.

de Giacomo and Lenzerini 1996. G. de Giacomo and M. Lenzerini. TBox and ABox reasoning in expressive description logics. In *Proceedings of the fifth Conference on Principles of Knowledge Representation and Reasoning*, Montreal, Canada, 1996. Morgan Kaufman.

de Giacomo 1995. G. de Giacomo. *Decidability of Class-Based Knowledge Representation Formalisms.* PhD thesis, Univ. di Roma, 1995.

Devanbu and Litman 1991. P. Devanbu and D. Litman. Plan-based terminological reasoning. In *Proceedings of the second Conference on Principles of Knowledge Representation and Reasoning*, pages 128–138, Montreal, Canada, 1991. Morgan Kaufman.

Fagin *et al.* 1995. R. Fagin, J. Halpern, Y. Moses, and M. Vardi. *Reasoning about Knowledge.* MIT Press, 1995.

Finger and Gabbay 1992. M. Finger and D. Gabbay. Adding a temporal dimension to a logic system. *Journal of Logic, Language and Information*, 2:203–233, 1992.

Gabbay and Shehtman 1998. D. Gabbay and V. Shehtman. Products of modal logics, part 1. *Journal of the IGPL*, 6:73–146, 1998.

Gabbay *et al.* 1994. D. Gabbay, I. Hodkinson, and M. Reynolds. *Temporal Logic.* Oxford University Press, 1994.

Halpern and Shoham 1991. J. Halpern and Y. Shoham. A propositional modal logic of time intervals. *Journal of the ACM*, 38:935–962, 1991.

Kröger 1990. F. Kröger. On the interpretability of arithmetic in temporal logic. *Theoretical Computer Science*, 73:47–60, 1990.

Marx and Venema 1997. M. Marx and Y. Venema. *Multi dimensional modal logic.* Kluwer Academic Publishers, 1997.

Marx 1997. M. Marx. Complexity of products of modal logics. Submitted, 1997.

Reynolds 1996. M. Reynolds. A decidable temporal logic of parallelism. Manuscript, 1996.

Schild 1993. K. Schild. Combining terminological logics with tense logic. In *Proceedings of the 6th Portuguese Conference on Artificial Intelligence*, pages 105–120, Porto, 1993.

Schmiedel 1990. A. Schmiedel. A temporal terminological logic. In *Proceedings of the 9th National Conference of the American Association for Artificial Intelligence*, pages 640–645, Boston, 1990.

Spaan 1993. E. Spaan. *Complexity of Modal Logics.* PhD thesis, Department of Mathematics and Computer Science, University of Amsterdam, 1993.

Szalas and Holenderski 1988. A. Szalas and L. Holenderski. Incompleteness of first-order temporal logic with until. *Theoretical Computer Science*, 57:317–325, 1988.

Weida and Litman 1992. R. Weida and D. Litman. Terminological reasoning with constraint networks and an application to plan recognition. In *Proceedings of the third Conference on Principles of Knowledge Representation and Reasoning*, pages 282–293, Montreal, Canada, 1992. Morgan Kaufman.

Weida and Litman 1994. R. Weida and D. Litman. Subsumption and recognition of heterogeneous constraint networks. In *Proceedings of CAIA 94*, 1994.

Wolter and Zakharyaschev 1998. F. Wolter and M. Zakharyaschev. Satisfiability problem in description logics with modal operators. In *Proceedings of the sixth Conference on Principles of Knowledge Representation and Reasoning*, Montreal, Canada, 1998. Morgan Kaufman.

Wright *et al.* 1993. G.T. Wright, E.S. Weixelbaum, G.T. Vesonder, K.E. Brown, S.R. Palmer, J.I. Berman, and H.H. Moore. A knowledge-based configurator that supports sales, engineering, and manufacturing at AT&T network systems. *AI Magazine*, 14:69–80, 1993.