Advanced Algorithms

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Lecture 4 – Greedy algorithms

Chess vs Scrabble

- In chess the player must **think ahead**
- A winning strategy must consider the **long term consequences of moves**

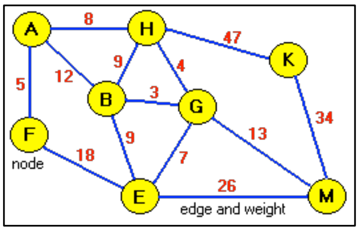
- In scrabble, it is possible to do well by simply making the **move that is best at the moment**
- The player can adopt a **greedy** strategy

Greedy algorithms

- **Greedy** algorithms build up a solution step by step, always choosing the next step that offers the **most obvious and immediate benefit**
- This approach can be **disastrous** for some computational problems
- But there are many other for which it is **optimal**
- In this lecture we will see algorithms that adopt a **greedy strategy**:
  - **Minimum spanning tree**
  - **Huffman coding**
  - **Set cover**

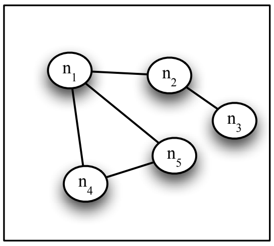
Graphs

- A graph is an abstract representation of a set of objects where some pairs of the objects are connected by links
- The interconnected objects are represented by mathematical abstractions called **nodes**
- The links that connect some pairs of nodes are called **edges**

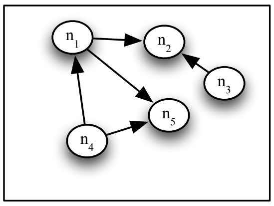


Graphs (cont.)

- **Undirected graph**
  - An ordered pair ***G* = (*V*,*E*)** where ***V*** is the set of **nodes** and ***E*** is the set **edges** (unordered pairs of nodes)



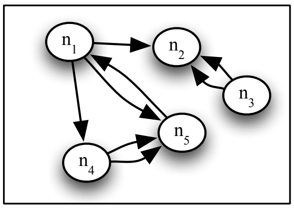
- **Directed graph or digraph**
  - An ordered pair ***D* = (*V*,*A*)** where ***V*** is the set of **nodes** and ***A*** is the set of **directed edges** (ordered pairs of nodes)



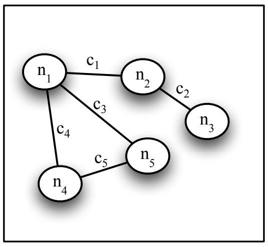
Graphs (cont.)

- **Multigraph**

  Multiple edges are allowed between the same pair of nodes

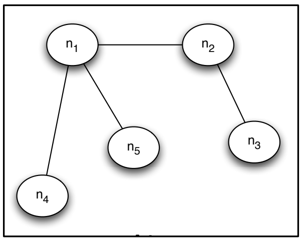


- **Weighted graph**
  - A **weight** is assigned to each edge
  - Weights represent, for example, **costs**, **lengths,** or **capacities**



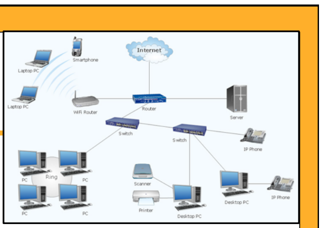
Trees

- A **tree** is an undirected graph that is **connected** and **acyclic**

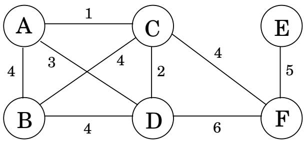


- **Properties**
  - A tree on *n* nodes has *n-1* edges
  - Any connected, undirected graph $G = (V, E)$ with $|E|=|V|-1$ is a tree
  - An undirected graph is a tree if and only if there is a unique path between any pair of nodes
    - i.e., no cycles

Building a network



- Suppose you are asked to **network a collection of computers** and other hardware **devices** by **linking selected pairs of them**
- This translates into a **graph problem** in which
  - **nodes** are hardware **devices**
  - undirected **edges** are potential **links**, each with a **maintenance cost** or **unit transfer cost**

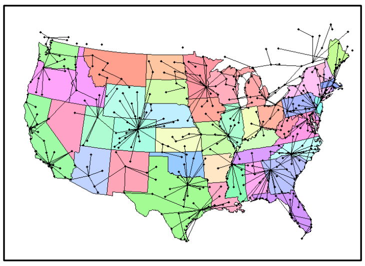


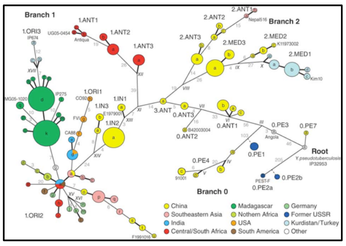
- The goal is to
  - Pick **enough edges** so that the **nodes** are **connected**
  - Keep the total **cost at minimum**
- Finding the cheapest possible network is an example of **minimum spanning tree** search

Minimum spanning tree examples

Urban walks

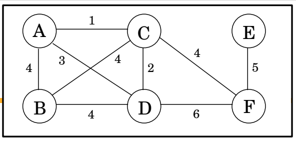
USA airport network





Bacterial agent diffusion

Network building (cont.)



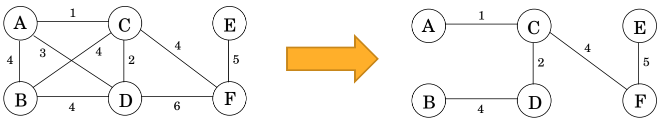
- The optimal set of edges **cannot contain a cycle**
  - removing an edge from this cycle would reduce the cost without compromising connectivity
- The **solution** is a connected acyclic graph, i.e., a **tree**
- The tree we want is the one with minimum total weight
  - The **minimum spanning tree**

Minimum spanning tree

- **Definition**
  - *Input:* An undirected graph $G = (V, E)$; edge weights w_e
  - *Output:* A tree $T = (V, E')$, with $E' \subseteq E$, that minimizes

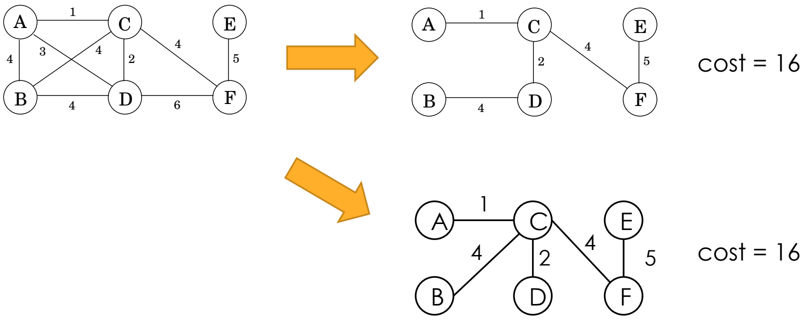
$$\text{weight}(T) = \sum_{e \in E'} w_e$$

- **Example**



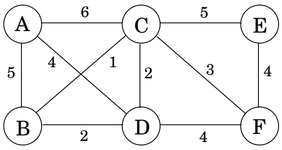
The cost is 1 + 4 + 2 + 4 + 5 =16

Is the solution unique? Can you spot another MST?

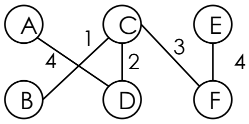


A greedy approach

- **Kruskal's minimum spanning tree algorithm** on *G = (V, E)*
  - Start with the **empty graph**
  - Repeatedly **add the next lightest edge** from *E* that **doesn't produce a cycle**
- **Example**



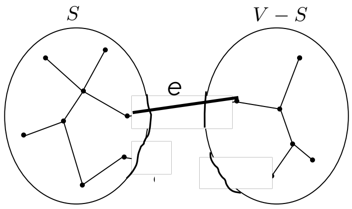
{B, C}	✔
{C, D}	✔
{B, D}	✖
{C, F}	✔
{D, F}	✖
{E, F}	✔
{A, D}	✔
{A, B}	✖
{C, E}	✖
{A, C}	✖



cost = 14

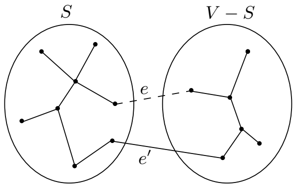
Cut property

- The correctness of Kruskal's method follows from the **cut property**
  - Suppose edges X are part of a MST of $G = (V, E)$
  - Pick any subset of nodes S for which X **does not cross** between S and V-S
  - Let e be the **lightest** edge across this partition
  - Then **$X \cup \{e\}$ is part of some MST**
- A cut is any partition of the nodes V into two groups, S and $V - S$
- The cut property says that **it is always safe to add the lightest edge across any cut, provided X has no edges across the cut**



Proof of the cut property

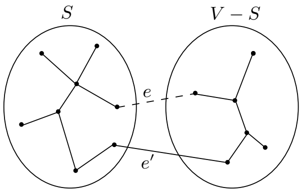
- Edges X are part of some MST T
- If the new edge e also is part of T, then there is nothing to prove
- If e is not in T, it is part of a different MST T' built as follows
  - Add edge e to T; since T is connected adding e creates a cycle
  - This cycle has another edge e' across the cut $(S, V$-$S)$
  - **$T' = T \cup \{e\} - \{e'\}$**



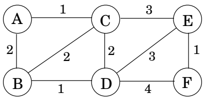
Proof of the cut property (cont.)

Is T' a minimum spanning tree?

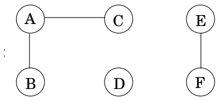
- T' is a tree (see properties of trees)
- T' is a minimum spanning tree
  - $weight(T') = weight(T) + w(e) - w(e')$
  - Both e and e' cross between S and $V - S$, and e is the lightest edge of this type
  - Therefore $w(e) \leq w(e')$, and $weight(T') \leq weight(T)$
- Since T is a *MST*, it must be the case that $weight(T') = weight(T)$ and T' is also a *MST*



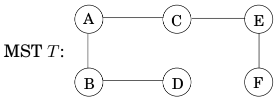
Cut property example

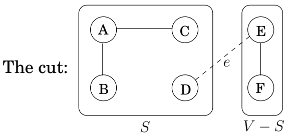


Undirected graph



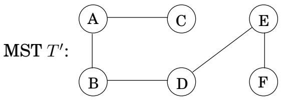
X has 3 edges and is part of MST *T*





S = {A,B,C,D} → one of the minimum-weight edges across the cut *(S, V—S)* is *e={D,E}*

X U {e} is part of MST *T´*



Kruskal’s algorithm

- At any moment**, the edges already chosen form a partial solution**
  - a collection of **connected components** each of which has a **tree structure**
- The **next edge** *e* to be added **connects two components** T_1 and T_2
  - *e* is the lightest edge that doesn’t produce a cycle
  - *e* is certain to be the lightest edge between T_1 and $V - T_1$
  - Therefore *e* is part of a *MST*, as it **satisfies the cut property**

Kruskal’s algorithm (cont.)

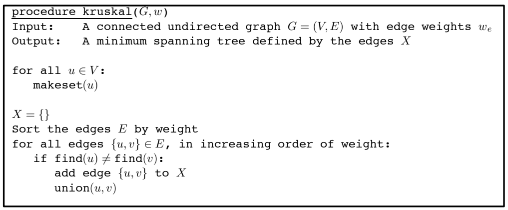
```
procedure kruskal($G, w$)
Input:    A connected undirected graph $G = (V, E)$ with edge weights $w_e$
Output:   A minimum spanning tree defined by the edges $X$

for all $u \in V$:
   makeset($u$)

$X = \{\}$
Sort the edges $E$ by weight
for all edges $\{u, v\} \in E$, in increasing order of weight:
   if find($u$) $\neq$ find($v$):
      add edge $\{u, v\}$ to $X$
      union($u, v$)
```

At each stage the algorithm chooses an edge to add to the partial solution

- test each candidate edge *{u,v}* to see whether the endpoints *u* and *v* lie in different components
- Once an edge is chosen, the two components are merged

Kruskal’s algorithm (cont.)

- The **state** is a collection of **disjoint sets**, containing the nodes of a particular component
- Initially each node is in a component by itself:
  - *makeset(x)*: **create a singleton** set containing just *x*
- Pairs of nodes are tested to see if they are in the same set
  - *find(x)*: **to which set does *x* belong?**
- Adding an edge means merging two components
  - union(x, y): **merge the sets containing x and y**
- **The algorithm uses |*V*| *makeset, 2*|E| find,* and |*V*|*-1 union***



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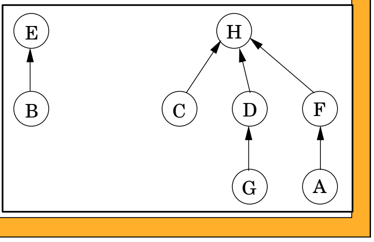
      union($u,v$)
```

A data structure for disjoint sets

We store a set by a **directed tree**

- **Nodes** of the tree are **elements** of the set, arranged in no particular order
- Each node has **parent pointers π** that eventually lead up to the **root** of the tree
- The **root** element is a convenient **representative**, or **name**, for the set
- Each node has a **rank**, the **height of the subtree** rooted in that node



A directed-tree representation of two sets {B, E} and {A, C, D, F, G, H}

rank(E)=1, rank(H)=2, rank(D)=1, rank(G)=0

Makeset and find

```
procedure makeset($x$)
$\pi(x) = x$
rank($x$) $= 0$
```

- *makeset* is a **constant-time operation**

```
function find($x$)
while $x \neq \pi(x)$:  $x = \pi(x)$
return $x$
```

- *find* follows pointers to the root and **takes time proportional to the height of the tree**

```
procedure kruskal($G, w$)
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for all $u \in V$:
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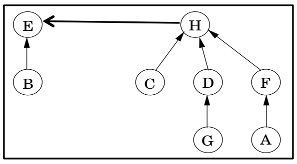
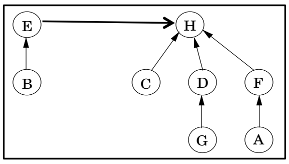
   if find($u$) $\neq$ find($v$):
      add edge $\{u, v\}$ to $X$
      union($u, v$)
```

The tree is built by *union,* which merges two sets of nodes. Any idea about how union can be implemented?

Merging two sets is make the root of one point to the root of the other



Option 1



Option 2

Which is the best option?

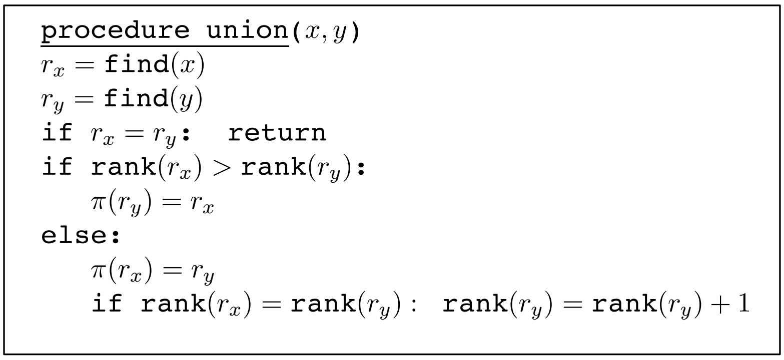
Option 2 is better than option 1, as it makes the three shallower

Union by rank

The tree is built by *union*

- **Merging** two sets is **make the root of one point to the root of the other**
- Since tree height is the main impediment to computational efficiency, a good strategy is to **make the root of the shorter tree point to the root of the taller tree**
- The overall **height increases** only if the **two trees being merged are equally tall**
- Instead of explicitly computing heights of trees, we use the *rank* numbers of their root nodes—which is why this scheme is called **union by rank**

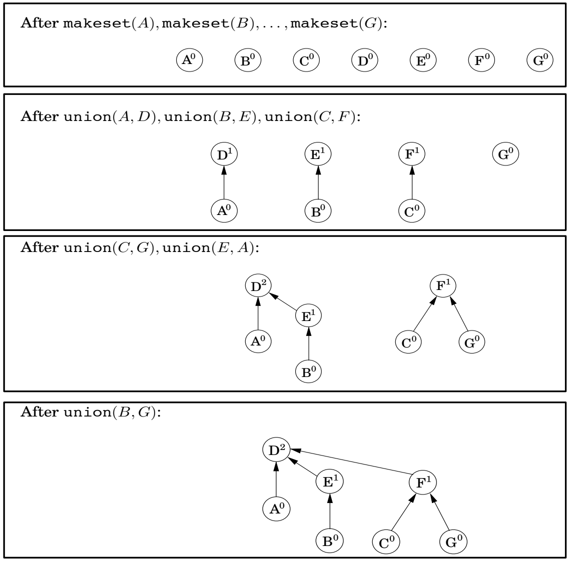
Union by rank (procedure)



```
procedure union($x, y$)
$r_x = $ find($x$)
$r_y = $ find($y$)
if $r_x = r_y$:  return
if rank($r_x$) $>$ rank($r_y$):
    $\pi(r_y) = r_x$
else:
    $\pi(r_x) = r_y$
    if rank($r_x$) $=$ rank($r_y$) :  rank($r_y$) $=$ rank($r_y$) $+ 1$
```

Union by rank (example)

```
procedure union(x, y)
$r_x = $ find$(x)$
$r_y = $ find$(y)$
if $r_x = r_y$:  return
if rank$(r_x) >$ rank$(r_y)$:
    $\pi(r_y) = r_x$
else:
    $\pi(r_x) = r_y$
    if rank$(r_x) =$ rank$(r_y)$:  rank$(r_y) =$ rank$(r_y) + 1$
```


After makeset(A), makeset(B), …, makeset(G):

After union(A, D), union(B, E), union(C, F):

After union(C, G), union(E, A):

After union(B, G):

A sequence of disjoint-set operations
Superscripts denote rank

Total time for Kruskal’s algorithm

```
procedure kruskal(G, w)
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```

```
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$\pi(x) = x$
rank$(x) = 0$
```

```
function find(x)
while $x \neq \pi(x)$:  $x = \pi(x)$
return $x$
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```
procedure union(x, y)
$r_x = $ find$(x)$
$r_y = $ find$(y)$
if $r_x = r_y$:  return
if rank$(r_x) >$ rank$(r_y)$:
    $\pi(r_y) = r_x$
else:
    $\pi(r_x) = r_y$
    if rank$(r_x) =$ rank$(r_y)$:  rank$(r_y) =$ rank$(r_y) + 1$
```

- *O(|V|)* for creating the initial trees
- *O(|E|*log|E|)=O(|E|*log|V|)*

  for sorting the edges
- ***O(|E|*log|V|)*** for the union and find

  operations that **dominate** the rest of the algorithm
  - A single *union* or *find* operation is *O(log|V|)*